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Original Article

An Inertial Forward-backward Splitting Method for Monotone Inclusions

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Abstract: In this work, we propose a splitting method for solving monotone inclusions in Hilbert spaces. Our method is a modification of the forward-backward algorithm by using the inertial effect. The weak convergence of the proposed algorithm is established under standard conditions.

Keywords: Monotone inclusion, Splitting method, Inertial effect, Forward-backward algorithm. Mathematics Subject Classifications (2020): 47H05, 49M29, 49M27, 90C25.

1. Introduction

Let consider the monotone inclusion of finding the zero points of the sum of a maximal monotone operator A and a monotone, L-Lipschitz operator B, acting on a real Hilbert space $\mathcal{H}$, i.e.,

$$\text{find } \bar{x} \in \mathcal{H} \text{ such that } 0 \in (A+B)\bar{x}\,. \tag{1}$$

Throughout this work, we assume that a solution $\bar{x}$ exists. This inclusion arises in numerous problems of fundamental importance in monotone operator theory, variational inequalities, convex optimization, equilibrium problems, image processing, and machine learning; see [1-4] and the references therein.

For solving problem (1), Tseng [5] proposed an algorithm called forward-backward-forward, namely:

$$\gamma \in]0,+\infty[, \qquad \begin{cases} y_k = J_{\gamma A}(x_k - \gamma B x_k), \\ x_{k+1} = y_k - \gamma B y_k + \gamma B x_k. \end{cases}$$

where $J_{\gamma A}$ denotes the resolvent of A, i.e.

$$J_{\gamma A} = (\mathrm{Id} + \gamma A)^{-1},$$

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where Id is identity operator on $\mathcal{H}$. A limitation of this method is that at each iteration step, one has to compute twice times the values of operator B. This issue was recently resolved in [6], the forward reflected backward splitting method was proposed, namely,

$$\gamma \in]0, +\infty[,\ x_{k+1} = J_{\gamma A}(x_k - 2\gamma B x_k + \gamma B x_{k-1}). \quad (2)$$

A method related to method (2) was suggested in [7], called the reflected forward-backward splitting method:

$$\gamma \in]0, +\infty[,\ x_{k+1} = J_{\gamma A}\big(x_k - \gamma B(2x_k - x_{k-1})\big). \quad (3)$$

The convenience of method (2) and (3) is that in each iteration, we only need to calculate the value of operator B once.

In [8], Polyak introduced the so-called heavy ball method in order to speed up the classical gradient method. For a differential function $f: \mathcal{H} \to \mathbb{R}$, the algorithm takes the following form:

$$x_{k+1} = x_k + \alpha_k(x_k - x_{k-1}) - \gamma_k \nabla f(x_k).$$

This idea then employed and refined by some authors [9-13]. In [13], Lorenz and Pock proposed the following inertial forward- backward algorithm for monotone operators' algorithm:

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}), \\ x_{k+1} = J_{\gamma_k A}(y_k - \gamma_k B y_k). \end{cases}$$

By using the inertial forward-backward algorithm above and the projection on a half space, in [10], the authors derived the strong convergence result of the proposed method. In this work, we propose a new method for solving problem (1). In our method a value of operator B is also used in each iteration as method (2) and (3). We also used the inertial effect to improve the performance of the algorithm. Under standard conditions, we also obtained the convergence of the proposed method. In some examples, our method gives better convergence rate in comparison with Tseng's method and the methods in [5, 7].

The rest of this work is organized as follows. After collecting preliminaries needed in Section 2, we present the proposed method and prove the convergence of the method in Section 3.

2. Preliminaries

The scalar product and the associated norm of the real Hilbert space $\mathcal{H}$ are denoted respectively by $\langle \cdot | \cdot \rangle$ and $\|\cdot\|$.

The symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence.

Let $A: \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The domain of A is denoted by $\mathrm{dom}(A)$ that is a set of all $x \in \mathcal{H}$ such that $Ax \neq \emptyset$. The range of A is $\mathrm{ran}(A) = \{u \in \mathcal{H} \mid (\exists x \in \mathcal{H}) u \in Ax\}$. The graph of A is denoted by $\mathrm{gra}(A) = \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}$. A^{-1} stands for the inverse of A, i.e., $A^{-1}: u \mapsto \{x \mid u \in Ax\}$. The zero set of A is $\mathrm{zer}(A) = A^{-1}0$.

Definition 2.1 We say that operator $A: \mathcal{H} \to 2^{\mathcal{H}}$ is (i) monotone if

$$(\forall (x, u) \in \mathrm{gra}(A))(\forall (y, v) \in \mathrm{gra}(A))\ \langle x - y \mid u - v \rangle \geq 0.$$

(ii) maximally monotone if it is monotone and there exists no monotone operator $B: \mathcal{H} \to 2^{\mathcal{H}}$ such that $\mathrm{gra}(B)$ properly contains $\mathrm{gra}(A)$, i.e., there is no monotone operator that properly contains it.

Definition 2.2 A mapping $T: \mathcal{H} \to \mathcal{H}$ is said to be L-Lipschitz continuous ($L > 0$) if

$$\| Tx - Ty \| \leq L \| x - y \| \quad \forall x, y \in \mathcal{H}.$$

Definition 2.3 For $A: \mathcal{H} \to 2^{\mathcal{H}}$, the resolvent of operator A is

$$J_A = (\mathrm{Id} + A)^{-1},$$

where Id denotes the identity operator on $\mathcal{H}$.

Note that, when A is maximally monotone, J_A is an everywhere single-valued operator [14].

3. Proposed Method and Convergence

We propose the following method for solving problem (1).

Algorithm 3.1. Let $\gamma > 0$, let $\alpha \geq 0$. Let $x_{-1}, x_0, x_1 \in \mathcal{H}$. Iterate $(\forall k \in \mathbb{N})$

$$x_{k+1} = J_{\gamma A}\left[x_k + \alpha(x_k - x_{k-1}) - \gamma\left(\frac{7}{2}Bx_k - 4Bx_{k-1} + \frac{3}{2}Bx_{k-2}\right)\right]. \quad (4)$$

Let us give some comments on the above algorithm.

i) In (4), in each iteration, we have to calculate three forward values. However, in the next iteration, we can use two forward values of the previous iteration. Therefore, we actually only compute one forward value in every iteration;

ii) When $\alpha = 0, B = 0$, Algorithm 3.1 becomes the proximal point algorithm as in [15].

To prove the convergence of Algorithm 3.1, we have the following lemma.

Lemma 3.2. Suppose that $(x_k)_{k\in\mathbb{N}}$ is the sequence generated by Algorithm 3.1. Then for $x \in \text{zer}(A + B)$, we get

$$\|x_{k+1} - x\|^2 - \alpha\|x_k - x\|^2 + 2\gamma t_{k+1} + \left(1 - \alpha - \frac{5\gamma L}{2}\right)\|x_{k+1} - x_k\|^2$$
$$\leq ||x_k - x||^2 - \alpha||x_{k-1} - x||^2 + 2\gamma t_k + (2\alpha + \gamma L)||x_k - x_{k-1}||^2 + \frac{3\gamma L}{2}||x_{k-1} - x_{k-2}||^2,$$

where $t_k = \langle Bx_{k-1} - Bx_k \mid x_k - x\rangle + \frac{3}{2}\langle Bx_{k-1} - Bx_{k-2} \mid x_k - x\rangle$.

Proof. From (4), we get

$$\frac{x_k + \alpha(x_k - x_{k-1}) - x_{k+1}}{\gamma} - \left(\frac{7}{2}Bx_k - 4Bx_{k-1} + \frac{3}{2}Bx_{k-2}\right) \in Ax_{k+1}. \quad (5)$$

For $x \in \text{zer}(A + B)$, then $-Bx \in Ax$. Hence,

$$\left\langle \frac{x_k + \alpha(x_k - x_{k-1}) - x_{k+1}}{\gamma} - \frac{7}{2}Bx_k + 4Bx_{k-1} - \frac{3}{2}Bx_{k-2} + Bx \,\middle|\, x_{k+1} - x\right\rangle \geq 0$$

which implies

$$\left\langle \frac{x_k - x_{k+1} + \alpha(x_k - x_{k-1})}{\gamma} \,\middle|\, x_{k+1} - x\right\rangle$$
$$\geq \left\langle \frac{7}{2}Bx_k - 4Bx_{k-1} + \frac{3}{2}Bx_{k-2} - Bx \,\middle|\, x_{k+1} - x\right\rangle. \quad (6)$$

For the left-hand side of (6), we have

$$\langle x_k - x_{k+1} + \alpha(x_k - x_{k-1}) \mid x_{k+1} - x\rangle$$
$$= \langle x_k - x_{k+1} \mid x_{k+1} - x\rangle + \alpha\langle x_k - x_{k-1} \mid x_{k+1} - x_k\rangle + \alpha\langle x_k - x_{k-1} \mid x_k - x\rangle$$
$$= \frac{1}{2}\left(\|x_k - x\|^2 - \|x_{k+1} - x\|^2 - \|x_{k+1} - x_k\|^2\right) - \frac{\alpha}{2}\left(\|x_{k-1} - x\|^2 - \|x_k - x\|^2 - \|x_{k-1} - x_k\|^2\right)$$
$$+\alpha\langle x_k - x_{k-1} \mid x_{k+1} - x_k\rangle. \quad (7)$$

Using the monotonicity of B, we estimate the right-hand side of (6) as:

$$
\begin{aligned}
&\left\langle \frac{7}{2}Bx_k - 4Bx_{k-1} + \frac{3}{2}Bx_{k-2} - Bx \,\middle|\, x_{k+1} - x \right\rangle \\
&= \left\langle \frac{7}{2}Bx_k - 4Bx_{k-1} + \frac{3}{2}Bx_{k-2} - Bx_{k+1} \,\middle|\, x_{k+1} - x \right\rangle + \langle Bx_{k+1} - Bx \mid x_{k+1} - x \rangle \\
&\geq \langle Bx_k - Bx_{k+1} \mid x_{k+1} - x \rangle + \frac{5}{2}\langle Bx_k - Bx_{k-1} \mid x_{k+1} - x \rangle \\
&-\frac{3}{2}\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x \rangle \\
&= \langle Bx_k - Bx_{k+1} \mid x_{k+1} - x \rangle + \frac{3}{2}\langle Bx_k - Bx_{k-1} \mid x_{k+1} - x \rangle \\
&+\langle Bx_k - Bx_{k-1} \mid x_{k+1} - x_k \rangle + \langle Bx_k - Bx_{k-1} \mid x_k - x \rangle \\
&-\frac{3}{2}\left(\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x_k \rangle + \langle Bx_{k-1} - Bx_{k-2} \mid x_k - x \rangle\right) \\
&= t_{k+1} - t_k + \langle Bx_k - \mathrm{B}x_{k-1} \mid x_{k+1} - x_k \rangle - \frac{3}{2}\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x_k \rangle. \qquad (8)
\end{aligned}
$$

Hence, from (6), (7) and (8), we deduce

$$
\begin{aligned}
&\left(\|x_k - x\|^2 - \|x_{k+1} - x\|^2 - \|x_{k+1} - x_k\|^2\right) - \alpha\left(\|x_{k-1} - x\|^2 - \|x_k - x\|^2 - \|x_{k-1} - x_k\|^2\right) \\
&\geq 2\gamma t_{k+1} - 2\gamma t_k + 2\gamma\langle Bx_k - Bx_{k-1} \mid x_{k+1} - x_k \rangle - 3\gamma\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x_k \rangle \\
&\quad -2\alpha\langle x_k - x_{k-1} \mid x_{k+1} - x_k \rangle. \qquad (9)
\end{aligned}
$$

Using Cauchy-Schwarz inequality and the Lipschitz property of B, we have

$$
\begin{cases}
2|\langle Bx_k - Bx_{k-1} \mid x_{k+1} - x_k \rangle| \leq L\left(\|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2\right) \\
2|\langle Bx_{k-1} - Bx_{k-2} \mid x_{k+1} - x_k \rangle| \leq L\left(\|x_{k-1} - x_{k-2}\|^2 + \|x_{k+1} - x_k\|^2\right) \\
2|\langle x_k - x_{k-1} \mid x_{k+1} - x_k \rangle| \leq \|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2
\end{cases}
$$

Therefore, (9) implies that

$$
\begin{aligned}
&\left(\|x_k - x\|^2 - \|x_{k+1} - x\|^2 - \|x_{k+1} - x_k\|^2\right) - \alpha\left(\|x_{k-1} - x\|^2 - \|x_k - x\|^2 - \|x_{k-1} - x_k\|^2\right) \\
&\geq 2\gamma t_{k+1} - 2\gamma t_k - \gamma L\left(\|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2\right) - \frac{3\gamma L}{2}\left(\|x_{k-1} - x_{k-2}\|^2 + \|x_{k+1} - x_k\|^2\right) \\
&-\alpha\left(\|x_k - x_{k-1}\|^2 + \|x_{k+1} - x_k\|^2\right)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\|x_{k+1} - x\|^2 - \alpha\|x_k - x\|^2 + 2\gamma t_{k+1} + \left(1 - \alpha - \frac{5\gamma L}{2}\right)\|x_{k+1} - x_k\|^2 \\
&\leq \|x_k - x\|^2 - \alpha\|x_{k-1} - x\|^2 + 2\gamma t_k + (2\alpha + \gamma L)\|x_k - x_{k-1}\|^2 + \frac{3\gamma L}{2}\|x_{k-1} - x_{k-2}\|^2
\end{aligned}
$$

The proof is completed. □

We have the result of the convergence of Algorithm 3.1.

Theorem 3.3. Let $(x_k)_{k\in\mathbb{N}}$ be generated by Algorithm 3.1 with $\alpha \in \left[0, \frac{1}{3}\right[$, and

$$
\gamma < \frac{1 - 3\alpha}{5L}. \qquad (10)
$$

Then $(x_k)_{k\in\mathbb{N}}$ converges weakly to $\bar{x} \in \operatorname{zer}(A + B)$.

Proof. For $x \in \operatorname{zer}(A + B)$, using Lemma 3.2, we obtain

$$\|x_{k+1}-x\|^2-\alpha\|x_k-x\|^2+2\gamma t_{k+1}+\left(2\alpha+\frac{5\gamma L}{2}\right)\|x_{k+1}-x_k\|^2+\frac{3\gamma L}{2}\|\| x_k-x_{k-1}\|^2$$
$$\leq\|x_k-x\|^2-\alpha\|x_{k-1}-x\|^2+2\gamma t_k+\left(2\alpha+\frac{5\gamma L}{2}\right)\|x_k-x_{k-1}\|^2+\frac{3\gamma L}{2}\|x_{k-1}-x_{k-2}\|^2$$
$$-(1-3\alpha-5\gamma L)\|x_{k+1}-x_k\|^2. \quad (11)$$

We denote

$$S_k=\|x_k-x\|^2-\alpha\|x_{k-1}-x\|^2+2\gamma t_k+\left(2\alpha+\frac{5\gamma L}{2}\right)\|x_k-x_{k-1}\|^2+\frac{3\gamma L}{2}\|x_{k-1}-x_{k-2}\|^2.$$

we rewrite (11) as

$$S_{k+1}\leq S_k-(1-3\alpha-5\gamma L)\|x_{k+1}-x_k\|^2. \quad (12)$$

We prove that $(\forall k\in\mathbb{N}), S_k\geq 0$. Indeed, from the formula of t_k, using Cauchy-Schwarz inequality and the Lipschitz property of B, we get

$$2|t_k|\leq L\left(\|x_{k-1}-x_k\|^2+\|x_k-x\|^2\right)+\frac{3L}{2}\left(\|x_{k-1}-x_{k-2}\|^2+\|x_k-x\|^2\right).$$

Hence

$$S_k\geq\|x_k-x\|^2-\alpha\|x_{k-1}-x\|^2+\left(2\alpha+\frac{5\gamma L}{2}\right)\|x_k-x_{k-1}\|^2$$
$$-\gamma L\left(\|x_{k-1}-x_k\|^2+\|x_k-x\|^2\right)-\frac{3\gamma L}{2}\|x_k-x\|^2$$
$$\geq\left(1-\frac{5\gamma L}{2}\right)\|x_k-x\|^2-\alpha\|x_{k-1}-x\|^2+2\alpha\|x_k-x_{k-1}\|^2$$
$$=\left(1-2\alpha-\frac{5\gamma L}{2}\right)\|x_k-x\|^2+\alpha\left(2\|x_k-x\|^2+2\|x_k-x_{k-1}\|^2-\|x_{k-1}-x\|^2\right)$$
$$\geq\left(1-2\alpha-\frac{5\gamma L}{2}\right)\|x_k-x\|^2\geq 0\,. \quad (13)$$

By combining (11), (12), and the condition (10), we get

$$\begin{cases}\lim\limits_{k\to+\infty}\|x_{k+1}-x_k\|=0,\\ \exists\lim\limits_{k\to+\infty}S_k=\xi\in\mathbb{R}.\end{cases} \quad (14)$$

It follows from (13) and (14) that $(x_k)_{k\in\mathbb{N}}$ is bounded and

$$\lim_{k\to\infty}S_k=\lim_{k\to\infty}\left(\|x_k-x\|^2-\alpha\|x_{k-1}-x\|^2\right)=\xi.$$

Let $x^\star\in\mathcal{H}$ be a weak sequential cluster point of $(x_k)_{k\in\mathbb{N}}$. Then there exists a subsequence $\left(x_{k_n}\right)_{n\in\mathbb{N}}$ that converges weakly to $x^\star$.

From (5), we have

$$\frac{x_k+\alpha(x_k-x_{k-1})-x_{k+1}}{\gamma}-\left(\frac{7}{2}Bx_k-4Bx_{k-1}+\frac{3}{2}Bx_{k-2}\right)+Bx_{k+1}\in(A+B)x_{k+1}$$

which is equivalent to

$$\frac{x_k-x_{k+1}}{\gamma}+\frac{\alpha(x_k-x_{k-1})}{\gamma}-(Bx_k-Bx_{k+1})-\frac{5}{2}(Bx_k-Bx_{k-1})$$
$$+\frac{3}{2}(Bx_{k-1}-Bx_{k-2})\in(A+B)x_{k+1}\,. \quad (15)$$

Using the Lipschitz condition of B and $\lim_{k\to\infty}\|x_{k+1}-x_k\|=0$, one can see that the left-hand side of (15) converges strongly to 0 . Using [3, Corollary 24.4], the sum $A+B$ is maximally monotone, and hence, its graph is closed in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$ [3, Proposition 20.33]. Therefore $x^\star \in \operatorname{zer}(A+B)$.

Assume that $\left(x_{k_n}\right)_{n\in\mathbb{N}} \rightharpoonup x, \left(x_{l_n}\right)_{n\in\mathbb{N}} \rightharpoonup y$. Then we have that

$$-\left(\|x_k-x\|^2-\alpha\|x_{k-1}-x\|^2\right)+\left(\|x_k-y\|^2-\alpha\|x_{k-1}-y\|^2\right)$$
$$+(\| x \|^2-\alpha \| x \|^2)-(\| y \|^2-\alpha \| y \|^2)=2(\langle x_k|x-y\rangle-\alpha\langle x_{k-1}|x-y\rangle). \tag{16}$$

Choosing $k=k_n$ and $k=l_n$ then taking limit both sides of (16) when $n\to\infty$, we get

$$\| x-y \|^2-\alpha \| x-y \|^2=0,$$

which implies $x=y$. Therefore $(x_k)_{k\in\mathbb{N}}$ converges weakly to a point in $\operatorname{zer}(A+B)$. The proof is completed.

Next, we consider some simple examples to illustrate the effectiveness of our method. We compare our method (P) to two methods: Tseng's method (T) and the forward-reflected-backward method (FRB) in [11]. The first example shows that in a particular case, with suitable initial values, our method is better than Tseng's and the forward-reflected-backward method even in the optimal case.

Example 1: We consider problem (1) with $\mathcal{H}=\mathbb{R}^n, Ax=0, Bx=x$ and the initial values $x_0\in\mathbb{R}^n, x_1=\frac{x_0}{3}, x_2=\frac{x_0}{9}$ for all three methods. Note that 0 is the unique solution of (1) and the operator B is 1 -Lipschitz.

Tseng's method (T): $x_{k+1}=(1-\gamma+\gamma^2)x_k$ for $\gamma<1$. The optimal stepsize is $\gamma=\frac{1}{2}$ which gives a rate of $\frac{3}{4}$.

FRB method: $x_{k+1}=(1-2\gamma)x_k+\gamma x_{k-1}$ for $\gamma<\frac{1}{2}$. The optimal stepsize is $\gamma\approx\frac{1}{2}$ which gives a rate of $\frac{1}{\sqrt{2}}$.

Proposed method (P): We Choose $\alpha=0, \gamma=\frac{2}{15}$, then (4) becomes

$$x_{k+1}=\frac{8}{15}x_k+\frac{8}{15}x_{k-1}-\frac{1}{5}x_{k-2}.$$

The proposed method is $x_{k+1}=\frac{x_0}{3^k}$ which gives a rate of $\frac{1}{3}$. We see that $\frac{1}{3}<\frac{1}{\sqrt{2}}<\frac{3}{4}$, therefore the proposed method converges faster than Tseng's method and FRB method in [11] for this particular problem.

The convergence of the three methods are illustrated in Figure 1. Note that, in this case, Tseng's and FRB methods are optimally selected, i.e., the stepsize is equal $\frac{1}{2}$ is optimal.

Example 2: Consider problem (1) with $\mathcal{H}=\mathbb{R}^2, A(z_1,z_2)=(0,0), B(z_1,z_2)=(-z_2,z_1)$. The convergence of Tseng's method, FRB method and Algorithm 3.1 are illustrated in Figure 2 and Figure 3. We see that the convergence of our method is the same FRB method and faster than Tseng's method.

To illustrate the effectiveness of the inertial techniques, we consider the following simple example.

Example 3: We consider problem (1) with $\mathcal{H}=\mathbb{R}, Ax=0, Bx=x$. The convergence of Algorithm 3.1 for different values of α is illustrated in Figure 4 with $\gamma=1/21$ and Figure 5 with $\gamma=1/15$.

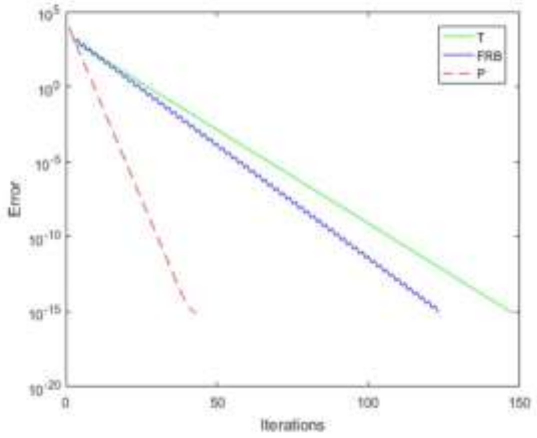

Figure 1. Convergence of the iteration of three methods.

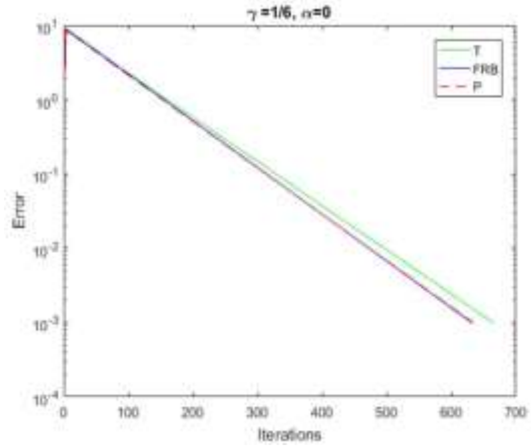

Figure 2. Convergence of the iteration of three methods.

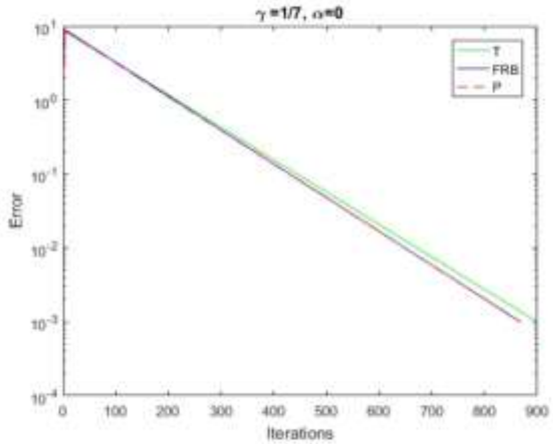

Figure 3. Convergence of the iteration of three methods.

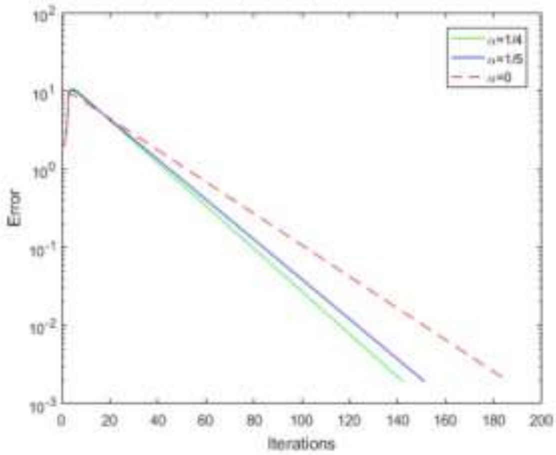

Figure 4. Convergence of proposed method.

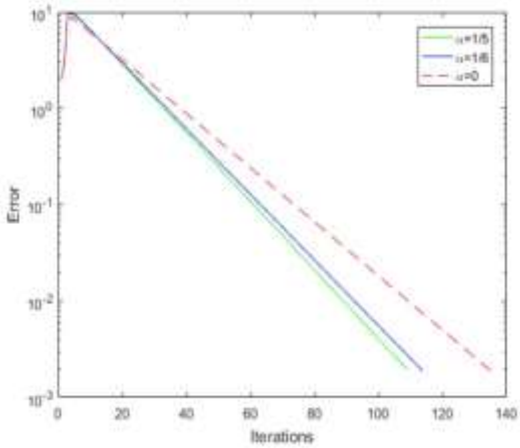

Figure 5. Convergence of proposed method.

4. Conclusions

We have proposed an inertial splitting method for finding a zero point of a sum of two operators, in which 1 is maximally monotone and 1 is monotone-Lipschitz. Under some conditions of the parameters, we have proved the weak convergence of the algorithm. In some special cases, the proposed method converges faster than some known methods.

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