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Original Article

Complete Convergence for Weighted Sums of Pairwise Negative Quadrant Dependent Random Variables

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Abstract: In this work, we developed Jajte's technique of the strong law of large numbers to the complete convergence for randomly weighted sums of pairwise negative quadrant dependent random variables.

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1. Introduction

In mathematical statistics, complete convergence plays an esential role. It can be used to obtain the almost sure convergence, to establish the convergence rate, and to yield consistency results. The concept of the complete convergence was first introduced in [1] as follows: a sequence $\{X_n, n \geq 1\}$ of random variables (r.v.s) is said to be completely convergent to a constant C if $\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. They proved that the sequence of arithmetic means of i.i.d r.v.s completely converges to the expected value of the variables, provided their variance is finite. The result of Hsu and Robbins that is considered as a fundamental theorem in probability theory was later generalized and extended during a process which led to the now classical paper by Baum and Katz [2]. After that, many other authors studied the convergence for dependent sequences such as martingale difference sequences, various types of mixing sequences, negatively associated (NA) sequences, negative quadrant dependent (NQD) random vectors, and so forth [3-10].

Jajte [11] gave the strong law of large numbers for general weighted sums of i.i.d r.v.s using the class of function ϕ statisfies the below conditions:

i) For some $d \geq 1$, $\phi(.)$ is a strictly increasing function on $[d, \infty)$ with range $[0, \infty)$;

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ii) There exist constant C and a positive integer $k_0 \geq d$ such that $\frac{\phi(y+1)}{\phi(y)} \leq C$ for all $y \geq k_0$;

iii) There exist constants a and b such that for all $s > d$ we have

$$\phi^2(s) \int_s^\infty \frac{1}{\phi^2(x)} dx \leq as + b.$$

Inspired by Jajte, in this work, we develop his technique to prove the complete convergence for randomly weighted sums of pairwise NQD r.v.s. In particular, we denote $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ an array of rowwise pairwise NQD r.v.s. We consider the classes $\mathcal{K}_r$ and $\mathcal{H}$ of function ϕ (see Section 2) to provide conditions for

$$\sum_{n=1}^\infty \frac{1}{n^{2-r}} P\left(\left|\sum_{i=1}^n A_{ni}X_{ni} - a_{ni}\right| > \varepsilon\phi(n)\right) < \infty \quad \text{for all } \varepsilon > 0 \tag{1}$$

where $\{A_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of r.v.s, $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real constants and $r \geq 1$.

Such limit theorems for randomly weighted sums have many extremely important applications in statistics. We refer readers to [12, 13] among others for more details.

Now, let us recall the definition of pairwise NQD r.v.s that was first introduced by Lehmann EL. (see [14]) as follows:

Definition 1. ([14]) Two r.v.s X_1 and X_2 are called NQD if

$$P(X_1 > x_1, X_2 > x_2) \leq P(X_1 > x_1)(X_2 > x_2)$$

for all real numbers x_1, x_2. An infinite sequence of r.v.s $\{X_n, n \geq 1\}$ is said to be pairwise NQD if every pair of r.v.s in the sequence is NQD.

It is easy to show that a pairwise NQD sequence of r.v.s is much weaker than a pairwise independent random sequence and some negatively dependent sequences, such as negatively orthant dependent sequences [3] and negatively associated sequences [15].

The following example shows a sequence of coordinatewise pairwise NQD r.v.s which is not pairwise independent (see Example 2.3. [8] or see Example 3.1 in [16]).

Example 2. Let $Z_n, n \geq 1$ be i.i.d $N(0,1)$ random variables. Then $\{Z_n - Z_{n+1}, n \geq 1\}$ are identically distributed $N(0,2)$ random variables. Let F be the $N(0,2)$ distribution function and $\{F_n, n \geq 1\}$ be a sequence of continuous distribution funtions. For $n \geq 1$, put

$$F_n^{-1}(t) = inf\{x: F_n(x) \geq t\} \text{ and } Y_n = F_n^{-1}\left(F(Z_n - Z_{n+1})\right).$$

Li et al. [17] showed that $\{Y_n, n \geq 1\}$ is a sequence of pairwise NQD random variables and for all $n \geq 1$, the distribution function of Y_n is F_n. Moreover, we have

$$cov(Z_n - Z_{n+1}, Z_{n+1} - Z_{n+2}) = -1.$$

Therefore, the Y_n and Y_{n+1} are not independent. Consequently, $\{Y_n, n \geq 1\}$ is not a sequence of pairwise independent random vectors.

We firstly give some lemmas which will be used to prove our main results.

Lemma 3. ([14]) Let X and Y be NQD r.v.s. Then,

i) $cov(X, Y) \leq 0$;

ii) If f and g are borel fuctions, both of which are nonincreasing (or both are nondecreasing), then $f(X)$ and $g(X)$ are NQD.

Lemma 4. [7] Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD r.v.s with mean zero. Then

$$E\left(\sum_{i=1}^n X_i\right)^2 \leq \sum_{i=1}^n E\left(X_i^2\right)$$

and

$$E\left(\max_{1\le k\le n}\left(\sum_{i=1}^{n} X_i\right)^2\right) \le \frac{\log^2(2n)}{\log^2(2)}\sum_{i=1}^{n} E\left(X_i^2\right).$$

Lemma 5. Let $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ be sequences of nonnegative pairwise NQD r.v.s. Assume that $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ are independent. Then $\{X_nY_n, n \ge 1\}$ is also a sequence of nonnegative pairwise NQD r.v.s.

Proof. For $t \ge 0, z \ge 0$ and for $i \ne j$, based on the proof of Lemma 1 in [12], we have that

$$\begin{aligned}
P\left(X_iY_i \le t, X_jY_j \le z\right) &= \int \dots \int I\left(x_iy_i \le t, x_jy_j \le z\right) dF_{X_i,X_j,Y_i,Y_j}\left(x_i,x_j,y_i,y_j\right)\\
&= \int \dots \int I\left(x_iy_i \le t, x_jy_j \le z\right) dF_{X_i,X_j}\left(x_i,x_j\right) dF_{Y_i,Y_j}\left(y_i,y_j\right)\\
&= \int \dots \int P\left(x_iY_i \le t, x_jY_j \le z\right) dF_{X_i,X_j}\left(x_i,x_j\right)\\
&\le \int \dots \int P\left(x_iY_i \le t\right) P\left(x_jY_j \le z\right) dF_{X_i,X_j}\left(x_i,x_j\right)\\
&\le E\left[F_{Y_i}\left(\frac{t}{X_i}\right) F_{Y_j}\left(\frac{z}{X_j}\right)\right] \le E\left[F_{Y_i}\left(\frac{t}{X_i}\right)\right] E\left[F_{Y_j}\left(\frac{z}{X_j}\right)\right]\\
&= \iint I(x_iy_i \le t) dF_{Y_i}(y_i) dF_{X_i}(x_i) \iint I\left(x_jy_j \le z\right) dF_{Y_j}\left(y_j\right) dF_{X_j}\left(x_j\right)\\
&= \iint I(x_iy_i \le t) dF_{X_i,Y_i}(x_i,y_i) \iint I\left(x_jy_j \le z\right) dF_{X_j,Y_j}\left(x_j,y_j\right)\\
&= P(X_iY_i \le t) P\left(X_jY_j \le z\right).
\end{aligned}$$

In this work, the notation $a_n = O(b_n)$ means that $a_n \le Cb_n$ for all $n \ge 1$ and we denote by I_A the indicator function of A. Throughout the paper, the symbol C is used for a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

In Section 2, we give our main results on complete convergence for pairwise NQD r.v.s.

2. Main Results

In this section, we give some sufficient conditions for (1) (see Theorem 6, Theorem 7 and Corollary 10). As consequences of these results, we obtain the almost sure convergence for randomly weighted of pairwise NQD r.v.s by taking $r = 2$. Based on the definition of the function ϕ in [11], for $r > 0$, we consider the class $\mathcal{K}_r$ of all functions $\phi(x)$ which satisfies the following conditions:

i) For $d \ge 0$, $\phi(x)$ is strictly increasing on $[d, +\infty)$ with range $[0, +\infty)$.

ii) There exist a constant C and a positive integer $n_0 \ge d$ such that $\frac{\phi(x+1)}{\phi(x)} \le C$ for all $x \ge n_0$.

iii) There exist constants a and b such that,

$$\phi^2(x)\int_s^{\infty} \frac{x^{r-1}\,dx}{\phi^2(x)} \le as^r + b, s > d.$$

The inverse function of $\phi(x)$ is denote by $\phi^{-1}(x)$.

Theorem 6. *Let $r \ge 1$, let $\{X, X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise pairwise NQD and identically distributed r.v.s, $\{A_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise pairwise NQD r.v.s satisfying*

$$\sum_{i=1}^{n} E\left(A_{ni}^2\right) = O(n). \tag{2}$$

Assume that $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ and $\{A_{ni}, 1 \le i \le n, n \ge 1\}$ are independent. If there exists $\phi \in \mathcal{K}_r$ such that $E\left(\phi^{-1}(|X|^r)\right) < \infty$, then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\left|A_{ni}X_{ni} - E\left(A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}\right)\right| > \varepsilon\phi(n)\right) < \infty. \quad (3)$$

Proof. When $0 < r < 1$, (3) is obvious therefore we only consider the case $r \ge 1$. Without loss of generality, we may assume that $A_{ni} \ge 0$ a.s and $X_{ni} \ge 0$ a.s. Otherwise, we may use A_{ni}^+, A_{ni}^- instead of A_{ni} and X_{ni}^+, X_{ni}^- instead of X_{ni}, respectively, and note that

$$A_{ni}X_{ni} = A_{ni}^+X_{ni}^+ - A_{ni}^-X_{ni}^+ - A_{ni}^+X_{ni}^- + A_{ni}^-X_{ni}^-.$$

Set

$$S_n = \sum_{i=1}^{n} \left[A_{ni}X_{ni} - E\left(A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}\right)\right].$$

For any fixed $\varepsilon > 0$,

$$\begin{aligned} P\left(|S_n| > \varepsilon\phi(n)\right) &\le \sum_{i=1}^{n} P\left(|X_{ni}| > \phi(n)\right) \\ &\quad + P\left(\left|\sum_{i=1}^{n}\left[A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))} - E\left(A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}\right)\right]\right| > \varepsilon\phi(n)\right). \end{aligned}$$

We have

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n^{2-r}}\sum_{i=1}^{n} P\left(|X_{ni}| > \phi(n)\right) &= \sum_{n=1}^{\infty}\frac{1}{n^{1-r}} P\left(|X| > \phi(n)\right) \\ &= \sum_{n=1}^{\infty}\frac{1}{n^{1-r}}\sum_{k=n}^{\infty} P\left(k < \phi^{-1}(|X|) \le k+1\right) \\ &= \sum_{k=1}^{\infty} P\left(k < \phi^{-1}(|X|) \le k+1\right)\sum_{n=1}^{k}\frac{1}{n^{1-r}} \\ &\le E\left(\phi^{-1}(|X|)\right)^r < \infty. \end{aligned} \quad (4)$$

Thus, it remains to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^{2-r}} P\left(\left|\sum_{i=1}^{n}\left[A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))} - E\left(A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}\right)\right]\right| > \varepsilon\phi(n)\right) < \infty.$$

For $n \ge 1, 1 \le i \le n$, denote

$$\begin{aligned} Y_{ni} &= X_{ni}I_{(|X_{ni}|\le\phi(n))} + \phi(n)I_{(X_{ni}>\phi(n))}, \\ Z_{ni} &= \phi(n)I_{(X_{ni}>\phi(n))}, \\ U_n &= \sum_{i=1}^{n}\left[A_{ni}Y_{ni} - E\left(A_{ni}Y_{ni}\right)\right], V_n = \sum_{i=1}^{n}\left[A_{ni}Z_{ni} - E\left(A_{ni}Z_{ni}\right)\right]. \end{aligned}$$

It follows by Lemma 3 and Lemma 5 that for each $n \ge 1$, $\{A_{ni}Y_{ni} - E(A_{ni}Y_{ni}); 1 \le i \le n\}$ and $\{A_{ni}Z_{ni} - E(A_{ni}Z_{ni});\ 1 \le i \le n\}$ are still sequences of pairwise NQD r.v.s. We see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\tfrac{1}{n^{2-r}}P\left(\left|\sum_{i=1}^{n}\left[A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}-E\left(A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}\right)\right]\right|>\varepsilon\phi(n)\right)\\
&=\sum_{n=1}^{\infty}\tfrac{1}{n^{2-r}}P\left(|U_n-V_n|>\varepsilon\phi(n)\right)\\
&\le\sum_{n=1}^{\infty}\tfrac{1}{n^{2-r}}P\left(|V_n|>\tfrac{\varepsilon\phi(n)}{2}\right)+\sum_{n=1}^{\infty}\tfrac{1}{n^{2-r}}P\left(|U_n|>\tfrac{\varepsilon\phi(n)}{2}\right)\\
&:=I_1+I_2.
\end{aligned}$$

For I_1, we have by Markov's inequality, Lemma 4 and (2) that

$$\begin{aligned}
I_1&\le\frac{4}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}E\left(|V_n|^2\right)\le\frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}\sum_{n=1}^{\infty}E\left(A_{ni}Z_{ni}\right)^2<\infty\\
&\le\frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{1-r}}P\left(|X|>\phi(n)\right)\le\frac{C}{\varepsilon^2}E\left(\phi^{-1}\left(|X|\right)\right)^r<\infty.
\end{aligned}$$

Next, we shall show that $I_2<\infty$. Again, applying Markov's inequality, Lemma 4 and (2), we get

$$\begin{aligned}
I_2&\le\frac{4}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}E\left(|U_n|^2\right)\le\frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}\sum_{i=1}^{n}E\left(A_{ni}^2Y_{ni}^2\right)\\
&\le\frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{1-r}}P\left(|X|>\phi(n)\right)+\frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}\sum_{i=1}^{n}E\left(A_{ni}^2X_{ni}^2I_{(|X_{ni}|\le\phi(n))}\right)\\
&\le\frac{C}{\varepsilon^2}E\left(\phi^{-1}\left(|X|\right)\right)^r+\frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}E\left(X^2I_{(|X|\le\phi(n))}\right)\sum_{i=1}^{n}E\left(A_{ni}^2\right)\\
&\le\frac{C}{\varepsilon^2}E\left(\left(\phi^{-1}\left(|X|\right)\right)^r\right)+\frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{E\left(X^2I_{(|X|\le\phi(n))}\right)}{n^{1-r}\phi^2(n)}.
\end{aligned}$$

To prove $I_2<\infty$, it remains to prove that

$$\sum_{n=1}^{\infty}\frac{E\left(X^2I_{(|X|\le\phi(n))}\right)}{n^{1-r}\phi^2(n)}\le\infty.$$

From $\phi\in\mathcal{K}_r$, together with the fact that $|X|=\phi\left(\phi^{-1}(|X|)\right)$, one has

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{X^2I_{(|X|\le\phi(n))}}{n^{1-r}\phi^2(n)}&\le n_0^r+C\sum_{n=n_0+1}^{\infty}\frac{X^2I_{(|X|\le\phi(n))}}{n^{1-r}\phi^2(n+1)}\\
&\le n_0^r+CX^2\int_{\phi^{-1}(|X|)}^{\infty}\frac{x^{r-1}dx}{\phi^2(x)}\\
&\le n_0^r+Ca\left(\phi^{-1}\left(|X|\right)\right)^r+Cb
\end{aligned}$$

which implies

$$\sum_{n=1}^{\infty}\frac{E\left(X^2I_{(|X|\le\phi(n))}\right)}{n^{1-r}\phi^2(n)}\le n_0^r+Cb+CaE\left(\left(\phi^{-1}(|X|)\right)^r\right)\le\infty.$$

The proof is completed.

Now, we consider the class $\mathcal{H}$ of all function $\phi(x)$ which satisfies the following condition: there exist constants a and b such that for $s > d$,

$$\phi(s)\int_d^s \frac{dx}{\phi(x)} \le \text{as} + b.$$

Theorem 7. Let $r \ge 1$, and $\{X, X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise pairwise NQD and identically distributed r.v.s with zero mean. Let $\{A_{ni}; 1 \le i \le n, n \ge 1\}$ be an array of rowwise pairwise NQD r.v.s satisfying (2). Assume that $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ and $\{A_{ni}; 1 \le i \le n, n \ge 1\}$ are independent. If there exists $\phi \in \mathcal{K}_r \cap \mathcal{H}$ such that

$$E\left(\left(\phi^{-1}(|X|)\right)^r\right) < \infty.$$

Then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\left|\sum_{i=1}^{n} A_{ni}X_{ni}\right| > \varepsilon\phi(n)\right) < \infty.$$

Proof. Using Theorem 6, we obtain:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\left|\sum_{i=1}^{n} A_{ni}X_{ni} - E\left(A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}\right)\right| > \varepsilon\phi(n)\right) < \infty.$$

Then, it is enough to prove that

$$\frac{\sum_{i=1}^{n}\left|E\left(A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}\right)\right|}{\phi(n)} \to 0 \quad as \quad n \to \infty. \tag{5}$$

For $n \ge 1$, by the Cauchy-Schwarz inequality and (2),

$$\left(\sum_{i=1}^{n} E|A_{ni}|\right)^2 \le n\left(\sum_{i=1}^{n} E\left(A_{ni}^2\right)\right) \le Cn^2. \tag{6}$$

This implies that

$$\begin{aligned}\frac{\sum_{i=1}^{n}\left|E\left(A_{ni}X_{ni}I_{(|X_{ni}|\le\phi(n))}\right)\right|}{\phi(n)} &= \frac{E\left(|X|I_{(|X|>\phi(n))}\right)}{\phi(n)}\sum_{i=1}^{n} E|A_{ni}| \\ &\le \frac{CnE\left(|X|I_{(|X|>\phi(n))}\right)}{\phi(n)}.\end{aligned} \tag{7}$$

Moreover, $\phi \in \mathcal{H}$ then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{E\left(|X|I_{(|X|>\phi(n))}\right)}{\phi(n)} &\le \sum_{n=1}^{\infty}\frac{1}{\phi(n)}\sum_{k=n}^{\infty} E\left(|X|I_{(\phi(k)<|X|<\phi(k+1))}\right) \\ &\le C\sum_{k=1}^{\infty} E\left(I_{(\phi(k)<|X|<\phi(k+1))}\right)\phi(k)\sum_{n=1}^{k}\frac{1}{\phi(n)} \\ &\le C\sum_{k=1}^{\infty} E\left(kI_{(k<\phi^{-1}(|X|)<k+1)}\right) \\ &\le C\sum_{k=1}^{\infty} E\left(\phi^{-1}(|X|)I_{(k<\phi^{-1}(|X|)<k+1)}\right) \\ &\le CE\left(\phi^{-1}(|X|)\right) < \infty.\end{aligned}$$

By Kronecker's lemma, we obtain:

$$\frac{nE\left(|X|I_{(|X|\leq\phi(n))}\right)}{\phi(n)} \leq \frac{\sum_{k=1}^{n} E\left(|X|I_{(|X|\leq\phi(k))}\right)}{\phi(n)} \to 0 \quad as \quad n\to\infty. \tag{8}$$

Combining (7) and (8), we obtain (5). This proof is completed.

Now, we present an example to illustrate Theorem 6 and Theorem 7.

Example 8. Let $\{X, X_n; n \geq 1\}$ be a sequence of pairwise NQD r.v.s. defined in Example 2 and $\{F_n, n \geq 1\}$ be the distribution functions of a common density function:

$$f(x) = \begin{cases} \dfrac{\alpha}{2|x|^{\alpha+1}} & for, \quad |x| > 1, \\ 0 & otherwise, \end{cases}$$

where $0 < \alpha < 2$.

For $0 < rp < \alpha$, put $\phi(x) = x^{1/p}$. It is easy to see that $\phi \in \mathcal{K}_r$ and $E(\phi^{-1}(|X|)^r) = E|X|^{rp} < \infty$. Therefore, for an array of r.v.s $\{A_{ni}; 1 \leq i \leq n, n \geq 1\}$ satisfying (2) and every $\varepsilon > 0$, we get:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\left|\sum_{i=1}^{n} A_{ni}X_{ni} - E\left(A_{ni}X_{ni}I_{(|X_{ni}|\leq n^{1/p})}\right)\right| > \varepsilon n^{1/p}\right) < \infty.$$

When $1 < \alpha < 2$ we can see $EX = 0$, using Holder's inequality and (2), we have

$$\begin{aligned} \frac{\sum_{i=1}^{n}\left|E\left(A_{ni}X_{i}I_{(|X|\leq n^{1/p})}\right)\right|}{n^{1/p}} &\leq \frac{E\left(|X|I_{(|X|>n^{1/p})}\right)}{n^{1/p}} \sum_{i=1}^{n} E(|A_{ni}|) \\ &\leq CE\left(n^{\frac{p-1}{p}}|X|I_{(|X|\leq n^{1/p})}\right) \\ &\leq CE\left(|X|^{p-1} I_{(|X|\leq n^{1/p})}\right) \\ &\leq CE\left(|X|^{rp} I_{(|X|\leq n^{1/p})}\right) \to 0 \quad \text{as} \quad n\to\infty. \end{aligned}$$

On the other hand, for $rp \geq \alpha$, applying Proposition 6.1.4 in [18], we have that for $0 < \delta < 1$ and n large,

$$2P\left(\left|\sum_{k=1}^{n} X_k\right| > x\right) \geq (1-\delta)n\frac{1}{(2x)^{\alpha}}, \quad x > 0.$$

This implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\left|\sum_{i=1}^{n} X_k\right| > n^{1/p}\varepsilon\right) \geq \frac{1-\delta}{2(2\varepsilon)^{\alpha}} \sum_{n=1}^{\infty} \frac{1}{n^{1-(rp-\alpha)/p}} = \infty.$$

Next, using Theorem 6, we give a result concerning the theory of regularly varying functions. We first recall the concept of slowly varying functions at infinity as follows.

Definition 9. Let $a \geq 0$. A positive measurable function $f(x)$ on $[a; \infty)$ is said to be slowly varying at infinity if

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = 1 \text{ for all } t > 0.$$

For $x > 0$, we denote $\log x = \max\{1, \ln x\}$, $\log_2 x = \log(\log x)$ where lnx is the natural logarithm function. Clearly, $\log x$ and $\log_2 x$ are slowly varying functions at infinity. If X is a random variable with $E(|X|^r) < \infty$ then $H(x) = E\big(|X|^r I(|X| \le x)\big)$ is a slowly varying function at infinity. For properties of slowly varying functions, the reader may refer to [19]. Let

$$\phi(x) = x^p \ell(x^p), \quad \frac{r}{2} < p < r, \quad 1 \le r \le 2, \tag{9}$$

where $\ell(x)$ is a slowly varying function at infinity. It is easy to check that $\phi(x) \in \mathcal{K}_r \cap \mathcal{H}$. Moreover, it follows from Theorem 1.5.12 and Proposition 1.5.15 in [19] that $\phi^{-1}(x) \sim \left(x\ell^{\#}(x)\right)^{1/p}$, where $\ell^{\#}(x)$ is the de Bruijn conjugate of $\ell(x)$. We obtain the following corollary.

Corollary 10. Let $1 \le r \le 2$, $\frac{r}{2} < p < r$ and let $\{X, X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise pairwise NQD and identically distributed r.v.s with mean zero, $\ell(x)$ be a slowly varying function at infinity. Let $\{A_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise pairwise NQD r.v.s satisfying (2). Assume that $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ and $\{A_{ni}, 1 \le i \le n, n \ge 1\}$ are independent. If

$$E\left(\left(|X|\ell^{\#}(|X|)\right)^{r/p}\right) < \infty,$$

then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\left|\sum_{i=1}^{n} A_{ni} X_{ni}\right| > \varepsilon n^p \ell(n^p)\right) < \infty.$$

Remark 11. In Theorem 6, Theorem 7 and Corollary 10, taking $r = 2$, we obtain the results of almost sure convergence for randomly weighted sums of pairwise NQD r.v.s.

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