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Original Article

# Interface of Two-component Bose-Einstein Condensates in Double-parabola Approximation

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**Abstract:** In this work we proposed a simple method to determine the coordinate of interface of two-component Bose-Einstein condensates (BECs) in double parabola approximation (DPA). Based on this idea, the static properties of BECs have been investigated totally in DPA

*Keywords:* Bose-Einstein condensates, Interface, Double parabola approximation.

## 1. Introduction

It is well-known that ground state of a two-component Bose-Einstein condensate (BECs) is determined by coupled Gross-Pitaevskii (GP) equations [1, 2]. In case of an immiscible BECs, an interface is formed and a phase of separation is established [3].

The interface of BECs is characterized by several quantities, such as, thickness, position, penetration length and so on. These characteristics strongly affect to both statistical and dynamic properties of the system. We start here with the simplest quantity of the interface, which is the position. A question arises naturally is that how to determine the position of the interface of BECs? In a homogeneous BECs, which is recently created by an uniform (flat-bottom) optical-box traps [4], the position of interface can be easily chosen at the origin. In case of inhomogeneous BECs, for example, BECs in the finite [5] or semi-infinite space [6], the position of interface becomes a difficult problem.

The reason for this fact is that the GP equations cannot be solve analytically. The first solution to this problem is numerical computation [7, 8]. The disadvantages of this method is, of course, we cannot obtain a equation for position of the interface and therefore other quantities is also calculated

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numerically. Two noticeable approximate approaches were proposed to deal with this problem, namely, interpolation [9] and double parabola approximation (DPA) developed by Joseph et al. [10]. However, position of the interface was only determined for the symmetric case. In general case, the numerical computation was invoked. In this work, we introduced a simple way to find position of the interface within DPA.

## 2. Interface of a Two Component Bose-Einstein Condensates Confined between Two Parallel Plates

To begin with, we consider a system of BECs confined between two parallel plates. These plates are separated at distance $2\ell$ perpendicular to *z*-axis. Along to (*x*,*y*)-directions, system is translational invariance. The stable criterion requires that the area *A* of each plate is very large compared with square of the distance between plates, i.e., $A \gg 4\ell^2$ [11]. As mentioned above, at zero temperature and without external field, the ground state can be described by coupled GP equations [2],

$$\begin{aligned} -\frac{\hbar^2}{2m_1}\frac{d^2\psi_1(z)}{dz^2} - \mu_1\psi_1(z) + g_{11}\psi_1^3(z) + g_{12}\psi_2^2(z)\psi_1(z) = 0, \\ -\frac{\hbar^2}{2m_2}\frac{d^2\psi_2(z)}{dz^2} - \mu_2\psi_2(z) + g_{22}\psi_2^3(z) + g_{12}\psi_1^2(z)\psi_2(z) = 0, \end{aligned} \tag{1}$$

in which $\hbar$ is reduced Planck constant, $m_j$ and $\mu_j$ are mass and chemical potential of component *j*, respectively. The coupling constants are defined as

$$g_{jj'} = 2\pi\hbar^2\left(\frac{1}{m_j} + \frac{1}{m_{j'}}\right)a_{jj'}, \tag{2}$$

with $a_{jj'}$ being the *s*-wave scattering length between components *j* and *j'* (*j*, *j'* = 1, 2). Let origin be at the middle point between two plates, boundary condition for wave function are imposed Dirichlet one

$$\psi_j(\pm\ell) = 0. \tag{3}$$

To seek simplification, we introduce dimensionless coordinate $\tilde{z} = z/\xi_1$ with $\xi_j = \hbar/\sqrt{2mg_{jj}n_{j0}}$ healing length, $n_{j0}$ is bulk density of component j. The reduced order parameter $\phi_j = \psi_j/\sqrt{n_{j0}}$ is used. At the two-phase coexistence, pressure in bulk of both components are equal, so that interspecies interaction can characterized by a control parameter

$$K = \frac{g_{12}}{\sqrt{g_{11}g_{22}}}. \tag{4}$$

Using Eqs. (2) and (4) one can rewrite GP equations (1) in dimensionless form

$$\begin{aligned} -\frac{d^2\phi_1}{d\tilde{z}^2} - \phi_1 + \phi_1^3 + K\phi_2^2\phi_1 = 0, \\ -\left(\frac{\xi_2}{\xi_1}\right)^2\frac{d^2\phi_2}{d\tilde{z}^2} - \phi_2 + \phi_2^3 + K\phi_1^2\phi_2 = 0, \end{aligned} \tag{5}$$

and the boundary condition (3) reduces to

$$\phi_j(\pm\tilde{\ell}) = 0, \tag{6}$$

in which $\tilde{\ell} = \ell / \xi_1$.

Now we move to solve this problem in DPA. The condensate 1 is assumed to occupy the right half-space and the remainder for component 2. Two components of condensate are separated by the interface locating at $z_0$. At this stage, we recall results for wave functions of ground state, which have found in [5]. Within DPA, Eqs. (5) have the form

$$\begin{aligned} &-\phi_1'' + 2(\phi_1 - 1) = 0, \\ &-\left(\frac{\xi_2}{\xi_1}\right)^2 \phi_2'' + \beta^2 \phi_2 = 0, \end{aligned} \tag{7}$$

in the right-hand side and in the left-hand side

$$\begin{aligned} &-\phi_1'' + \beta^2 \phi_1 = 0, \\ &-\left(\frac{\xi_2}{\xi_1}\right)^2 \phi_2'' + 2(\phi_2 - 1) = 0, \end{aligned} \tag{8}$$

where $\beta = \sqrt{K-1}$. It is obvious that Eqs. (7) and (8) are not couple. Solutions for Eqs. (7) with constraint of the boundary condition (6) have the form

$$\begin{aligned} \phi_1 &= 1 - \exp\left[\frac{\sqrt{2}(\ell - z)}{\xi_1}\right] \\ &\quad - 2A_1 \exp\left(\frac{\sqrt{2}\ell}{\xi_1}\right)\sinh\left[\frac{\sqrt{2}(\ell - z)}{\xi_1}\right], \\ \phi_2 &= -2B_1 \exp\left(\beta\frac{\ell}{\xi_2}\right)\sinh\left[\beta\frac{\ell - z}{\xi_2}\right]. \end{aligned} \tag{9}$$

Similarly, Eqs. (8) gives

$$\begin{aligned} \phi_1 &= A_2 \exp\left(\beta\frac{2\ell - z}{\xi_1}\right)\left[\exp\left(\beta\frac{2(\ell - z)}{\xi_1}\right) - 1\right], \\ \phi_2 &= \exp\left(-\sqrt{2}\,\frac{2\ell - z}{\xi_2}\right)\left\{\exp\left[\frac{\sqrt{2}(\ell + z)}{\xi_2}\right] - 1\right\} \\ &\quad \times\left\{B_2 + \exp\left(\frac{\sqrt{2}\ell}{\xi_2}\right) + B_2\exp\left[\frac{\sqrt{2}(\ell + z)}{\xi_1}\right]\right\}. \end{aligned} \tag{10}$$

Note that $A_1, A_2, B_1$ and $B_2$ in Eqs. (9) and (10) are integral constants. In principle, these constants can be evaluated by request that both the wave functions and their first derivatives to be continuous at $z = z_0$. The results are shown in Appendix.

We now focus on herein main objective, which is how to determine position of the interface. No later than proposed, the DPA has been widely applied to investigate BEC(s) in both homogenous and inhomogenous systems. In the homogenous BECs, the interface can always be chosen at the origin [10]. In a semi-infinite system of Bose gases, the position of interface was studied in both GP theory [12] and DPA [6]. In our previous work [5], this position was pointed out by solving numerically the coupled GP equations (5). It is obvious that those problems were not thoroughly solved. A simple method is proposed by Joseph et al., [13]. In this method, the wave functions are requested to be the same at the matching point $\tilde{z}_0$,

$$\phi_1(\tilde{z}_0) = \phi_2(\tilde{z}_0). \tag{11}$$

Plugging Eqs. (9) and (10) in to (11) one arrives at

$$\exp\left[\frac{\sqrt{2}(\ell - z_0)}{\xi_1}\right] + 2A_1 e^{\sqrt{2}\ell/\xi_1} \sinh\left[\frac{\sqrt{2}(\ell - z_0)}{\xi_1}\right] - 2B_1 e^{\beta\ell/\xi_2} \sinh\left[\frac{\beta(\ell - z_0)}{\xi_2}\right] - 1 = 0. \tag{12}$$

Equation (12) shows that position of the interface is not universal, which depends on the coupling constant, healing lengths and distance between two plates.

Position of the interface as a function of the inverse coupling constant is showed in Fig. 1 at several values of ratio of the healing lengths. The black, dashed and dotted curves at $\tilde{\ell} = 10$ correspond to $\xi_2 / \xi_1 = 1, 2$ and 3, respectively. In weak segregation region, Fig. 1 shows that when coupling constant tends to unity $(1/K \to 1)$ and $\xi_2 / \xi_1 > 1$, value of coordinate of the matching point increases. The component 2 will be extended to the left and component 1 will be compressed in the right half-space. Approaching to demix state, two components approach to the miscible state. In this case, the interface moves to the rightmost.

In opposite direction, at fixed value of the healing lengths, the deviation of matching point from the origin decreases when the coupling constant increases and $z_0$ approaches a constant when $K$ is large enough, which is expressed by flat region in Fig. 1. There are two cases depending on the healing length ratio:

- The symmetric case: this case is defined by unity value of the healing length ratio. The solid curve in Fig. 1 shows that the interface locates at the middle with $z_0 = 0$.

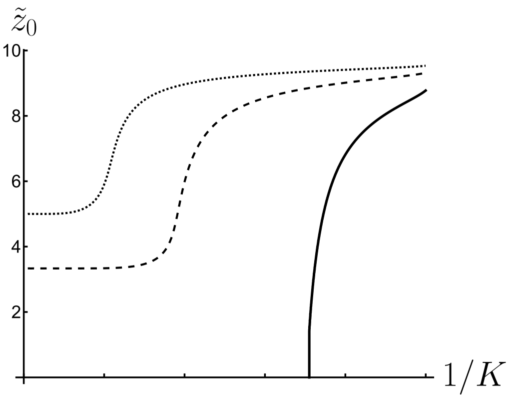


Figure 1. (Color online) Position of the interface at $\tilde{\ell} = 10$ as a function of 1/*K*. The solid, dashed and dotted curves correspond to $\xi_2 / \xi_1 = 1, 2$ and 3, respectively.

- The asymmetric case: this happens for $\xi_1 \neq \xi_2$. the interface moves to the half-space corresponding to the component with smaller healing length (the dashed and dotted curves in Fig. 1).

## 4. Conclusion

In forgoing section, the ground state of two-component BECs is totally determined by DPA. A progress is archived by DPA is that the position of interface by requiring that at the matching point, not only the wave functions and their first derivative are continuous, but also the wave functions are the same value. By this way, the ground state of a two-component BECs can be completely considered by DPA. Compared with the same problem investigated beforewhere the position of interface was investigated by numerical computation, our work can be seen as an achievement. By the way, we should note that in our previous work, the position of interface was found by numerical calculation, in which the particle number is fixed, i.e. the canonical ensemble was employed, whereas only grand canonical ensemble was invoked in this work.

In the case illustrated in Fig. 1, coordinate of the interface is positive, which is consequence of assumption that the ratio of the healing lengths is assumed to be larger than unity. The interface will move to the left if we consider for $\xi_2 < \xi_1$.

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## Appendix: Integral constants

The continuity of the wave functions and their first derivative is

$$\phi_j(\tilde{z}_0+)=\phi_j(\tilde{z}_0-),$$
$$\left.\frac{\partial\phi_j}{\partial\tilde{z}}\right|_{\tilde{z}=\tilde{z}_0+}=\left.\frac{\partial\phi_j}{\partial\tilde{z}}\right|_{\tilde{z}=\tilde{z}_0-}. \tag{A1}$$

Substituting Eqs. (8) and (9) into Eqs. (A1) one finds

$$A_1=\frac{(\sqrt{2}-\beta)e^{\ell/\xi_1}+\beta e^{[2\beta(\ell+z_0)+\sqrt{2}z_0]/\xi_1}-(\beta+\sqrt{2})e^{[2\beta(\ell+z_0)+\sqrt{2}\ell]/\xi_1}+\beta e^{\sqrt{2}z_0/\xi_1}}{\sqrt{2}(e^{2\sqrt{2}\ell/\xi_1}+e^{2\sqrt{2}z_0/\xi_1})[e^{2\beta(\ell+z_0)/\xi_1}-1]+2\beta[e^{(2\beta+\sqrt{2})(\ell+z_0)/\xi_1}+e^{\sqrt{2}(\ell+z_0)/\xi_1}]\sinh\frac{\sqrt{2}(\ell-z_0)}{\xi_1}}.$$

$$A_2=\frac{\sqrt{2}(e^{\sqrt{2}\ell/\xi_1}-e^{\sqrt{2}z_0})^2e^{\beta(2\ell+z_0)/\xi_1}}{\sqrt{2}(e^{2\sqrt{2}\ell/\xi_1}+e^{2\sqrt{2}z_0/\xi_1})[e^{2\beta(\ell+z_0)/\xi_1}-1]+2\beta[e^{(2\beta+\sqrt{2})(\ell+z_0)/\xi_1}+e^{\sqrt{2}(\ell+z_0)/\xi_1}]\sinh\frac{\sqrt{2}(\ell-z_0)}{\xi_1}}.$$

$$B_1=\frac{e^{-\beta\ell/\xi_2}\left(e^{\sqrt{2}(\ell+z_0)/\xi_2}-1\right)^2}{2\left(e^{2\sqrt{2}(\ell+z_0)/\xi_2}+1\right)\sinh\frac{\beta(\ell-z_0)}{\xi_2}+\sqrt{2}\beta[e^{2\sqrt{2}(\ell+z_0)/\xi_2}-1]\cosh\frac{\beta(\ell-z_0)}{\xi_2}}.$$

$$B_2=-\frac{e^{\sqrt{2}\ell/\xi_1}\left\{\sqrt{2}\sinh\frac{\beta(\ell-z_0)}{\xi_2}+\beta\left[e^{2\sqrt{2}(\ell+z_0)/\xi_2}-1\right]\cosh\frac{\beta(\ell-z_0)}{\xi_2}\right\}}{\sqrt{2}\left[e^{2\sqrt{2}(\ell+z_0)/\xi_2}+1\right]\sinh\frac{\beta(\ell-z_0)}{\xi_2}+\beta\left[e^{2\sqrt{2}(\ell+z_0)/\xi_2}-1\right]\cosh\frac{\beta(\ell-z_0)}{\xi_2}}.$$