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Original Article

Recovering the Heat Distribution on the Surface of a Two-dimensional Finite Slab from Interior Data in the Nonhomogeneous Case

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Abstract: We considered the two – dimensional problem of reconstructing the historical distribution on the surface of a finite slab from interior temperature data in the nonhomogeneous case. The problem is ill – posed. So, a regularization is essential. Using the integration truncation method, we have got the estimation of the error between the regularized solution and the exact solution in the nonhomogeneous case. Then, we provided a numerical experiment for illustration ofthe theoretically obtained results.

Keywords: Heat distribution, interior data, ill – posed problem, regularization, finite slab.

1. Introduction

In this work, we investigate the problem of recovering the surface heat distribution

$$u(x,0,t) = v(x,t) \tag{1}$$

such that

$$\Delta u - u_t = F(x,y,t),\ x \in \mathbb{R},\ 0 < y < 2,\ t > 0, \tag{2}$$

$$u(x,1,t) = f(x,t),\ x \in \mathbb{R},\ t > 0, \tag{3}$$

$$u(x,2,t) = g(x,t),\ x \in \mathbb{R},\ t > 0, \tag{4}$$

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$$u(x,y,0)=0,\ x\in\mathbb{R},\ 0<y<2. \quad (5)$$

where f,g are measured data and F is the heat source.

The problem (1) – (5) has been researched by many scientists because of its applications in many fields, such as physics and geology. As we know, this problem has been treated widely in one - dimensional case [1, 2] and two – dimensional case [3, 4].

For instance, in [2], Le et al. solved the following problem:

Finding $w(t)=u_x(1,t)$ such that

$$\begin{cases} u_{xx}-u_t=0,\ 1<x<a,\ t>0,\ a>2, \\ u(1,t)=f(t), t>0, \\ u(2,t)=g(t), t>0. \end{cases}$$

Moreover, the authors also considered the problem:

Finding $u(0,t)=v(t)$ such that

$$\begin{cases} u_{xx}-u_t=0,\ 0<x<1,\ t>0, \\ u(1,t)=f(t),\ t>0, \\ u_x(1,t)=w(t),\ t>0. \end{cases}$$

To regularize these problems, the authors use the Tikhonov method.

In addition, in [3], Dinh Alain et al. considered the problem (1) – (5) in the homogeneous case:

Finding $u(x,0,t)=v(x,t)$ such that

$$\begin{aligned} &\Delta u-u_t=0,\ x\in\mathbb{R},\ 0<y<2, \mathrm{t}>0, \\ &u(x,1,t)=f(x,t),\ x\in\mathbb{R},\ t>0, \\ &u(x,2,t)=g(x,t),\ x\in\mathbb{R},\ t>0, \\ &u(x,y,0)=0,\ x\in\mathbb{R},\ 0<y<2. \end{aligned}$$

Moreover, in [4], the authors solved the homogenous problem as follows

Finding $u_y(x,1,t)=w(x,t)$ such that

$$\begin{aligned} &\Delta u-u_t=0,\ x\in\mathbb{R}, 1<y<2,\ t>0, \\ &u(x,1,t)=f(x,t),\ x\in\mathbb{R},\ t>0, \\ &u(x,2,t)=g(x,t),\ x\in\mathbb{R},\ t>0, \\ &u(x,y,0)=0,\ x\in\mathbb{R},\ 1<y<2. \end{aligned}$$

By using the truncation method, the authors regularize the problem.

As is known, such problem is ill – posed. In fact, a small change in the data may lead to a large change in the solution. Hence, a regularization is in order and that is the main goal of this paper.

To the best of our knowledge, papers related to the problem (1) – (5) in the nonhomogeneous case are not much. Therefore, in this work, we regularize this problem by using truncated integration method. With different conditions on the exact solution, we will get the error estimates between the regularized solution and the exact solution.

If $v_0 \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $0 < \varepsilon < e^{-5}$ then $\|v_\varepsilon - v_0\|_2 \le C_4 \dfrac{1}{\ln(1/\varepsilon)}$,

where $C_4 > 0$ depends on v_0.

If $v_0 \in L^1(\mathbb{R}^2)$, $e^{(5/2)\sqrt{z^2+r^2}} |\hat{v}_0(z,r)| \in L^2(\mathbb{R}^2)$ and $0 < \varepsilon < e^{-7}$ then $\|v_\varepsilon - v_0\|_2 \le C_6 \varepsilon^{1/2}$,

where $C_6 > 0$ depends on v_0.

The rest of our paper is divided into three sections. In Section 2, we will find the exact solution and construct the regularized solution. Then we give the main results of the regularization method. In Section 3, we will give the proof of main results. Finally, a numerical example will be given in Section 4, which proves the effectiveness of our method.

2. Main Results

2.1. The Solution of the Problem (1) - (5)

Lemma 1 (see [5]). *Let $a > 0$. Put*

$$H(x,t) = \begin{cases} \dfrac{1}{t^2} e^{-\frac{x^2+a^2}{4t}} & (x,t) \in \mathbb{R} \times [0,+\infty) \\ 0 & (x,t) \in \mathbb{R} \times (-\infty,0) \end{cases},$$

$$K(x,t) = \begin{cases} \dfrac{1}{t} e^{-\frac{x^2+a^2}{4t}} & (x,t) \in \mathbb{R} \times [0,+\infty) \\ 0 & (x,t) \in \mathbb{R} \times (-\infty,0) \end{cases}.$$

Then

$$\hat{H}(z,r) = \frac{2}{a} e^{(-a/\sqrt{2})\sqrt{\sqrt{z^4+r^2}+z^2}} \left[\cos\left((a/\sqrt{2})\sqrt{\sqrt{z^4+r^2}-z^2} \right) - i\,\mathrm{sgn}(r) \sin\left((a/\sqrt{2})\sqrt{\sqrt{z^4+r^2}-z^2} \right) \right],$$

$$\begin{aligned} \hat{K}(z,r) = \frac{e^{(-a/\sqrt{2})\sqrt{\sqrt{z^4+r^2}+z^2}}}{\sqrt{2}\sqrt{z^4+r^2}} \Bigg[& \sqrt{\sqrt{z^4+r^2}+z^2} \cos\left((a/\sqrt{2})\sqrt{\sqrt{z^4+r^2}-z^2} \right) \\ & - \sqrt{\sqrt{z^4+r^2}-z^2} \sin\left((a/\sqrt{2})\sqrt{\sqrt{z^4+r^2}-z^2} \right) \\ & - i\,\mathrm{sgn}(r)\sqrt{\sqrt{z^4+r^2}+z^2} \sin\left((a/\sqrt{2})\sqrt{\sqrt{z^4+r^2}-z^2} \right) \\ & - i\,\mathrm{sgn}(r)\sqrt{\sqrt{z^4+r^2}-z^2} \cos\left((a/\sqrt{2})\sqrt{\sqrt{z^4+r^2}-z^2} \right) \Bigg]. \end{aligned}$$

To get the exact solution, we will transform problem (1) – (5) into a convolution integral equation.

We put

$$\Gamma(x,y,t,\xi,\eta,\tau) = \frac{1}{4\pi(t-\tau)} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4(t-\tau)}}$$

and

$$G(x,y,t,\xi,\eta,\tau)=\Gamma(x,y,t,\xi,\eta,\tau)-\Gamma(x,4-y,t,\xi,\eta,\tau).$$

It gives

$$G_{\xi\xi}+G_{\eta\eta}+G_{\tau}=0.$$

Taking integration of the identity $div(G\nabla u-u\nabla G)-(uG)_{\tau}=G.F$ over the domain $(-n,\,n)\times(1,\,2)\times(0,\,t-\varepsilon)$ and letting $\varepsilon\to 0$, after some rearrangements, we get,

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_{0}^{t}g(\xi,\tau)G_{\eta}(x,y,t,\xi,2,\tau)d\tau d\xi+\int_{-\infty}^{+\infty}\int_{0}^{t}G(x,y,t,\xi,1,\tau)u_{\eta}(\xi,1,\tau)d\tau d\xi\\&-\int_{-\infty}^{+\infty}\int_{0}^{t}f(\xi,\tau)G_{\eta}(x,y,t,\xi,1,\tau)d\tau d\xi+u(x,y,t)+\int_{-\infty}^{+\infty}\int_{0}^{t}\int_{1}^{2}G(x,y,t,\xi,\eta,\tau).F(\xi,\eta,\tau)d\eta d\tau d\xi=0.\end{aligned}\tag{6}$$

Letting $y\to 1^{+}$ in (6), we obtain

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_{0}^{t}\left[\frac{1}{2\pi(t-\tau)}e^{-\frac{(x-\xi)^2}{4(t-\tau)}}-\frac{1}{2\pi(t-\tau)}e^{-\frac{(x-\xi)^2+4}{4(t-\tau)}}\right]u_{\eta}(\xi,1,\tau)d\tau d\xi+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{0}^{t}\frac{1}{(t-\tau)^2}e^{-\frac{(x-\xi)^2+4}{4(t-\tau)}}f(\xi,\tau)d\tau d\xi\\&-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{0}^{t}\frac{1}{(t-\tau)^2}e^{-\frac{(x-\xi)^2+1}{4(t-\tau)}}g(\xi,\tau)d\tau d\xi+f(x,t)+2\int_{-\infty}^{+\infty}\int_{0}^{t}\int_{1}^{2}G(x,1,t,\xi,\eta,\tau).F(\xi,\eta,\tau)d\eta d\tau d\xi=0.\end{aligned}\tag{7}$$

We put

$$N(x,y,t,\xi,\eta,\tau)=\Gamma(x,y,t,\xi,\eta,\tau)-\Gamma(x,-y,t,\xi,\eta,\tau)$$

satisfying

$$N_{\xi\xi}+N_{\eta\eta}+N_{\tau}=0.$$

Taking integration of the identity $div(N\nabla u-u\nabla N)-(uN)_{\tau}=N.F$ over the domain $(-n,n)\times(0,\,1)\times(0,\,t-\varepsilon)$ and letting $n\to\infty,\ \varepsilon\to 0$, we have

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_{0}^{t}N(x,y,t,\xi,1,\tau)u_{\eta}(\xi,1,\tau)d\tau d\xi-\int_{-\infty}^{+\infty}\int_{0}^{t}f(\xi,\tau)N_{\eta}(x,y,t,\xi,1,\tau)d\tau d\xi\\&+\int_{-\infty}^{+\infty}\int_{0}^{t}v(\xi,\tau)N_{\eta}(x,y,t,\xi,0,\tau)d\tau d\xi-u(x,y,t)-\int_{-\infty}^{+\infty}\int_{0}^{t}\int_{0}^{1}N(x,y,t,\xi,\eta,\tau).F(\xi,\eta,\tau)d\eta d\tau d\xi=0.\end{aligned}\tag{8}$$

Letting $y\to 1^{-}$ in (8), after some computations, we get

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_{0}^{t}\frac{1}{2\pi(t-\tau)}\left[e^{-\frac{(x-\xi)^2}{4(t-\tau)}}-e^{-\frac{(x-\xi)^2+4}{4(t-\tau)}}\right]u_{\eta}(\xi,1,\tau)d\tau d\xi-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{0}^{t}\frac{1}{(t-\tau)^2}e^{-\frac{(x-\xi)^2+4}{4(t-\tau)}}f(\xi,\tau)d\tau d\xi\\&+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{0}^{t}\frac{1}{(t-\tau)^2}e^{-\frac{(x-\xi)^2+1}{4(t-\tau)}}v(\xi,\tau)d\tau d\xi-f(x,t)-2\int_{-\infty}^{+\infty}\int_{0}^{t}\int_{0}^{1}N(x,1,t,\xi,\eta,\tau).F(\xi,\eta,\tau)d\eta d\tau d\xi=0.\end{aligned}\tag{9}$$

From (7) and (9), we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t \frac{1}{(t-\tau)^2} e^{-\frac{(x-\xi)^2+1}{4(t-\tau)}} v(\xi,\tau)d\tau d\xi = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_0^t \frac{1}{(t-\tau)^2} e^{-\frac{(x-\xi)^2+4}{4(t-\tau)}} f(\xi,\tau)d\tau d\xi$$
$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t \frac{1}{(t-\tau)^2} e^{-\frac{(x-\xi)^2+1}{4(t-\tau)}} g(\xi,\tau)d\tau d\xi + 2f(x,t)$$
$$+\int_{-\infty}^{+\infty}\int_0^t\int_1^2 \frac{1}{2\pi(t-\tau)}\left[e^{-\frac{(x-\xi)^2+(1-\eta)^2}{4(t-\tau)}} - e^{-\frac{(x-\xi)^2+(3-\eta)^2}{4(t-\tau)}}\right].F(\xi,\eta,\tau)d\eta d\tau d\xi$$
$$+\int_{-\infty}^{+\infty}\int_0^t\int_0^1 \frac{1}{2\pi(t-\tau)}\left[e^{-\frac{(x-\xi)^2+(1-\eta)^2}{4(t-\tau)}} - e^{-\frac{(x-\xi)^2+(1+\eta)^2}{4(t-\tau)}}\right].F(\xi,\eta,\tau)d\eta d\tau d\xi.$$

This implies that

$$S*v(x,t) = 2R_1*f(x,t) - S*g(x,t) + 2f(x,t) + \int_1^2 (R_2-R_3)*F(x,\eta,t)d\eta$$
$$+\int_0^1 (R_2-R_4)*F(x,\eta,t)d\eta\,, \tag{10}$$

in which we define that $v(x,t) = f(x,t) = g(x,t) = 0$ as $t<0$,

$$S(x,t) = \begin{cases} \frac{1}{t^2} e^{-\frac{x^2+1}{4t}} & (x,t)\in\mathbb{R}\times[0,+\infty) \\ 0 & (x,t)\in\mathbb{R}\times(-\infty,0) \end{cases},$$

$$R_1(x,t) = \begin{cases} \frac{1}{t^2} e^{-\frac{x^2+4}{4t}} & (x,t)\in\mathbb{R}\times[0,+\infty) \\ 0 & (x,t)\in\mathbb{R}\times(-\infty,0) \end{cases},$$

$$R_2(x,\eta,t) = \begin{cases} \frac{1}{t}\left[e^{-\frac{x^2+(1-\eta)^2}{4t}}\right] & (x,\eta,t)\in\mathbb{R}\times[0,2]\times[0,+\infty) \\ 0 & (x,\eta,t)\in\mathbb{R}\times[0,2]\times(-\infty,0) \end{cases},$$

$$R_3(x,\eta,t) = \begin{cases} \frac{1}{t}\left[e^{-\frac{x^2+(3-\eta)^2}{4t}}\right] & (x,\eta,t)\in\mathbb{R}\times[0,\,2]\times[0,+\infty) \\ 0 & (x,\eta,t)\in\mathbb{R}\times[0,\,2]\times(-\infty,\,0) \end{cases},$$

$$R_4(x,\eta,t) = \begin{cases} \frac{1}{t}\left[e^{-\frac{x^2+(1+\eta)^2}{4t}}\right] & (x,\eta,t)\in\mathbb{R}\times[0,\,2]\times[0,+\infty) \\ 0 & (x,\eta,t)\in\mathbb{R}\times[0,\,2]\times(-\infty,0) \end{cases}.$$

Put

$$M(x,t)=2R_1*f(x,t)-S*g(x,t)+2f(x,t)+\int_1^2\left(R_2-R_3\right)*F(x,\eta,t)d\eta$$
$$+\int_0^1\left(R_2-R_4\right)*F(x,\eta,t)d\eta. \quad (11)$$

Taking the Fourier transform of (10), we get

$$\hat{S}(z,r).\hat{v}(z,r)=\hat{M}(z,r), \quad (12)$$

where

$$\hat{v}(z,r)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}v(x,t)e^{-i(xz+tr)}dxdt.$$

From (12), we get the exact solution of the problem (1) – (5)

$$v(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\hat{M}(z,r)}{\hat{S}(z,r)}e^{i(xz+tr)}dzdr. \quad (13)$$

In the following main results, we denote $\|\,.\,\|_2$ the $L^2(\mathbb{R}^2)$ - norm.

2.2. The Regularization of Problem (1) – (5) by Truncated Integral Method

We construct the regularized solution for the problem (1) – (5) as follows

$$v_\varepsilon(x,t)=\frac{1}{2\pi}\int_{D_\varepsilon}\frac{\hat{M}(z,r)}{\hat{S}(z,r)}e^{i(xz+tr)}dzdr, \quad (14)$$

where $D_\varepsilon=\left\{(z,r)\,/\,a_\varepsilon\le|z|\le b_\varepsilon \ and\ a_\varepsilon^2\le|r|\le b_\varepsilon^2\right\}$ with $a_\varepsilon, b_\varepsilon$ positive will be chosen later such that $\lim_{\varepsilon\to0}a_\varepsilon=0$ and $\lim_{\varepsilon\to0}b_\varepsilon=+\infty$.

Lemma 2 (The stability of regularized solution given by (14)).

Assume that $v_k\in L^2(\mathbb{R}^2)$ is the regularized solution given by (14) corresponding to the data $f_k,g_k\in L^2(\mathbb{R}^2),F_k\in L^2(0,2,L^2(\mathbb{R}^2)),k=1,2.$

Then

$$\left\|v_1-v_2\right\|_2\le C_1e^{(\sqrt{2}/2)\sqrt{\sqrt{2}+1}b_\varepsilon}\sqrt{\|f_1-f_2\|_2^2+\|g_1-g_2\|_2^2+\frac{1}{a_\varepsilon^2}\|F_1-F_2\|_{L^2(0,2,L^2(\mathbb{R}^2))}^2},$$

where $C_1>0$ is constant.

Theorem 1.

Let $\gamma\in\left(0,\dfrac{1}{2}\right)$ and $\varepsilon\in(0,\,e^{-1/\gamma})$.

Assume that $v_0\in L^2(\mathbb{R}^2)$ is the (unique) solution of (1) – (5) corresponding to the exact data $f_0,g_0\in L^2(\mathbb{R}^2),F_0\in L^2(0,2,L^2(\mathbb{R}^2))$ and $v_\varepsilon\in L^2(\mathbb{R}^2)$ is the solution given by (14) corresponding to the measured data $f_\varepsilon, g_\varepsilon \in L^2(\mathbb{R}^2), F_\varepsilon \in L^2(0,2,L^2(\mathbb{R}^2))$ satisfying $\| f_\varepsilon - f_0\|_{L^2(\mathbb{R}^2)} \le \varepsilon$, $\|g_\varepsilon - g_0\|_{L^2(\mathbb{R}^2)} \le \varepsilon$ and $\|F_\varepsilon - F_0\|_{L^2(0,2,L^2(\mathbb{R}^2))} \le \varepsilon$.

Then

$$\|v_\varepsilon - v_0\|_2 \le C_2 \varepsilon^{1-2\gamma} + \eta(\varepsilon),$$

where $C_2 > 0$ and $\eta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Remark 1.

In the above theorem, the error estimate is not good because the condition of the exact solution is not strong enough. In two following theorems, we will give some explicit error estimates with different conditions on the exact solution.

Theorem 2.

Let v_ε, v_0 be as in Theorem *1*.

Assume that $v_0 \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

If $0 < \varepsilon < e^{-5}$ then

$$\|v_\varepsilon - v_0\|_2 \le C_4 \frac{1}{\ln(1/\varepsilon)},$$

where $C_4 > 0$ depends on v_0.

Remark 2.

When $F = 0$, the error estimate has the same order with the result in [3]. To get the error estimate of order ε^p $(0 < p < 1)$, we need a stronger condition on the exact solution.

Theorem 3.

Let v_ε, v_0 be as in Theorem 1.

We assume that $v_0 \in L^1(\mathbb{R}^2)$ and $e^{(5/2)\sqrt{z^2+r^2}} |\hat{v}_0(z,r)| \in L^2(\mathbb{R}^2)$.

If $0 < \varepsilon < e^{-7}$ then

$$\|v_\varepsilon - v_0\|_2 \le C_6 \varepsilon^{1/2},$$

where $C_6 > 0$ depends on v_0.

Remark 3.

As we can see, the error estimate in theorem 3 is of the Holder type. However, the condition on the exact solution is so strong and that is one disadvantage of our method.

3. Proof of the Main Results

Proof of Lemma 2.

From Lemma 1, we have

$$\left|\hat{S}(z,r)\right|^2 = 4e^{-\sqrt{2}\sqrt{\sqrt{z^4+r^2}+z^2}},$$

$$\left|\hat{R}_2(z,r)\right|^2 = \frac{e^{-\sqrt{2}|\eta-1|\sqrt{\sqrt{z^4+r^2}+z^2}}}{\sqrt{z^4+r^2}},$$

$$\left|\hat{R}_3(z,r)\right|^2 = \frac{e^{-\sqrt{2}|3-\eta|\sqrt{\sqrt{z^4+r^2}+z^2}}}{\sqrt{z^4+r^2}},$$

$$\left|\hat{R}_4(z,r)\right|^2 = \frac{e^{-\sqrt{2}|1+\eta|\sqrt{\sqrt{z^4+r^2}+z^2}}}{\sqrt{z^4+r^2}}.$$

For $(z,r)\in D_\varepsilon$, we have

$$\begin{aligned} &\left|\hat{S}(z,r)\right|^2 \ge 4e^{-\sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon}, \\ &\left|\hat{R}_2\right|^2 + \left|\hat{R}_3\right|^2 \le \frac{\sqrt{2}}{a_\varepsilon^2}, \\ &\left|\hat{R}_2\right|^2 + \left|\hat{R}_4\right|^2 \le \frac{\sqrt{2}}{a_\varepsilon^2}. \end{aligned} \tag{15}$$

Put

$$M_1(x,t) = 2R_1 * f_1(x,t) - S * g_1(x,t) + 2f_1(x,t) + \int_1^2 \left(R_2 - R_3\right) * F_1(x,\eta,t)d\eta$$
$$+\int_0^1 \left(R_2 - R_4\right) * F_1(x,\eta,t)d\eta,$$

$$M_2(x,t) = 2R_1 * f_2(x,t) - S * g_2(x,t) + 2f_2(x,t) + \int_1^2 \left(R_2 - R_3\right) * F_2(x,\eta,t)d\eta$$
$$+\int_0^1 \left(R_2 - R_4\right) * F_2(x,\eta,t)d\eta.$$

We get

$$\|v_1 - v_2\|_2^2 = \|\hat{v}_1 - \hat{v}_2\|_2^2 = \int_{\mathbb{R}^2} \left|\hat{v}_1(z,r) - \hat{v}_2(z,r)\right|^2 dzdr = \int_{D_\varepsilon} \left|\frac{\hat{M}_1(z,r) - \hat{M}_2(z,r)}{\hat{S}(z,r)}\right|^2 dzdr.$$

Applying the inequality $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, we have

$$\|v_1 - v_2\|_2^2 \le e^{\sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon} \int_{D_\varepsilon} \Bigg(\left|\left(2+2\hat{R}_1\right)\left(\hat{f}_1 - \hat{f}_2\right)\right|^2 + \left|\hat{S}\left(\hat{g}_1 - \hat{g}_2\right)\right|^2$$
$$+\int_1^2 \left|\left(\hat{R}_2 - \hat{R}_3\right)\left(\hat{F}_1(.,\eta,.) - \hat{F}_2(.,\eta,.)\right)\right|^2 d\eta + \int_0^1 \left|\left(\hat{R}_2 - \hat{R}_4\right)\left(\hat{F}_1(.,\eta,.) - \hat{F}_2(.,\eta,.)\right)\right|^2 d\eta \Bigg) dzdr.$$

Applying (15) and the inequality $(a+b)^2 \le 2(a^2+b^2)$, we get

$$\|v_1-v_2\|_2^2 \le e^{\sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon}\Big((8+8\|R_1\|_1^2)\|f_1-f_2\|_2^2+\|S\|_1^2\|g_1-g_2\|_2^2$$
$$+\int_1^2\int_{D_\varepsilon} 2\left(\left|\hat{R}_2\right|^2+\left|\hat{R}_3\right|^2\right)\left|\left(\hat{F}_1(.,\eta,.)-\hat{F}_2(.,\eta,.)\right)\right|^2 dzdrd\eta$$
$$+\int_0^1\int_{D_\varepsilon} 2\left(\left|\hat{R}_2\right|^2+\left|\hat{R}_4\right|^2\right)\left|\left(\hat{F}_1(.,\eta,.)-\hat{F}_2(.,\eta,.)\right)\right|^2 dzdrd\eta\Big)$$
$$\le e^{\sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon}\left((8+8\|R_1\|_1^2)\|f_1-f_2\|_2^2+\|S\|_1^2\|g_1-g_2\|_2^2+\frac{2\sqrt{2}}{a_\varepsilon^2}\|F_1-F_2\|^2_{L^2(0,2,L^2(\mathbb{R}^2))}\right).$$

Therefore, it leads to

$$\|v_1-v_2\|_2^2 \le C_1^2 e^{\sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon}\left[\|f_1-f_2\|_2^2+\|g_1-g_2\|_2^2+\frac{1}{a_\varepsilon^2}\|F_1-F_2\|^2_{L^2(0,2,L^2(\mathbb{R}^2))}\right],$$

where

$$C_1^2=\max\left\{8+8\|R_1\|_1^2,\|S\|_1^2\right\}.$$

Thus, we obtain

$$\|v_1-v_2\|_2 \le C_1 e^{(\sqrt{2}/2)\sqrt{\sqrt{2}+1}b_\varepsilon}\sqrt{\|f_1-f_2\|_2^2+\|g_1-g_2\|_2^2+\frac{1}{a_\varepsilon^2}\|F_1-F_2\|^2_{L^2(0,2,L^2(\mathbb{R}^2))}}. \tag{16}$$

This completes the proof of lemma 2.

Proof of Theorem 1.

We put

$$a_\varepsilon=\varepsilon^\gamma,$$
$$b_\varepsilon=\frac{\sqrt{2}}{\sqrt{\sqrt{2}+1}}\ln\left(\frac{1}{\varepsilon^\gamma}\right).$$

Applying triangle inequality, we have

$$\left\|v_{\varepsilon_{(f_\varepsilon,g_\varepsilon,F_\varepsilon)}}-v_0\right\|_2=\left\|\hat{v}_{\varepsilon_{(f_\varepsilon,g_\varepsilon,F_\varepsilon)}}-\hat{v}_0\right\|_2\le\left\|\hat{v}_{\varepsilon_{(f_\varepsilon,g_\varepsilon,F_\varepsilon)}}-\hat{v}_{\varepsilon_{(f_0,g_0,F_0)}}\right\|_2+\left\|\hat{v}_{\varepsilon_{(f_0,g_0,F_0)}}-\hat{v}_0\right\|_2. \tag{17}$$

Applying Lemma 2 and the inequality $\sqrt{a^2+b^2+c^2}\le a+b+c$ for $a,b,c\ge 0$, we get

$$\left\|\hat{v}_{\varepsilon_{(f_\varepsilon,g_\varepsilon,F_\varepsilon)}}-\hat{v}_{\varepsilon_{(f_0,g_0,F_0)}}\right\|_2\le C_1\varepsilon^{-\gamma}\left(2\varepsilon+\varepsilon^{-\gamma}\varepsilon\right).$$

Hence

$$\left\|\hat{v}_{\varepsilon_{(f_\varepsilon,g_\varepsilon,F_\varepsilon)}}-\hat{v}_{\varepsilon_{(f_0,g_0,F_0)}}\right\|_2\le C_2\varepsilon^{1-2\gamma}, \tag{18}$$

where $C_2=3C_1$.

Moreover, putting $\eta(\varepsilon)=\sqrt{\int_{\mathbb{R}^2\setminus D_\varepsilon}|\hat{v}_0(z,r)|^2dzdr}$, we have

$$\left\|\hat{v}_{\varepsilon_{(f_0,g_0,F_0)}}-\hat{v}_0\right\|_2=\eta(\varepsilon)\to 0 \text{ as } \varepsilon\to 0. \tag{19}$$

From (17), (18) and (19), it implies that

$$\|v_\varepsilon-v_0\|_2\le C_2\varepsilon^{1-2\gamma}+\eta(\varepsilon). \tag{20}$$

This completes the proof of theorem 1.

Proof of Theorem 2.

We put

$$T_\varepsilon=[-b_\varepsilon,b_\varepsilon]\times\left[-b_\varepsilon^2,b_\varepsilon^2\right],$$
$$K_\varepsilon=[-a_\varepsilon,a_\varepsilon]\times\left[-a_\varepsilon^2,a_\varepsilon^2\right],$$
$$D_\varepsilon=T_\varepsilon\setminus K_\varepsilon.$$

From Lemma 2, it implies that

$$\left\|\hat{v}_{\varepsilon_{(F_\varepsilon,f_\varepsilon,g_\varepsilon)}}-\hat{v}_{\varepsilon_{(F_0,f_0,g_0)}}\right\|_2^2\le C_1^2e^{\sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon}\left[2\varepsilon^2+\frac{\varepsilon^2}{a_\varepsilon^2}\right]. \tag{21}$$

$$\left\|\hat{v}_{\varepsilon_{(F_0,f_0,g_0)}}-\hat{v}_0\right\|_2^2=\int_{\mathbb{R}^2\setminus D_\varepsilon}|\hat{v}_0(z,r)|^2dzdr=\int_{\mathbb{R}^2\setminus T_\varepsilon}\frac{(z^2+r^2)|\hat{v}_0(z,r)|^2}{z^2+r^2}dzdr+\int_{K_\varepsilon}|\hat{v}_0(z,r)|^2dzdr$$
$$\le\left\|\sqrt{z^2+r^2}\hat{v}_0(z,r)\right\|_2^2\frac{1}{2b_\varepsilon^2}+4\|\hat{v}_0\|_{L^\infty(\mathbb{R}^2)}^2a_\varepsilon^3.$$

Hence

$$\left\|\hat{v}_{\varepsilon_{(F_0,f_0,g_0)}}-\hat{v}_0\right\|_2^2\le\left\|\sqrt{z^2+r^2}\hat{v}_0(z,r)\right\|_2^2\frac{1}{2b_\varepsilon^2}+4\|v_0\|_1^2a_\varepsilon^3. \tag{22}$$

From (17), (21) and (22), it results in

$$\|v_\varepsilon-v_0\|_2^2\le C_3^2\left(\varepsilon^2e^{\sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon}+\frac{1}{b_\varepsilon^2}+\frac{\varepsilon^2}{a_\varepsilon^2}e^{\sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon}+a_\varepsilon^3\right), \tag{23}$$

where

$$C_3^2=\max\left\{2C_1^2,\frac{1}{2}\left\|\sqrt{z^2+r^2}\hat{v}_0(z,r)\right\|_2^2,4\|v_0\|_1^2\right\}.$$

Let b_ε be the positive solution of the following equation

$$e^{\sqrt{2}\sqrt{\sqrt{2}+1}\,b_\varepsilon}b_\varepsilon^{10/3}=\frac{1}{\varepsilon^2}. \tag{24}$$

The function $h(s)=s^{10/3}e^{\sqrt{2}\sqrt{\sqrt{2}+1}s}$ is strictly increasing in $(0;+\infty)$ and $h(\mathbb{R}^+)=\mathbb{R}^+$, so that equation (24) has a unique solution b_ε and $b_\varepsilon\to+\infty$ as $\varepsilon\to 0.$

For ε sufficiently small, it implies that

$$6b_\varepsilon \ge \sqrt{2}\sqrt{\sqrt{2}+1}b_\varepsilon+\frac{10}{3}\ln(b_\varepsilon)=2\ln\left(\frac{1}{\varepsilon}\right).$$

Therefore

$$\frac{1}{b_\varepsilon}\le\frac{3}{\ln(1/\varepsilon)}. \tag{25}$$

Putting

$$a_\varepsilon=\frac{\sqrt[3]{2}}{b_\varepsilon^{2/3}},$$

we have $a_\varepsilon\to 0$ as $\varepsilon\to 0.$

From (23) and (25), it leads to

$$\begin{aligned}\|v_\varepsilon-v_0\|_2^2 &\le C_3^2\left(\frac{1}{b_\varepsilon^{10/3}}+\frac{1}{b_\varepsilon^2}+\frac{1}{\sqrt[3]{4}b_\varepsilon^2}+\frac{2}{b_\varepsilon^2}\right)\\ &\le C_3^2\left(4+\frac{1}{\sqrt[3]{4}}\right)\frac{1}{b_\varepsilon^2}\\ &\le C_4^2\frac{1}{\ln^2(1/\varepsilon)},\end{aligned}$$

where

$$C_4^2=9C_3^2\left(4+\frac{1}{\sqrt[3]{4}}\right).$$

Thus, we obtain

$$\|v_\varepsilon-v_0\|_2\le C_4\frac{1}{\ln(1/\varepsilon)}. \tag{26}$$

This completes the proof of theorem 2.

Proof of theorem 3.

We put

$$b_\varepsilon=\frac{1}{3\sqrt{2(\sqrt{2}+1)}}\ln\left(\frac{1}{\sqrt[3]{2}\varepsilon}\right),$$
$$a_\varepsilon=2^{4/9}\varepsilon^{1/3}.$$

We have

$$\left\|\hat{v}_{\varepsilon_{(F_0,f_0,g_0)}}-\hat{v}_0\right\|_2^2=\int_{\mathbb{R}^2\setminus D_\varepsilon}\left|\hat{v}_0(z,r)\right|^2dzdr=\int_{\mathbb{R}^2\setminus T_\varepsilon}\frac{e^{5\sqrt{z^2+r^2}}\left|\hat{v}_0(z,r)\right|^2}{e^{5\sqrt{z^2+r^2}}}dzdr+\int_{K_\varepsilon}\left|\hat{v}_0(z,r)\right|^2dzdr$$

$$\le\left\|e^{(5/2)\sqrt{z^2+r^2}}\hat{v}_0(z,r)\right\|_2^2\frac{1}{e^{5\sqrt{2}b_\varepsilon}}+4\left\|\hat{v}_0\right\|_\infty^2a_\varepsilon^3.$$

$$\le\left\|e^{(5/2)\sqrt{z^2+r^2}}\hat{v}_0(z,r)\right\|_2^2\frac{1}{e^{3\sqrt{2(\sqrt{2}+1)}b_\varepsilon}}+4\left\|v_0\right\|_1^2a_\varepsilon^3. \tag{27}$$

From (23) and (27), it gives

$$\left\|v_\varepsilon-v_0\right\|_2^2\le C_5^2\left(\varepsilon^2e^{\sqrt{2(\sqrt{2}+1)}b_\varepsilon}+\frac{1}{e^{3\sqrt{2(\sqrt{2}+1)}b_\varepsilon}}+\frac{\varepsilon^2}{a_\varepsilon^2}e^{\sqrt{2(\sqrt{2}+1)}b_\varepsilon}+a_\varepsilon^3\right),$$

where

$$C_5^2=\max\left\{2C_1^2,\left\|e^{(5/2)\sqrt{z^2+r^2}}\left|\hat{v}_0(z,r)\right|\right\|_2^2,4\left\|v_0\right\|_1^2\right\}.$$

This implies that

$$\left\|v_\varepsilon-v_0\right\|_2^2\le C_5^2\left(\varepsilon^2\left(\frac{1}{2^{1/3}\varepsilon}\right)^{1/3}+2^{1/3}\varepsilon+\frac{\varepsilon^2}{2^{8/9}\varepsilon^{2/3}}\left(\frac{1}{2^{1/3}\varepsilon}\right)^{1/3}+2^{4/3}\varepsilon\right)$$

$$\le C_5^2\left(\frac{1}{2^{1/9}}\varepsilon^{5/3}+2^{1/3}\varepsilon+\frac{1}{2}\varepsilon+2^{4/3}\varepsilon\right).$$

Therefore

$$\left\|v_\varepsilon-v_0\right\|_2\le C_6\varepsilon^{1/2}, \tag{28}$$

where

$$C_6=C_5\sqrt{\left(\frac{1}{2^{1/9}}+2^{1/3}+\frac{1}{2}+2^{4/3}\right)}.$$

This completes the proof of theorem 3.

4. Numerical Example

We consider the problem:

$$\Delta u-u_t=F(x,y,t)=\left(\frac{t^2}{2}+\frac{x^2t}{4}+t-1\right)e^{-\frac{x^2-t^2-4y}{4}},\ x\in\mathbb{R},\ 0<y<2,\ t>0, \tag{29}$$

$$u(x,1,t) = f(x,t) = e^{-1}te^{\frac{-x^2-t^2}{4}},\ x \in \mathbb{R},\ t > 0, \tag{30}$$

$$u(x,2,t) = g(x,t) = e^{-2}te^{\frac{-x^2-t^2}{4}},\ x \in \mathbb{R},\ t > 0, \tag{31}$$

$$u(x,y,0) = 0,\ x \in \mathbb{R},\ 0 < y < 2. \tag{32}$$

that the unknown is

$$v(x,t) = u(x,0,t). \tag{33}$$

The exact solution of the equation is

$$v_{ex}(x,t) = te^{\frac{-x^2-t^2}{4}}. \tag{34}$$

The Fourier transform of exact solution is

$$\hat{v}_{ex}(z,r) = -4ire^{-z^2-r^2}. \tag{35}$$

Let $\varepsilon > 0$ and $\hat{f}_\varepsilon = \left(1+\frac{\varepsilon}{12}\right)\hat{f},\ \hat{g}_\varepsilon = \left(1+\frac{\varepsilon}{12}\right)\hat{g},\ \hat{F}_\varepsilon = \left(1+\frac{\varepsilon}{12}\right)\hat{F}.$

We get

$$\|f_\varepsilon - f\|_{L^2(\mathbb{R}^2)} < \varepsilon,\ \|g_\varepsilon - g\|_{L^2(\mathbb{R}^2)} < \varepsilon \text{ and } \|F_\varepsilon - F\|_{L^2(0,2,L^2(\mathbb{R}^2))} < \varepsilon.$$

We have

$$\hat{M}_\varepsilon(z,r) = \left((2\hat{R}_1+2).\hat{f}_\varepsilon\right)(z,r) - \left(\hat{S}.\hat{g}_\varepsilon\right)(z,r) - \int_1^2 \left(\left(\hat{R}_2-\hat{R}_3\right).\hat{F}_\varepsilon\right)(z,\eta,r)d\eta$$
$$+\int_0^1 \left(\left(\hat{R}_2-\hat{R}_4\right).\hat{F}_\varepsilon\right)(z,\eta,r)d\eta$$

Let

$$a_\varepsilon = 2^{4/9}\varepsilon^{1/3},$$
$$b_\varepsilon = \frac{1}{3\sqrt{2(\sqrt{2}+1)}}\ln\left(\frac{1}{\sqrt[3]{2}\varepsilon}\right). \tag{36}$$

We obtain the Fourier transform of the regularized solution

$$\hat{v}_\varepsilon(z,r) = \frac{\hat{M}_\varepsilon(z,r)}{\hat{S}(z,r)}\chi_{D_\varepsilon} = \left(1+\frac{\varepsilon}{12}\right)\hat{v}_{ex}(z,r)\chi_{D_\varepsilon} \tag{37}$$

where $D_\varepsilon = \left\{(z,r)\,/\,a_\varepsilon \le |z| \le b_\varepsilon \text{ and } a_\varepsilon^2 \le |r| \le b_\varepsilon^2\right\}.$

We consider $\varepsilon_1 = 10^{-5},\ \varepsilon_2 = 10^{-10},\ \varepsilon_3 = 10^{-15},\ \varepsilon_4 = 10^{-20},\ \varepsilon_5 = 10^{-30}$. Then we get the error estimate between the exact solution and the regularized solutions corresponding to $\varepsilon_i,\ i = 1,...,5.$ From (35), (36) and (37), we get the following table which expresses the error estimate.

Table 1. The errors between the error estimate between the regularized solution and the exact solution corresponding to $\varepsilon_i,\ i=1,...,5.$

ε	a_ε	b_ε	$\|v_\varepsilon - v_{ex}\|_2$
10^{-5}	0.0293	1.7115	0.5456
10^{-10}	$6.3163.10^{-4}$	3.458	0.0794
10^{-15}	$1.3608.10^{-5}$	5.2045	0.0117
10^{-20}	$2.9318.10^{-7}$	6.9510	0.0017
10^{-30}	$1.3608.10^{-10}$	10.444	$3.6938.10^{-5}$

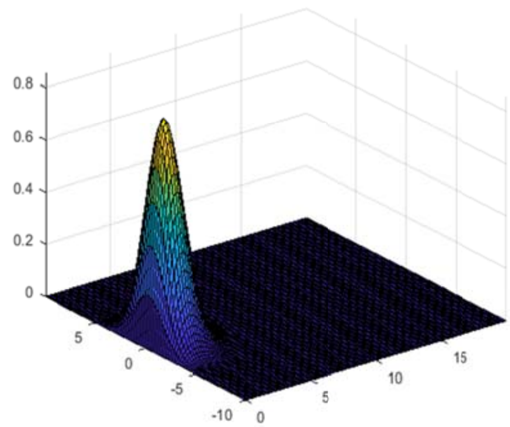

Figure 1. The 3D graph of the exact solution.

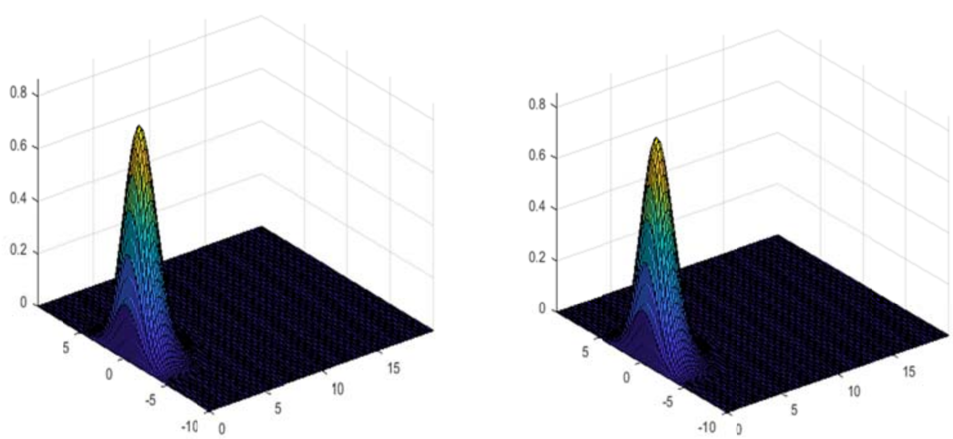

Figure 2. The 3D graph of the regularized solutions corresponding to $\varepsilon_i,\ i=1,2.$

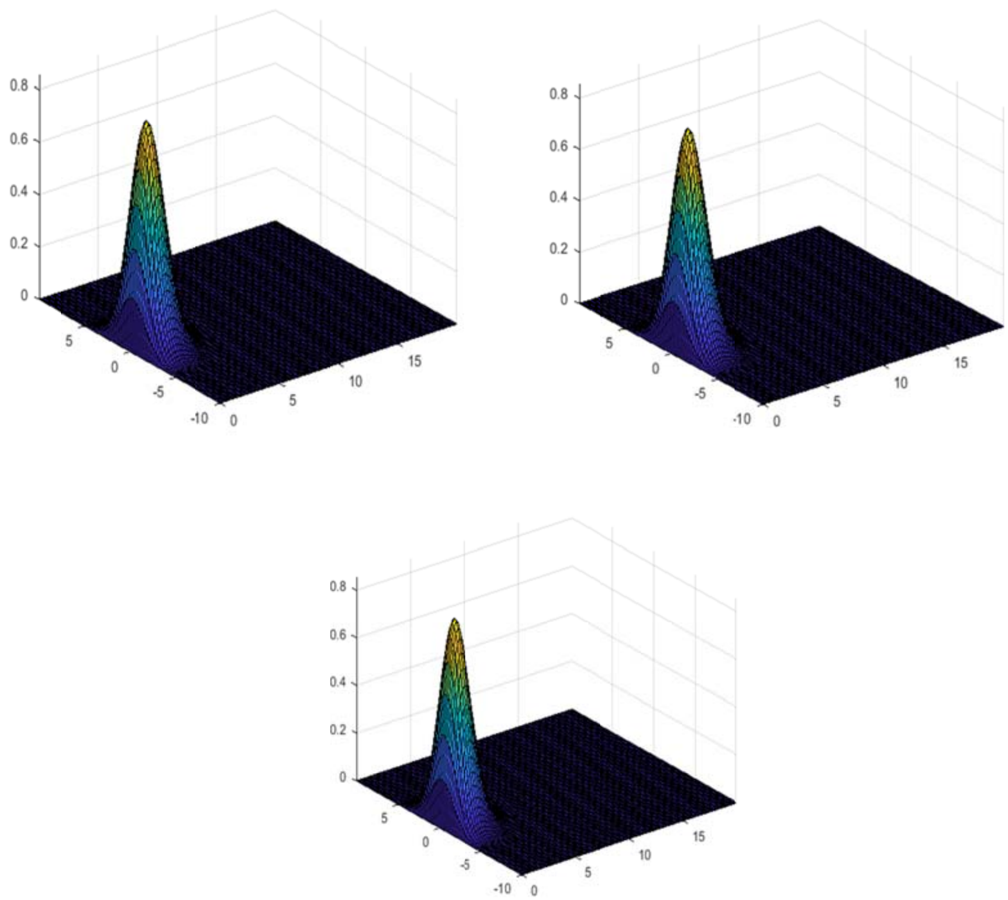

Figure 3. The 3D graph of the regularized solutions corresponding to $\varepsilon_i,\ i = 3,4,5.$

5. Conclusion

In this work, we regularized the problem of reconstructing the historical distribution from interior data in the nonhomogeneous case. We proposed the truncation method to establish the regularized solution and get the error estimate of the regularization. Moreover, we provided a numerical example to illustrate results obtained by our theoretical method. In the future, we will consider the problem in the nonlinear case.

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