VNU Journal of Science: Mathematics – Physics, Vol. 40, No. 2 (2024) 56-69

VNU Journal of Science: Mathematics - Physics

Journal homepage: https://js.vnu.edu.vn/MaP

Original Article

Three Regularization Methods for A Homogeneous Time Fractional Diffusion Equation with Space - Dependent Diffusivity Coefficient

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Received 24 November 2023
Revised 06 February 2024; Accepted 22 April 2024

Abstract: We studied an inverse problem for a time fractional diffusion equation with space - dependent diffusivity coefficient in the nonhomogeneous case. This problem is ill – posed in Hadamard's sense. So, a regularization is essential. Using three regularization methods, we get the estimation of the error between the regularized solution and the exact solution

Keywords: Time fractional diffusion equation, inverse problem, regularization, space - dependent coefficient, error estimate.

1. Introduction

Fractional derivaties were introduced in the 17^{th} century and since then, they have become more popular because they can be used to model natural and engineering phenomena that integer derivatives can not be used. Fractional mathematical models have been studied much due to their wide range of applications in various fields such as in biology, physics, chemistry, medical imaging, control theory, finance, population dynamics, ecology, engineering, signal processing, etc.

Time - fractional diffusion equation is commonly used to describe anomalous diffusion such as in flow of viscoelastic fluid, movement of some kinds of biological individuals, diffusive phenonmenon of substances causing the pollution in the environment.

Inverse problem for time fractional diffusion equations has been considered much by many mathematicians in previous studies [1-4].

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https//doi.org/10.25073/2588-1124/vnumap.4898

Inspired by the various applications of time fractional partial differential equation, in this work, we investigate the problem of recovering the temperature $u(x,t)$ satisfying the following time - fractional diffusion equation:

$$\begin{cases} \dfrac{D^{\beta}u(x,t)}{Dt^{\beta}} + a(x)u_x(x,t) = 0, \ \ x>0,\ t>0,\ 0<\beta<1, \\ u(1,t) = g(t),\ t \ge 0, \\ \lim\limits_{x\to+\infty} u(x,t) = u(x,0) = 0, \end{cases} \tag{1}$$

where $g(t)$ is the given data and $a(x)$ is a space – dependent diffusion coefficient such that $0 < p \le a(x) \le q$, $\beta \in (0,1)$ is the fractional order and $\dfrac{D^{\beta}u(t)}{Dt^{\beta}}$ denotes the Caputo fractional derivative with respect to t :

$$\frac{D^{\beta}u(t)}{Dt^{\beta}} = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}u'(s)ds,$$

where Γ is the Gamma function: $\Gamma(p) = \int_0^{+\infty} x^{p-1}e^{-x}dx$.

The problem (1) in various cases has been studied by many scientists. As is known, the problem is ill – posed and a regularization is necessary. For example, in [5], Cheng and Fu considered the problem (1) in the homogeneous case, in the case $a(x)=1$. The authors regularized the problem by using a new iteration method. Moreover, in [6, 7], Zheng and Wei considered the problem (1) in the case $a(x)=a$. The authors used a spectral regularization method and a new regularization method to deal with the problem. In the inhomogeneous case, Hoan and co-authors presented the truncation method to regularize this problem [8]. In addition, Tuan et al. considered the problem with constant coefficient in the case the source function is nonlinear [9, 10]. The authors used the quasi - boundary value method to solve this problem.

Until now, to the best of our knowledge, the regularization results concerning for the time fractional diffusion equation with space dependent diffusivity coefficient are not much. Motivated by this reason, in this work, we propose three regularization methods for the problem and get the error estimates between the regularized solution and the exact solution under different conditions.

The remainder of the present work is divided into three sections. In section 2, we will find the exact solution of the problem (1). In section 3, we prove the ill-posedness of the problem. In section 4, we give the main results of the regularization methods.

2. The Solution of the Problem

We denote the Fourier transform of a function:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-it\xi}dt,$$

and its inversion

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{it\xi}d\xi.$$

We define that $u(x,t) = g(t) = 0$ as $t < 0$.

By taking Fourier transform of the problem (1) with respect to *t*, we get

$$(i\xi)^{\beta}\hat{u}(x,\xi) + a(x)\frac{d}{dx}\hat{u}(x,\xi) = 0,$$

$$\frac{d}{dx}\hat{u}(x,\xi) + \frac{(i\xi)^{\beta}}{a(x)}\hat{u}(x,\xi) = 0.$$

Multiplying both sides with $e^{(i\xi)^{\beta}\int_0^x \frac{1}{a(r)}dr}$, we get

$$e^{(i\xi)^{\beta}\int_0^x \frac{1}{a(r)}dr}\frac{d}{dx}\hat{u}(x,\xi) + e^{(i\xi)^{\beta}\int_0^x \frac{1}{a(r)}dr}\frac{(i\xi)^{\beta}}{a(x)}\hat{u}(x,\xi) = 0.$$

Then we have

$$\left[\hat{u}(x,\xi)e^{(i\xi)^{\beta}\int_0^x \frac{1}{a(r)}dr}\right]_x = 0,$$

$$\hat{u}(x,\xi)e^{(i\xi)^{\beta}\int_0^x \frac{1}{a(r)}dr} = C,$$

$$\hat{u}(x,\xi) = Ce^{-(i\xi)^{\beta}\int_0^x \frac{1}{a(r)}dr}.$$

Since $\hat{u}(1,\xi) = \hat{g}(\xi)$, we obtain

$$C = \hat{g}(\xi)e^{(i\xi)^{\beta}\int_0^1 \frac{1}{a(r)}dr}.$$

Then we get

$$\hat{u}(x,\xi) = \hat{g}(\xi)e^{(i\xi)^{\beta}(F(1)-F(x))},$$

where

$$F(s) = \int_0^s \frac{1}{a(r)}dr.$$

By taking the inversion of the Fourier transform, we obtain the exact solution of the problem as following

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(\xi)e^{(i\xi)^{\beta}(F(1)-F(x))}e^{it\xi}d\xi, \tag{2}$$

where $(i\xi)^{\beta}$ is given by

$$(i\xi)^{\beta} = R((i\xi)^{\beta}) + i\,I((i\xi)^{\beta}),$$

$$R((i\xi)^{\beta}) = |\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right),$$

$$I((i\xi)^{\beta}) = |\xi|^{\beta}\,sign(\xi)\sin\left(\frac{\beta\pi}{2}\right).$$

Noting that $(i\xi)^{\beta}$ has the positive real part $|\xi|^{\beta}\cos\left(\dfrac{\beta\pi}{2}\right)$, therefore $\left|e^{(i\xi)^{\beta}(F(1)-F(x))}\right|$ tends to $+\infty$ for $x\in(0,1)$ as $\xi\to+\infty$. So, a small change in the input data $g(t)$ may result in a large change in the solution. This causes the instability and leads to the ill – posedness of the problem. Next, we will give an example to show the ill – posedness of the problem.

3. The Ill – posedness of the Problem

If we choose the exact data $g=0$ then we get the solution $u=0$.

We choose the measured data g_n with the Fourier transform

$$\hat{g}_n(\xi)=\begin{cases}0, & \xi\le n,\\ \dfrac{\sqrt{n}}{\xi^{5/4}}, & \xi>n,\end{cases}\qquad n\in\mathbb{N}.$$

We get the Fourier transform of the solution u_n corresponding to the measured data g_n

$$\hat{u}_n(x,\xi)=\begin{cases}0, & \xi\le n,\\ \dfrac{\sqrt{n}}{\xi^{5/4}}e^{(i\xi)^{\beta}(F(1)-F(x))}, & \xi>n,\end{cases}\qquad n\in\mathbb{N}.$$

The error between the measured data g_n and the exact data g is

$$\|g_n-g\|_2=\|\hat{g}_n-\hat{g}\|_2=\left(\int_n^{+\infty}\frac{n}{\xi^{5/2}}d\xi\right)^{1/2}=\sqrt{\frac{2}{3n}}\to 0\ as\, n\to\infty.$$

The error between u_n and u is

$$\|u_n-u\|_2=\|\hat{u}_n-\hat{u}\|_2=\left(\int_n^{+\infty}\frac{n}{\xi^{5/2}}e^{2(i\xi)^{\beta}(F(1)-F(x))}d\xi\right)^{1/2}.$$

Since $F(1)-F(x)=\int_x^1\dfrac{1}{a(r)}ds\ge\dfrac{1-x}{q}\quad\forall x\in(0,1)$, we get

$$\|u_n-u\|_2=\|\hat{u}_n-\hat{u}\|_2\ge\sqrt{\frac{2}{3}}\frac{e^{\frac{n^{\beta}(1-x)}{q}\cos\left(\frac{\beta\pi}{2}\right)}}{\sqrt{n}}\to\infty\ as\, n\to\infty.$$

While the error of the input data tends to zero, the error of the solution tends to infinity. It follows that the solution is not stable and the problem (1) is ill – posed. Hence, the regularization is in order and in the next section we will give three methods to regularize the problem.

4. Regularization Methods

4.1. The Truncation Method

We establish the regularized solution for the problem (1) as follows

$$u_\varepsilon(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\hat{g}(\xi)e^{(i\xi)^\beta F(1)-F(x))}e^{it\xi}d\xi, \tag{3}$$

where $a=a(\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0}a(\varepsilon)=+\infty$.

Lemma 1. (The stability of the regularized solution given by (3))

Let $g_1,g_2\in L^2(\mathbb{R})$ and $u_\varepsilon(g_1),u_\varepsilon(g_2)$ be two solutions corresponding to the final values g_1,g_2, respectively. Then we obtain

$$\left\|u_\varepsilon(g_1)(x,.)-u_\varepsilon(g_2)(x,.)\right\|_{L^2(\mathbb{R})}\le e^{a^\beta\cos\left(\frac{\beta\pi}{2}\right)\left(\frac{1-x}{p}\right)}\left\|g_1-g_2\right\|_{L^2(\mathbb{R})}\ \forall x\in(0,1).$$

Proof. We have

$$\begin{aligned}\left\|u_\varepsilon(g_1)-u_\varepsilon(g_2)\right\|^2_{L^2(\mathbb{R})}&=\left\|\hat{u}_\varepsilon(g_1)-\hat{u}_\varepsilon(g_2)\right\|^2_{L^2(\mathbb{R})}\\&=\int_{-a}^{a}\left|e^{2(i\xi)^\beta(F(1)-F(x))}\right|\left|\hat{g}_1-\hat{g}_2\right|^2d\xi\\&=\int_{-a}^{a}\left|e^{2|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)(F(1)-F(x))}\right|\left|\hat{g}_1-\hat{g}_2\right|^2d\xi.\end{aligned}$$

Then we get

$$\left\|u_\varepsilon(g_1)(x,.)-u_\varepsilon(g_2)(x,.)\right\|^2_{L^2(\mathbb{R})}\le e^{2a^\beta\cos\left(\frac{\beta\pi}{2}\right)\left(\frac{1-x}{p}\right)}\left\|g_1-g_2\right\|^2_{L^2(\mathbb{R})}.$$

It implies that

$$\left\|u_\varepsilon(g_1)(x,.)-u_\varepsilon(g_2)(x,.)\right\|_{L^2(\mathbb{R})}\le e^{a^\beta\cos\left(\frac{\beta\pi}{2}\right)\left(\frac{1-x}{p}\right)}\left\|g_1-g_2\right\|_{L^2(\mathbb{R})}.$$

Suppose that $u_{ex}(x,\xi)$ is the exact solution of the problem (1) corresponding to the exact data $g_{ex}\in L^2(\mathbb{R})$ and $u_\varepsilon(g_\varepsilon)(x,\xi)$ is the regularized solution given by (3) corresponding to the measured data $g_\varepsilon\in L^2(\mathbb{R})$. Next, we give the error estimate between the regularized solution and the exact solution under different conditions.

Theorem 1. Suppose that $u_{ex}(x,.)\in L^2(\mathbb{R})$, $g_\varepsilon,g_{ex}\in L^2(\mathbb{R})$ such that $\left\|g_\varepsilon-g_{ex}\right\|_{L^2(\mathbb{R})}\le\varepsilon$ and $\left\|u_{ex}(0,.)\right\|_{L^2(\mathbb{R})}\le A$. Then we get

$$\left\|u_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}\le\varepsilon\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1-x}{p}}+\sqrt{2}A\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-x}{q}}\quad\forall x\in(0,1).$$

Proof.

Applying the triangle inequality, we have

$$\begin{aligned}&\left\|u_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}\\&\quad\le\left\|u_\varepsilon(g_\varepsilon)(x,.)-u_\varepsilon(g_{ex})(x,.)\right\|_{L^2(\mathbb{R})}+\left\|u_\varepsilon(g_{ex})(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}.\end{aligned}\tag{4}$$

From lemma 1, we have

$$\begin{aligned}\|u_\varepsilon(g_\varepsilon)(x,.)-u_\varepsilon(g_{ex})(x,.)\|_{L^2(\mathbb{R})} &\le e^{a^\beta\cos\left(\frac{\beta\pi}{2}\right)\left(\frac{1-x}{p}\right)}\|g_\varepsilon-g_{ex}\|_{L^2(\mathbb{R})}\\ &\le e^{a^\beta\cos\left(\frac{\beta\pi}{2}\right)\left(\frac{1-x}{p}\right)}\varepsilon.\end{aligned} \tag{5}$$

Using Plancherel ‘s theorem, we have

$$\begin{aligned}\|u_\varepsilon(g_{ex})-u_{ex}\|^2_{L^2(\mathbb{R})} &= \|\hat{u}_\varepsilon(g_{ex})-\hat{u}_{ex}\|^2_{L^2(\mathbb{R})}\\ &= \int_{-\infty}^{-a}\left|e^{2(i\xi)^\beta(F(1)-F(x))}\right||\hat{g}_{ex}|^2\,d\xi+\int_a^{+\infty}\left|e^{2(i\xi)^\beta(F(1)-F(x))}\right||\hat{g}_{ex}|^2\,d\xi.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|u_\varepsilon(g_{ex})-u_{ex}\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{-a}\left|e^{2R((i\xi)^\beta)(F(1)-F(x))}\right||\hat{g}_{ex}|^2\,d\xi+\int_a^{+\infty}\left|e^{2R((i\xi)^\beta)(F(1)-F(x))}\right||\hat{g}_{ex}|^2\,d\xi\\ &\le 2\int_{-\infty}^{-\infty}\left|e^{-2R((i\xi)^\beta)F(x)}\right||\hat{g}_{ex}|^2\,e^{2R((i\xi)^\beta)F(1)}d\xi.\end{aligned}$$

It implies that

$$\begin{aligned}\|u_\varepsilon(g_{ex})-u_{ex}\|^2_{L^2(\mathbb{R})} &\le 2\int_{-\infty}^{-\infty}e^{-2a^\beta\cos\left(\frac{\beta\pi}{2}\right)\frac{x}{q}}|\hat{g}_{ex}|^2\,e^{2R((i\xi)^\beta)F(1)}d\xi\\ &\le 2e^{-2a^\beta\cos\left(\frac{\beta\pi}{2}\right)\frac{x}{q}}\|\hat{u}_{ex}(0,.)\|^2_{L^2(\mathbb{R})}\,d\xi.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|u_\varepsilon(g_{ex})-u_{ex}\|_{L^2(\mathbb{R})} &\le \sqrt{2}e^{-a^\beta\cos\left(\frac{\beta\pi}{2}\right)\frac{x}{q}}\|u_{ex}(0,.)\|_{L^2(\mathbb{R})}\\ &\le \sqrt{2}Ae^{-a^\beta\cos\left(\frac{\beta\pi}{2}\right)\frac{x}{q}}.\end{aligned} \tag{6}$$

From (4), (5) and (6), we get

$$\|u_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\|_{L^2(\mathbb{R})} \le e^{a^\beta\cos\left(\frac{\beta\pi}{2}\right)\left(\frac{1-x}{p}\right)}\varepsilon+\sqrt{2}Ae^{-a^\beta\cos\left(\frac{\beta\pi}{2}\right)\frac{x}{q}}.$$

We choose

$$a(\varepsilon)=\left(\frac{\ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)}{\cos\left(\frac{\beta\pi}{2}\right)}\right)^{\frac{1}{\beta}}.$$

Then we obtain

$$\|u_\varepsilon(g_\varepsilon)(x,.)-u(g)(x,.)\|_{L^2(\mathbb{R})}\le\varepsilon\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1-x}{p}}+\sqrt{2}A\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-x}{q}}.$$

Theorem 2. Suppose that $u_{ex}(x,.)\in L^2(\mathbb{R})$, $g_\varepsilon, g_{ex}\in L^2(\mathbb{R})$ such that $\|g_\varepsilon-g_{ex}\|_{L^2(\mathbb{R})}\le\varepsilon$ and $\int_{-\infty}^{\infty}e^{2\gamma|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)}|\hat{u}_{ex}(x,\xi)|^2d\xi\le B^2$. Then we get

$$\|u_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\|_{L^2(\mathbb{R})}\le\varepsilon^x+\sqrt{2}B\varepsilon^{p\gamma}\quad\forall x\in(0,1).$$

Proof. Using Plancherel ‘s theorem, we obtain

$$\begin{aligned}\|u_\varepsilon(g_{ex})(x,.)-u_{ex}(x,.)\|^2_{L^2(\mathbb{R})}&=\|\hat{u}_\varepsilon(g_{ex})(x,.)-\hat{u}_{ex}(x,.)\|^2_{L^2(\mathbb{R})}\\&=\int_{-\infty}^{-a}|\hat{u}_{ex}(x,\xi)|^2d\xi+\int_a^{+\infty}|\hat{u}_{ex}(x,\xi)|^2d\xi.\end{aligned}$$

Thus, we get

$$\begin{aligned}&\|u_\varepsilon(g_{ex})(x,.)-u_{ex}(x,.)\|^2_{L^2(\mathbb{R})}\\&=\int_{-\infty}^{-a}e^{-2\gamma|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)}e^{2\gamma|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)}|\hat{u}_{ex}(x,\xi)|^2d\xi+\int_a^{+\infty}e^{-2\gamma|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)}e^{2\gamma|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)}|\hat{u}_{ex}(x,\xi)|^2d\xi\\&\le e^{-2\gamma a^\beta\cos\left(\frac{\beta\pi}{2}\right)}\int_{-\infty}^{-a}e^{2\gamma|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)}|\hat{u}_{ex}(x,\xi)|^2d\xi+e^{-2\gamma a^\beta\cos\left(\frac{\beta\pi}{2}\right)}\int_a^{+\infty}e^{2\gamma|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)}|\hat{u}_{ex}(x,\xi)|^2d\xi.\end{aligned}$$

It implies that

$$\begin{aligned}&\|u_\varepsilon(g_{ex})(x,.)-u_{ex}(x,.)\|^2_{L^2(\mathbb{R})}\\&\le e^{-2\gamma a^\beta\cos\left(\frac{\beta\pi}{2}\right)}\int_{-\infty}^{+\infty}e^{2\gamma\xi^2}|\hat{u}_{ex}(g)(x,\xi)|^2d\xi+e^{-2\gamma a^\beta\cos\left(\frac{\beta\pi}{2}\right)}\int_{-\infty}^{+\infty}e^{2\gamma\xi^2}|\hat{u}_{ex}(g)(x,\xi)|^2d\xi\\&\le 2e^{-2\gamma a^\beta\cos\left(\frac{\beta\pi}{2}\right)}B^2.\end{aligned}$$

Therefore, we obtain

$$\|u_\varepsilon(g_{ex})(x,.)-u_{ex}(x,.)\|_{L^2(\mathbb{R})}\le\sqrt{2}Be^{-\gamma a^\beta\cos\left(\frac{\beta\pi}{2}\right)}. \tag{7}$$

From (5) and (7) we have

$$\begin{aligned}&\|u_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\|_{L^2(\mathbb{R})}\\&\le\|u_\varepsilon(g_\varepsilon)(x,.)-u_\varepsilon(g_{ex})(x,.)\|_{L^2(\mathbb{R})}+\|u_\varepsilon(g_{ex})(x,.)-u_{ex}(x,.)\|_{L^2(\mathbb{R})}\\&\le e^{a^\beta\cos\left(\frac{\beta\pi}{2}\right)\left(\frac{1-x}{p}\right)}\varepsilon+\sqrt{2}Be^{-\gamma a^\beta\cos\left(\frac{\beta\pi}{2}\right)}.\end{aligned}$$

We choose

$$a(\varepsilon)=\left(\frac{p\ln\left(\frac{1}{\varepsilon}\right)}{\cos\left(\frac{\beta\pi}{2}\right)}\right)^{\frac{1}{\beta}}.$$

Then we obtain

$$\left\|u_{\varepsilon}(g_{\varepsilon})(x,.)-u(g)(x,.)\right\|_{L^2(\mathbb{R})}\le\varepsilon^{x}+\sqrt{2}B\varepsilon^{p\gamma}.$$

4.2. The Quasi Boundary Value Method

We construct the regularized solution for the problem (1) as follows

$$v_{\varepsilon}(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{(i\xi)^{\beta}(F(1)-F(x))}\hat{g}(\xi)}{1+\delta e^{|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)F(1)}}e^{it\xi}d\xi, \tag{8}$$

where $\delta=\delta(\varepsilon)$ such that $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0.$

Lemma 2. For $x\in(0,1)$,

$$\left|\frac{e^{(i\xi)^{\beta}(F(1)-F(x))}}{1+\delta e^{|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)F(1)}}\right|\le\delta^{\frac{F(x)-F(1)}{F(1)}}.$$

Proof. In fact, we have

$$\left|\frac{e^{(i\xi)^{\beta}(F(1)-F(x))}}{1+\delta e^{|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)F(1)}}\right|=\frac{e^{|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)(F(1)-F(x))}}{1+\delta e^{|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)F(1)}}\le\frac{e^{-|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)F(x)}}{\left[\delta+e^{-|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)F(1)}\right]^{\frac{F(1)-F(x)}{F(1)}}\left[\delta+e^{-|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)F(1)}\right]^{\frac{F(x)}{F(1)}}}$$

$$\le\frac{1}{\left[\delta+e^{\left(-|\xi|^{\beta}\cos\left(\frac{\beta\pi}{2}\right)F(1)\right)}\right]^{\frac{F(1)-F(x)}{F(1)}}}$$

$$\le\delta^{\frac{F(x)-F(1)}{F(1)}}.$$

Lemma 3. (The stability of the regularized solution given by (8))

Let $g_1,g_2\in L^2(\mathbb{R})$ and $v_{\varepsilon}(g_1),v_{\varepsilon}(g_2)$ be two solutions corresponding to the final values g_1,g_2, respectively. Then we obtain

$$\left\|v_{\varepsilon}(g_1)-v_{\varepsilon}(g_2)\right\|_{L^2(\mathbb{R})}\le\delta^{\frac{F(x)-F(1)}{F(1)}}\left\|g_1-g_2\right\|_{L^2(\mathbb{R})}\quad\forall x\in(0,1).$$

Proof.

From Lemma 2, we get

$$\left|\hat{v}_\varepsilon(g_1)-\hat{v}_\varepsilon(g_2)\right| = \left|\frac{e^{(i\xi)^\beta(F(1)-F(x))}}{1+\delta e^{|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)F(1)}}\right|\left|\hat{g}_1(\xi)-\hat{g}_2(\xi)\right|$$
$$\le \delta^{\frac{F(x)-F(1)}{F(1)}}\left|\hat{g}_1(\xi)-\hat{g}_2(\xi)\right|.$$

Using Plancherel‘s theorem, we obtain

$$\left\|v_\varepsilon(g_1)-v_\varepsilon(g_2)\right\|_{L^2(\mathbb{R})} = \left\|\hat{v}_\varepsilon(g_1)-\hat{v}_\varepsilon(g_2)\right\|_{L^2(\mathbb{R})}$$
$$\le \delta^{\frac{F(x)-F(1)}{F(1)}}\left\|g_1-g_2\right\|_{L^2(\mathbb{R})}.$$

Theorem 3. Suppose that $u_{ex}(x,.)\in L^2(\mathbb{R})$, $g_\varepsilon, g_{ex}\in L^2(\mathbb{R})$ such that $\left\|g_\varepsilon-g_{ex}\right\|_{L^2(\mathbb{R})}\le\varepsilon$ and $\int_{\mathbb{R}}\left|e^{|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)F(1)}\hat{g}_{ex}(\xi)\right|^2 d\xi\le Q^2$. Then we get

$$\left\|v_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}\le(1+Q)\varepsilon^{\frac{qx}{p}} \quad \forall x\in(0,1).$$

Proof.
Applying the triangle inequality, we have

$$\left\|v_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}$$
$$\le\left\|v_\varepsilon(g_\varepsilon)(x,.)-v_\varepsilon(g_{ex})(x,.)\right\|_{L^2(\mathbb{R})}+\left\|v_\varepsilon(g_{ex})(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}. \tag{9}$$

From Lemma 2, we have

$$\left\|v_\varepsilon(g_\varepsilon)(x,.)-v_\varepsilon(g_{ex})(x,.)\right\|_{L^2(\mathbb{R})}\le\delta^{\frac{F(x)-F(1)}{F(1)}}\left\|g_\varepsilon-g_{ex}\right\|_{L^2(\mathbb{R})}$$
$$\le\delta^{\frac{F(x)-F(1)}{F(1)}}\varepsilon. \tag{10}$$

We have

$$\left|\hat{v}_\varepsilon(g_{ex})(x,\xi)-\hat{u}_{ex}(x,\xi)\right| = \left|\frac{e^{(i\xi)^\beta(F(1)-F(x))}}{1+\delta e^{|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)F(1)}}\hat{g}_{ex}(\xi)-e^{(i\xi)^\beta(F(1)-F(x))}\hat{g}_{ex}(\xi)\right|$$
$$=\left|\frac{e^{(i\xi)^\beta(F(1)-F(x))}}{1+\delta e^{|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)F(1)}}\delta e^{|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)F(1)}\hat{g}_{ex}(\xi)\right|$$
$$\le\left|\delta^{\frac{F(x)-F(1)}{F(1)}}\delta e^{|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)F(1)}\hat{g}_{ex}(\xi)\right|$$
$$\le\delta^{\frac{F(x)}{F(1)}}\left|e^{|\xi|^\beta\cos\left(\frac{\beta\pi}{2}\right)F(1)}\hat{g}_{ex}(\xi)\right|. \tag{11}$$

Then we obtain:

$$\left\|\hat{v}_\varepsilon(g_{ex})(x,\xi)-\hat{u}_{ex}(x,\xi)\right\|_{L^2(\Omega)} \leq \delta^{\frac{F(x)}{F(1)}}\left[\int_{\mathbb{R}}\left|e^{|\xi|^\beta \cos\left(\frac{\beta\pi}{2}\right)F(1)}\hat{g}_{ex}(\xi)\right|^2 d\xi\right]^{\frac{1}{2}}.$$

Thus, from Plancherel's theorem, we get

$$\left\|v_\varepsilon(g_{ex})(x,\xi)-u_{ex}(x,\xi)\right\|_{L^2(\Omega)} \leq Q\delta^{\frac{F(x)}{F(1)}}. \tag{12}$$

From (10), by choosing $\delta=\varepsilon$, we have:

$$\left\|v_\varepsilon(g_\varepsilon)(x,.)-v_\varepsilon(g_{ex})(x,.)\right\|_{L^2(\mathbb{R})} \leq \varepsilon^{\frac{F(x)}{F(1)}}.$$

From (9), (11) and (12), we obtain:

$$\left\|v_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})} \leq (1+Q)\varepsilon^{\frac{F(x)}{F(1)}}.$$

Since $\dfrac{x}{p} \geq F(x)=\displaystyle\int_0^x \frac{1}{a(r)}ds \geq \frac{x}{q} \quad \forall x\in(0,1)$, we get:

$$\left\|v_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})} \leq (1+Q)\varepsilon^{\frac{qx}{p}}. \tag{13}$$

Theorem 4. Suppose that $\dfrac{d}{dx}(u_{ex})(x,.)\in L^2(\mathbb{R})\,\forall x\in(0,1)$, $g_\varepsilon, g_{ex}\in L^2(\mathbb{R})$ such that $\left\|g_\varepsilon-g_{ex}\right\|_{L^2(\mathbb{R})}\leq\varepsilon$ and $\left\|\dfrac{d}{dx}u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}\leq C$. Then for $\varepsilon\in(0,1)$ there exists a $x_\varepsilon>0$ such that

$$\left\|v_\varepsilon(g_\varepsilon)(x_\varepsilon,.)-u_{ex}(0,.)\right\|_{L^2(\mathbb{R})} \leq 2M\sqrt{\frac{p}{q}}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-\frac{1}{2}}.$$

Proof.

Applying the triangle inequality, we have:

$$\left\|v_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(0,.)\right\|_{L^2(\mathbb{R})} \leq \left\|v_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}+\left\|u_{ex}(x,.)-u_{ex}(0,.)\right\|_{L^2(\mathbb{R})}. \tag{14}$$

By applying the fundamental theorem of calculus, we get:

$$u_{ex}(x,t)-u_{ex}(0,t)=\int_0^x \frac{d}{ds}u_{ex}(s,t)ds.$$

Using Holder inequality, we obtain:

$$\begin{aligned}\left\|u_{ex}(x,.)-u_{ex}(0,.)\right\|^2_{L^2(\Omega)} &= \int_{-\infty}^{+\infty}\left|\int_0^x \frac{d}{ds}u_{ex}(s,t)ds\right|^2 dt \\ &\leq x\int_0^x\left\|\frac{d}{ds}u_{ex}(s,.)\right\|^2 ds \\ &\leq x^2C^2.\end{aligned}$$

Hence, we get:

$$\left\| u_{ex}(x,t) - u_{ex}(0,t) \right\|_{L^2(\Omega)} \le Cx. \tag{15}$$

From (13), (14) and (15), we get:

$$\left\| v_\varepsilon(g_\varepsilon)(x,.) - u_{ex}(0,.) \right\|_{L^2(\mathbb{R})} \le M\left(\varepsilon^{\frac{qx}{p}} + x \right),$$

where $M = \max\left\{1+Q, C\right\}$.

For $\varepsilon \in (0,1)$, there exists a unique $x_\varepsilon > 0$ such that $x = \varepsilon^{\frac{qx}{p}}$.

By using the inequality $\ln x > -\dfrac{1}{x}, x > 0$, we obtain:

$$\left\| v_\varepsilon(g_\varepsilon)(x_\varepsilon,.) - u_{ex}(0,.) \right\|_{L^2(\mathbb{R})} \le 2M\sqrt{\frac{p}{q}}\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-\frac{1}{2}}.$$

4.3. The New Regularization Method (the Association of the Quasi – Boundary Value Method and the Truncation Method)

We construct the regularized solution for the problem (1) as follows:

$$w_\varepsilon(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-m}^{m} \frac{e^{(i\xi)^\beta (F(1)-F(x))} \hat{g}(\xi)}{1+\varepsilon\xi^2 e^{|\xi|^\beta \cos\left(\frac{\beta\pi}{2}\right) F(1)}} e^{it\xi} d\xi, \tag{16}$$

where $m = m(\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0} m(\varepsilon) = +\infty$.

Lemma 4. (The stability of the regularized solution given by (16))

Let $g_1, g_2 \in L^2(\mathbb{R})$ and $w_\varepsilon(g_1), w_\varepsilon(g_2)$ be two solutions corresponding to the final values g_1, g_2, respectively. Then we obtain

$$\left\| w_\varepsilon(g_1)(x,.) - w_\varepsilon(g_2)(x,.) \right\|_{L^2(\mathbb{R})} \le \varepsilon^{\frac{qx}{p}} \left\| g_1 - g_2 \right\|_{L^2(\mathbb{R})} \quad \forall x \in (0,1).$$

Proof.

We have:

$$\begin{aligned} \left\| \hat{w}_\varepsilon(g_1) - \hat{w}_\varepsilon(g_2) \right\|^2_{L^2(\mathbb{R})} &= \int_{-m}^{m} \left| \frac{e^{(i\xi)^\beta (F(1)-F(x))}}{1+\varepsilon e^{|\xi|^\beta \cos\left(\frac{\beta\pi}{2}\right) F(1)}} \right| \left| \hat{g}_1 - \hat{g}_2 \right|^2 d\xi \\ &= \int_{-m}^{m} \varepsilon^{\frac{F(x)-F(1)}{F(1)}} \left| \hat{g}_1 - \hat{g}_2 \right|^2 d\xi. \end{aligned}$$

It implies that

$$\left\| w_\varepsilon(g_1)(x,.) - w_\varepsilon(g_2)(x,.) \right\|_{L^2(\mathbb{R})} \le \varepsilon^{\frac{qx}{p}} \left\| g_1 - g_2 \right\|_{L^2(\mathbb{R})}.$$

Theorem 5. Suppose that $u_{ex}(x,.) \in L^2(\mathbb{R})$, $g_\varepsilon, g_{ex} \in L^2(\mathbb{R})$ such that $\|g_\varepsilon - g_{ex}\|_{L^2(\mathbb{R})} \le \varepsilon$ and $\int_{-\infty}^{\infty} \left|(1+\xi^2)\hat{u}_{ex}(x,\xi)\right|^2 d\xi \le N^2$. Then we get:

$$\|w_\varepsilon(g_\varepsilon)(x,.) - u_{ex}(x,.)\|_{L^2(\mathbb{R})} \le \varepsilon^x + \sqrt{2}B\varepsilon^{p\gamma} \quad \forall x \in (0,1).$$

Proof.
Using Plancherel's theorem and the triangle inequality, we have:

$$\begin{aligned}&\|w_\varepsilon(g_\varepsilon)(x,.) - u_{ex}(x,.)\|_{L^2(\mathbb{R})} \\ &\qquad \le \|w_\varepsilon(g_\varepsilon)(x,.) - w_\varepsilon(g_{ex})(x,.)\|_{L^2(\mathbb{R})} + \|w_\varepsilon(g_{ex})(x,.) - u_{ex}(x,.)\|_{L^2(\mathbb{R})}.\end{aligned} \tag{17}$$

From lemma 2, we get:

$$\begin{aligned}\|w_\varepsilon(g_\varepsilon)(x,.) - w_\varepsilon(g_{ex})(x,.)\|_{L^2(\mathbb{R})} &\le \varepsilon^{\frac{F(x)-F(1)}{F(1)}} \|g_\varepsilon - g_{ex}\|_{L^2(\mathbb{R})} \\ &\le \varepsilon^{\frac{qx}{p}}.\end{aligned} \tag{18}$$

We have:

$$\begin{aligned}&\|w_\varepsilon(g_{ex})(x,.) - u_{ex}(x,.)\|^2_{L^2(\mathbb{R})} \\ &\quad = \|\hat{w}_\varepsilon(g_{ex})(x,.) - \hat{u}_{ex}(x,.)\|^2_{L^2(\mathbb{R})} \\ &\quad = \int_{-m}^{m} \left|\hat{w}_\varepsilon(g_{ex})(x,\xi) - \hat{u}_{ex}(x,\xi)\right|^2 d\xi + \int_{-\infty}^{-m} \left|\hat{u}_{ex}(x,\xi)\right|^2 d\xi + \int_{m}^{+\infty} \left|\hat{u}_{ex}(x,\xi)\right|^2 d\xi.\end{aligned}$$

It leads to

$$\begin{aligned}&\int_{-m}^{m} \left|\hat{w}_\varepsilon(g_{ex})(x,\xi) - \hat{u}_{ex}(x,\xi)\right|^2 d\xi \\ &\qquad = \int_{-m}^{m} \left| \frac{1}{1+\varepsilon e^{|\xi|^\beta \cos\left(\frac{\beta\pi}{2}\right)F(1)}} \hat{u}_{ex}(x,\xi) - \hat{u}_{ex}(x,\xi) \right|^2 d\xi \\ &\qquad = \int_{-m}^{m} \left| \frac{-\varepsilon e^{|\xi|^\beta \cos\left(\frac{\beta\pi}{2}\right)F(1)}}{1+\varepsilon e^{|\xi|^\beta \cos\left(\frac{\beta\pi}{2}\right)F(1)}} \hat{u}_{ex}(x,\xi) \right|^2 d\xi \\ &\qquad = \int_{-m}^{m} \left| \varepsilon e^{|\xi|^\beta \cos\left(\frac{\beta\pi}{2}\right)F(1)} \hat{u}_{ex}(x,\xi) \right|^2 d\xi \\ &\qquad \le \varepsilon^2 e^{2m^\beta F(1)} \int_{-\infty}^{+\infty} \left|(1+\xi^2)\hat{u}_{ex}(x,\xi)\right|^2 d\xi\end{aligned}$$

$$\le N^2\varepsilon^2 e^{\frac{2m^\beta}{p}}. \tag{19}$$

On the other hand, we get:

$$\int_{-\infty}^{-m}\left|\hat{u}_{ex}(x,\xi)\right|^2 d\xi+\int_{m}^{+\infty}\left|\hat{u}_{ex}(x,\xi)\right|^2 d\xi$$

$$=\int_{-\infty}^{-m}(1+\xi^2)^{-2}\left|(1+\xi^2)\hat{u}_{ex}(x,\xi)\right|^2 d\xi+\int_{m}^{+\infty}(1+\xi^2)^{-2}\left|(1+\xi^2)\hat{u}_{ex}(x,\xi)\right|^2 d\xi$$

$$\le\frac{2N^2}{m^4}. \tag{20}$$

From (17), (18), (19) and (20), it implies that

$$\left\|w_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}\le\varepsilon^{\frac{qx}{p}}+N\varepsilon e^{\frac{m^\beta}{p}}+\frac{\sqrt{2}N}{m^2}.$$

By choosing $m=\left(\frac{p}{2}\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{\beta}}$, we get:

$$\left\|w_\varepsilon(g_\varepsilon)(x,.)-u_{ex}(x,.)\right\|_{L^2(\mathbb{R})}\le H\left(\varepsilon^{\frac{qx}{p}}+\varepsilon\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{\beta}}+\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-2}{\beta}}\right),$$

where $\max\left\{1,N\left(\frac{p}{2}\right)^{\frac{1}{\beta}},\sqrt{2}N\left(\frac{p}{2}\right)^{\frac{-2}{\beta}}\right\}$.

5. Conclusion

In this work, we regularized the problem of finding the temperature from the given data for a time fractional diffusion equation with space - dependent coefficient in the homogeneous case. We propose the truncation method, the quasi boundary value method and a new method to establish the regularized solution and get the error estimate of the regularization. Further, we will consider problems in the nonlinear case with the perturbed space dependent coefficient.

Acknowledgments

The author would like to thank reviewers for their helpful comments and Ho Chi Minh UTE for financial support.

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