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Original Article

Two Modes of the Pion Decay in Weak Interaction

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Abstract: This work presents traditional methods and new results for calculation of decay rates of positive pion in the Standard Model. Both the phenomenological and theoretical knowledge of the decays were presented. The calculation of the decays of the pion into three particles from isotopic spin conservation and the principle of vector current conservation was updated. The comparison was made with exclusive branching ratio data and inclusive charged particle distribution measurements in pion decays; The difficulty in accounting for all the one charged-particle decays as a sum of exclusive decay modes has been discussed.

Keywords: Weak interaction, pion decay, decay rate.

1. Introduction

In the field of particle physics, the process of decay refers to the transformation of one particle into two or more different particles. In the case of electro-weak interactions, which involve both electromagnetic and weak interactions, the decay process can occur between certain particles.

One specific example is the decay of a pion. Pions are pseudoscalar meson (spin 0, odd parity), which is weakly interacting particle and can decay into two or three other particles. The most common decay modes of pions are the pion decaying into a muon and a neutrino ($\pi^+ \rightarrow \mu^+\nu$) or into an electron and a neutrino ($\pi^- \rightarrow e^-\nu$). These decays occur through the weak interaction.

The specific decay modes and probabilities depend on the properties of the particles involved, as well as the energy available in the system. The study of these decays processes and their respective branching ratios is an important aspect of understanding the fundamental particles and their interactions within the framework of the electro-weak theory.

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The decay process of pions in general is of great interest in the secondary interactions of the generated particles. For example, the decay process of a pion (π) into three particles [1] typically involves the pion transforming into two photons (γ) and a neutral lepton-antilepton pair, such as an electron-positron pair (e^+e^-) or a muon-antimuon pair ($\mu^+\mu^-$). This decay process, typically occurs through the electromagnetic interaction (of a particle – antiparticle pair). However, in this paper we are interested in the weak interaction, so we consider the beta decay process of charged pions which can occur under certain circumstances.

The law of conservation of vector current is an important principle in physics, particularly in the fields of quantum mechanics and field theory. This law states that the total rate of change of the vector current is constant in an interacting process. It consists of two parts, namely the conservation of current density and electromagnetic current conservation.

Here are a few papers that study the process of pion decay into two or three particles, based on the law of conservation of vector current (CVC). Its contents concentrated in discussing the pion decay process into two photons [1], the experimental results on the measurement of the branching ratio for pion decay into an electron and a neutrino, providing insights into the vector current conservation and the determination of the pion decay constant [2], the pion decay process into two and three photons, discussing the role of chiral anomaly and vector-meson dominance in understanding the decay mechanism and constraints on the vector current conservation [3, 4] and investigates the pion decay into three charged particles [5-6].

In this paper, we study the decay of a pion particle into two or three particles associated with the conservation of vector current. The conserving conjecture, which was proposed by Zeldovich and Gerstein, Feynman and Gell-Mann in the 1950s, is the phenomenological basis of the (V-A) weak interaction theory. In Section 2, we consider the decay of a pion into two particles in the weak interaction. This is a classic process in the weak interaction of gauge theory. The results obtained are not new but are recalculated on positively charged pions. Section 3 is devoted to the calculation of the decay of a positive pion into 3 particles ($\pi^+ \to \pi^0 + e^+ + \nu_e$) with conservation of the vector current. The discussion of the obtained results is presented in section 4.

2. Pion Decays in Weak Interaction

The study of pion decay is an important source of information about weak interactions. In this section, we consider pion decay in the following mode

$$\pi^+ \to \mu^+ + \nu_\mu \tag{1}$$

In the standard model of weak interaction, Hamiltonian for the mode (1) has form

$$H_{\text{int}} = \frac{G}{\sqrt{2}}\left(\bar{\nu}_\mu \gamma_\alpha \left(1+\gamma_5\right)\mu\right) J^\alpha + h.c., \tag{2}$$

where J^α is the weak charge current of hadrons (quarks) in the Heisenberg picture; G is the Fermi constant; $c.h$ is the Hermitian conjugate symbol.

The matrix element of the decay process is

$$\begin{aligned}\langle f|S|i\rangle = -i\frac{G}{\sqrt{2}} N_{k'} N_k \bar{u}(k')\gamma_\alpha(1+\gamma_5)u(-k)\times \\ \times\langle 0|J^\alpha(0)|q\rangle(2\pi)^4\delta(k'+k-q)\end{aligned}, \tag{3}$$

where k, k' and q are the momentum of the muon, neutrino and pion, respectively; $N_k = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k^0}}$ is the normalization constant. Process (1) is schematically shown in the Fig. 1.

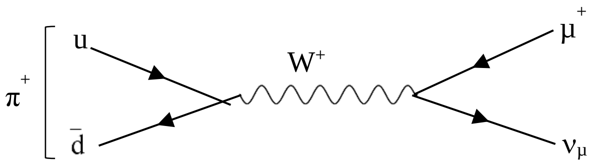

Figure 1. Pion decay $\pi^+ \to \mu^+ + \nu_\mu$.

Consider the hadron matrix element $\langle 0|J^\alpha(0)|q\rangle$ where J^α is the sum of the vector current V^α and axial-vector current A^α (V-A interaction):

$$J^\alpha = V^\alpha + A^\alpha \tag{4}$$

Both matrix elements $\langle 0|V^\alpha(0)|q\rangle$ and $\langle 0|A^\alpha(0)|q\rangle$ depend only on momentum q^α. Due to the Lorentz invariant, we can write

$$\langle 0|V^\alpha(0)|q\rangle = N_q f_V(q^2)q^\alpha; \quad \langle 0|A^\alpha(0)|q\rangle = N_q f_A(q^2)q^\alpha, \tag{5}$$

here $q^2 = m_\pi^2$ (m_π is the pion mass), f_V and f_A are decay constants, expressing the effect of the pion into vacuum.

Since the strong interaction preserves parity, the matrix elements $\langle 0|V^\alpha(0)|q\rangle$ and $\langle 0|A^\alpha(0)|q\rangle$ will transform as follows with the inverse

$$U_P V^\alpha(0) U_P^{-1} = \eta^\alpha V^\alpha(0); \; U_p A^\alpha(0) U_P^{-1} = -\eta^\alpha A^\alpha(0), \tag{6}$$

where U_P is the inverse operator, $\eta^0 = 1$; $\eta^i = -1$. We have

$$U_P|0\rangle = |0\rangle; \; U_P|q\rangle = I_\pi |q'\rangle \tag{7}$$

where $q'^\alpha = (q^0, -q)$ and π meson is pseudoscalar, its intrinsic parity is -1, $I_\pi = -1$

Using (6) and (7) we obtained

$$\langle 0|V^\alpha(0)|q\rangle = \langle 0|U_P^{-1}U_P V^\alpha(0) U_P^{-1} U_P|q\rangle = -\eta^\alpha \langle 0|V^\alpha(0)|q'\rangle \;, \tag{8}$$

$$\langle 0|A^\alpha(0)|q\rangle = \eta^\alpha \langle 0|A^\alpha(0)|q'\rangle \tag{9}$$

Substitute (5) into (8), (9) and notice that $q'^\alpha \eta^\alpha = q^\alpha$ we get

$$f_V = -f_V; \; f_A = f_A \Rightarrow f_V = 0 \tag{10}$$

Thus, only the axial current[1] contributes to the matrix element $\langle 0 | J^{\alpha}(0) | q \rangle$.

In gauge theory, the current that we are interested in has the form $J^{\alpha} = d\gamma^{\alpha}(1+\gamma_5)uU_{ud}$, U_{ud} is the Cabibbo - Kobayashi –Maskawa (CKM) matrix element.

If we write $f_A = if_{\pi}U_{ud}$, f_{π} is the real constant (experiment $f_{\pi} = 0,932m_{\pi}$). It contains all strong interaction and has mass dimensional. Then

$$\langle 0 | J^{\alpha}(0) | q \rangle = iN_q f_{\pi} U_{ud} q^{\alpha} \tag{11}$$

From (3) and (10), we get the matrix element of the decay process

$$\begin{aligned} \langle f | S | i \rangle = \frac{G}{\sqrt{2}} f_{\pi} U_{ud} N_{k'} N_k N_q \bar{u}(k')q(1+\gamma_5)u(-k) \times \\ \times (2\pi)^4 \delta(k+k'-q) \end{aligned} \tag{12}$$

To simple (12), we used both the law of conservation of momentum (replace q with $k+k'$) and the Dirac equation[2]

From (12), it is indicated that the matrix element of the decay process is proportional to the muon mass and becomes zero when $m_{\mu} \to 0$. This involves a charge current consisting of only the left components of the field. When $m_{\mu} \to 0$, the helicity of μ^+ is 1 (similarly to the helicity of the anti-neutrino). Considering the decay process a in the rest frame of the pion, the momenta of μ^+ and ν_{μ} will be equal in magnitude and opposite in direction. Clearly, the projection of the momenta of μ^+ and ν_{μ} onto the neutrino's motion direction is when $m_{\mu} \to 0$ is -1 (Fig. 2). Due to the pion's spin being zero, the decay of $\pi^+ \to \mu^+ + \nu_{\mu}$ is forbidden within the limit $m_{\mu} \to 0$, as conservation of momentum law holds. Therefore, in the gauge theory, the decay process (1) is allowed only when μ^+ can be found in a state with negative helicity (due to finite mass).

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[1] This can be proven otherwise. Specifically, the matrix element $N_q^{-1} \langle 0 | V^{\alpha}(0) | 0 \rangle$ is transformed as a pseudo-vector. Since it is not possible to construct a pseudo-vector from a vector, $\langle 0 | V^{\alpha}(0) | q \rangle = 0$. The matrix element is transformed as a vector and has the form $f_A(q^2)q^{\alpha}$.

[2] If the Dirac equation allows to reduce the number of matrices γ in the matrix element, then when we calculate the trace, it will reduce the number of matrices γ .

Figure 2. Represent the momentum and spins of μ^+ and ν_μ in the stationary system of the pion when $m_\mu \to 0$. The direction of the spin is plotted with double arrows.

Now, we calculate the decay rate of the pion decay process $\pi^+ \to \mu^+ + \nu_\mu$. In the rest frame of pion $p = \left(m_\pi, 0, 0, 0\right)$, the differential decay rate of this process can be written as [7, 8]:

$$d\Gamma_{\pi\to\mu\nu_\mu} = \frac{G^2}{2}\frac{1}{(2\pi)^2}\frac{1}{2m_\pi}m_\mu{}^2 U_{ud}{}^2 f_\pi{}^2 \times$$
$$Tr(1-\gamma_5)(\hat{k}-m_\mu)(1+\gamma_5)\hat{k}'\delta(k'+k-q)\frac{d^3k'}{2k_0'}\frac{d^3k}{2k_0} \tag{13}$$

The trace of this expression is equal to

$$Tr(1-\gamma_5)(\hat{k}-m_\mu)(1+\gamma_5)\hat{k}' = 8kk' \tag{14}$$

Use the 4- momentum conservation law to deduce

$$2kk' = m_\pi{}^2 - m_\mu{}^2. \tag{15}$$

Integrating with respect to the momentum of the particles in the final state and using δ - function

$$\int \delta(k'+k-q)\frac{d^3k'}{2k_0'}\frac{d^3k}{2k_0} = \int \delta(k'+k-q)\delta(k'^2)d^4k'\frac{d^3k}{2k_0}$$
$$= \int \delta(m_\pi{}^2 + m_\mu{}^2 - 2m_\pi k_0)|k|\frac{1}{2}dk_0 d\Omega = \frac{m_\pi{}^2 - m_\mu{}^2}{8m_\pi{}^2}d\Omega \tag{16}$$

From (16), the expression of the differential decay rate is derived

$$d\Gamma_{\pi\to\mu\nu_\mu} = \frac{G^2}{8\pi} f_\pi{}^2 U^2{}_{ud} m_\mu{}^2\left(1-\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2\frac{d\Omega}{4\pi} \tag{17}$$

Obviously, we see that the decay rate does not depend on the rotation angle of the muon. This is related to the pion's spin being zero.

Integrating (17) over solid angle Ω, we find the total decay rate of the decay process

$$\Gamma_{\pi\to\mu\nu_\mu} = \frac{G^2}{8\pi} f_\pi{}^2 U^2{}_{ud} m_\mu{}^2 m_\pi\left(1-\frac{m_\mu{}^2}{m_\pi{}^2}\right)^2 \tag{18}$$

In a similar way when studying the decay branch

$$\pi^+ \to e^+ + \nu_e \tag{19}$$

On the concept of $\mu \to e$, $\nu_\mu \to \nu_e$ universality, the decay rate of the process (19) can be obtained from expression (18) by just replacing μ by e and ν_μ by ν_e and we get

$$\Gamma_{\pi\to e\nu_e} = \frac{G^2}{8\pi} f_\pi^{\ 2} U^2_{\ ud} m_e^{\ 2} m_\pi \left(1 - \frac{m_e^{\ 2}}{m_\pi^{\ 2}}\right)^2, \tag{20}$$

where m_e is the electron mass.

From (18) and (20), the theory ratio of branching ratios (1) and (19) is determined by

$$R_\pi = \frac{\Gamma_{\pi\to e\nu_e}}{\Gamma_{\pi\to\mu\nu_\mu}} = \frac{m_e^{\ 2}}{m_\mu^{\ 2}} \frac{\left(1 - \frac{m_e^{\ 2}}{m_\pi^{\ 2}}\right)^2}{\left(1 - \frac{m_\mu^{\ 2}}{m_\pi^{\ 2}}\right)^2}. \tag{21}$$

Thus, this rate ratio depends only on the mass of electron, muon, and pion. From the mass of these particles $m_\pi \approx 139 MeV, m_e \approx 0,511 MeV, m_\mu \approx 105,1 MeV$, we get

$$R_\pi \approx 1.288 \times 10^{-4} \tag{22}$$

The above ratio is derived from experimentally observed by [9].

$$R_\pi = 1.233 \times 10^{-4} \tag{23}$$

The concordance between (22) and (23) is one of the important proofs of the universality of the weak interaction. It should also be emphasized that comparing the decay rates and allowing us to check the universality of the weak interaction of pairs of hadrons has only their axial contribution.

3. Pion Decay Into 3 – Particles and the Law of Conservation of Vector Current

In this section, we consider the decay of a pion into three particles due to the weak interaction, but take into account the strong interaction and the structure of the particles through the form factor. The form factor is determined through Lorentz invariance and the vector current conservation law.

The β - decay process of positive pion (as shown in Fig. 3) is described by equation:

$$\pi^+ \to \pi^0 + e^+ + \nu_e \tag{24}$$

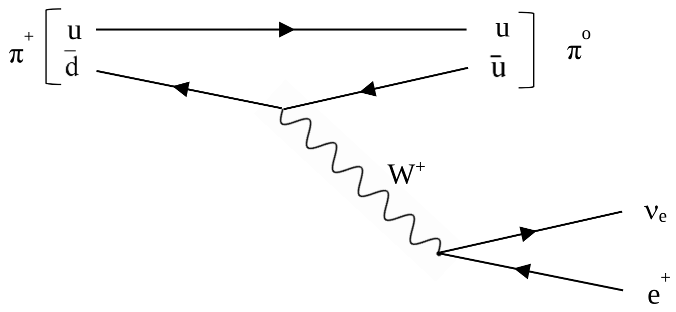

Figure 3. Feynman pion decay diagram $\pi^+ \to \pi^0 + e^+ + \nu_e$.

Studying the decay process experimentally will allow us to test the current vector conservation. For this purpose, we consider the weak charged current, theoretically constructed from u-quarks and d-quarks, as the sum of the vector and axial currents respectively, similar to (4).

Hamiltonian for the mode (24) has form

$$H_I = \frac{G}{\sqrt{2}}\left(\overline{\nu_e}\gamma_\alpha\left(1+\gamma_5\right)e\right)J^\alpha + h.c$$

and matrix elements of the decay is read

$$\begin{aligned}\left\langle f \mid S \mid i\right\rangle = -i\frac{G}{\sqrt{2}}N_k N_{k'}\bar{u}\left(k'\right)\gamma_\alpha\left(1+\gamma_5\right)u\left(-k\right) \\ \times\left\langle p' \mid J^\alpha \mid p\right\rangle\left(2\pi\right)^4\delta\left(k+k'+p'-p\right)\end{aligned}, \quad (25)$$

where k and k’ are momentums of electron and neutrino; p and p’ are momentums of π^+ and π^0, respectively. J^α is the current of weakly charged hadrons in the Heisenberg representation

$${}_{\pi^0}\left\langle p' \mid J^\alpha \mid p\right\rangle_{\pi^+} = U_{ud}\left\langle p' \mid V^\alpha \mid p\right\rangle.$$

Since the strong interaction is invariant under isotopic changes, the vector current satisfies the continuity equation.

$$\partial_\alpha V^\alpha = 0 \quad (26)$$

Because the axial current is considered to have no contribution to the matrix element of the decay, it is calculated according to the formula

$$\left\langle p' \mid V^\alpha\left(0\right) \mid p\right\rangle = N_p N_{p'}\left[f_+\left(q^2\right)\left(p+p'\right)^\alpha + f_-\left(q^2\right)\left(p-p'\right)^\alpha\right] \quad (27)$$

where $f_+\left(q^2\right)$ and $f_-\left(q^2\right)$ are form factors, they are real functions of the square of momentum $q^2 = \left(p'-p\right)^2 = \left(k+k'\right)^2$

From (27), neglecting the small term in the matrix element of the electroweak vector current in the pion decay process, with the electromagnetic form factor $f_+ = \sqrt{2}$, $f_- = 0$ [8], we can write

$$U_{ud\,\pi^0}\left\langle p' \mid V^\alpha \mid p\right\rangle_{\pi^+} \cong \mathrm{N}_{p'}N_p\sqrt{2}\left(p+p'\right)^\alpha U_{ud}. \quad (28)$$

We start again with Fermi’s golden rule [7] to calculate the total probability of the β -decay of pion

$$\begin{aligned}\frac{d\Gamma_{\pi^+\to\pi^0 e^+\nu_e}}{dq^2} = \frac{G^2}{\left(2\pi\right)^3}U_{\mathrm{ud}}^2\frac{\lambda\left(q^2, m_{\pi^+}^2, m_{\pi^0}^2\right)}{48m_{\pi^+}^3}\left(1-\frac{m_e^2}{q^2}\right) \\ \times\left[4\left(1+\frac{m_e^2}{2q^2}\right)\lambda^2\left(q^2, m_{\pi^+}^2, m_{\pi^0}^2\right) + 6\frac{m_e^2}{2q^2}\left(m_{\pi^+}^2 - m_{\pi^0}^2\right)^2\right]\end{aligned} \quad (29)$$

where

$$\lambda\left(q^2, m_{\pi^+}^2, m_{\pi^+}^2\right) = \Delta\left(m_{\pi^+} + m_{\pi^0}\right)\sqrt{1-x}\left(1 - x\frac{\Delta^2}{\left(m_{\pi^+} + m_{\pi^0}\right)^2}\right)^{\frac{1}{2}} \cong \Delta\left(m_{\pi^+} + m_{\pi^0}\right)\sqrt{1-x} \quad (30)$$

x is a dimensionless variable, $x = q^2/\Delta^2$ and $\Delta = m_{\pi^+} - m_{\pi^0}$. Clearly, $\varepsilon \le x \le 1$, with $\varepsilon = m_e^2/\Delta^2 \approx 1,24.10^{-2}$.

Integrating (29) with respect to x, we obtain

$$\Gamma_{\pi^+\to\pi^0 e^+\nu_e} = \frac{G^2}{(2\pi)^3}\frac{\Delta^5}{24}U_{ud}^2\left(1 + \frac{m_{\pi^0}}{m_{\pi^+}}\right)^3 \times\int_c^1 \frac{dx}{x}\sqrt{1-x}\left\{\left[(2x+\varepsilon)(1-x)+3\varepsilon\right]\left(1 - \frac{m_{\pi^0}}{m_{\pi^+}}\right)\times \times\left[\frac{2}{5}\sqrt{1-e}\left(2-9\varepsilon+8\varepsilon^2\right)+6\varepsilon^2\ln\frac{1+\sqrt{1-e}}{\varepsilon}\right]\right\} \quad (31)$$

From (31), the ratio of the probability of β - decay to the probability of decay of π^+ into mesons in section 2 is approximately equal

$$R = (1.048 \pm 0.005)\times 10^{-8}.$$

and the average value of the β - decay probability over all measured pion decay types is equal

$$R = (1.025 \pm 0.034)\times 10^{-8}$$

Within the framework of the V −A theory of the weak interactions, the pion beta decay rate can be expressed in terms of the leading-order width Γ_0 and the radiative and loop corrections δ_π as [8]

$$\Gamma = \Gamma_0(1+\delta_\pi) = \frac{G^2}{30\pi^3}\frac{\Delta^5}{24}U_{ud}^2 f(\varepsilon,\Delta)\left(1 - \frac{\Delta}{2m_{\pi^+}}\right)^3(1+\delta_\pi), \quad (32)$$

up to leading order in $\frac{\Delta^2}{(m_{\pi^+} + m_{\pi^0})^2}$ □ 2.8×10^{-4}, the function f(ε;Δ) has the value of *f(ε; Δ)* ≈ 0.94104. The overall uncertainty of the rate in Eq. (32) is dominated by two comparable contributions, one from the Δ^5 factor, and the other from the δ_π radiative/loop corrections. Thus, the pion beta decay rate provides a direct means to determine the CKM matrix element $|U_{ud}|$. In fact, being free of nuclear structure corrections present in super allowed nuclear beta decays, and free of tree-level axial corrections present in neutron beta decay, $\pi^+ \to \pi^0 e^+ \nu_e$ decay offers the theoretically cleanest path to measuring U_{ud} and, hence, testing quark-lepton universality. However, the extremely low branching ratio for the process has so far limited the experimental accuracy [9].

Experimentally, $\pi^+ \to \pi^0 e^+ \nu_e$ decay is observed primarily through detection of the final decay particles produced in the near-instantaneous neutral pion decay ($\tau \approx 8.5 \times 10^{-17}$ s). The positron is generally harder to detect with precise efficiency, except for π^+ decays in flight, since its kinetic energy ranges from 0 to only 4 MeV.

In 1967, the CERN apparatus had an acceptance of ~ 22.4%, and enabled the experimenters to record pion beta decays from ~ 1,5 × 10^{11} pion stops in the target. The reported branching ratio value was

$$R = \frac{\Gamma_{(\pi^+ \to \pi^0 e^+ \nu)}}{\Gamma_{(\pi^+ \to \mu^+ \nu)}} \approx (1.00 \pm 0.08) \times 10^{-8}$$

in very good agreement with CVC theory predictions above.

It should be noted that, according to reference [9], the primary goal of the PIBETA experiment conducted at PSI (the Swiss Institute for Nuclear Research, now part of PSI) was to improve by an order of magnitude the existing experimental precision of the pion beta decay branching ratio. During three runs in 1999, 2000 and 2001, the PIBETA collaboration acquired over 64,000 pion beta decay events, which led to an improvement of the experimental precision of the $\pi^+ \to \pi^0 e^+ \nu_e$ branching ratio of about an order of magnitude. The PIBETA result was evaluated in two ways and given values [9]:

R= [1.036 ± 0.004(stat) ± 0.004(syst)] × 10^{-8};

and R= [1.040 ± 0.004(stat) ± 0.004(syst)] × 10^{-8}

There was every incentive to pursue a higher precision result in the pion beta decay rate or branching ratio. For more accurate results on the beta pion decay rate or branching ratio, it is theoretically necessary to correct the loop correction and renormalize the quantity δ_π in Eq. 32.

4. Conclusion

In summary, for charged pions only, is a very rare "pion beta decay" (with branching fraction of about 10^{-8}) into a neutral pion, an electron and an electron antineutrino (or for positive pions, a neutral pion, a positron, and electron neutrino). The rate at which pions decay is a prominent quantity in many sub-fields of particle physics, such as chiral perturbation theory. This rate is parametrized by the pion decay constant ($f\pi$), related to the wave function overlap of the quark and antiquark, which is about 130MeV as reported in [10].

A close look at the recent record of study of the decays of the charged pion reveals a great deal of activity and continued strong relevance. The extraordinary precision of the theoretical description of the pion decays remains unmatched by the available experimental results. There will remain considerable room for improvement of experimental precision with limits on physics not included in the present Standard Model. This work remains relevant and complementary to the direct searches on the energy frontier currently under way at particle colliders, with considerable theoretical significance from [9].

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