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Original Article

The Rank of Matrices Over Positive Cones of Commutative Semirings

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Abstract: In this work, we introduced and studied the PC rank notion of a commutative semiring, and compared the PC rank with the nonnegative rank and the factor ranks of matrices, and gived also some sufficient conditions for these ranks to be the same.

Keywords: Ring, Commutative semiring, Positive cone, Idempotent matrix, Nonnegative matrix.

1. Introduction

In recent times, rank functions such as factor rank, no-nnegative rank and rank of completely positive matrices have attracted considerable attention from scientists when investigating conditions for a matrix to be factorizable into the product of matrices over a given commutative semiring [1-7]. In 1993, Cohen and Rothblum [1] proved some characteristic properties of the non-negative rank of nonnegative matrices over the semiring of real numbers ¡ , they also constructed a procedure for computing the non-negative rank of any non-negative matrix over this semiring. Moreover, according to [1, Theorem 4.1], if a non-negative matrix *A* has factor rank less than or equal to 2 then the factor rank and nonnegative rank of *A* coincide. In [2], Beasley and Laffey provided upper bounds on the nonnegative rank of nonnegative real matrices A and A^2. In [3], Mohindru initiated research on completely positive matrices over commutative semirings. Accordingly, a non-negative matrix over a commutative semiring *S* is a matrix with entries in the positive sub-semiring $P(S)$ of *S*, see [3, Definition 7.5.1]. According to [3, Proposition 7.5.2], the factor rank of a non-negative matrix is always less than or equal to its non-negative rank. In [4], Mohindru and Pereira provided necessary and sufficient conditions for a symmetric matrix over a regular incline to be completely positive, see [4,

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Theorem 2.2]. In [5], Shitov constructed an algorithm for computing the non-negative rank of tridiagonal matrices over specific commutative semirings [5, Theorem 12]. In [6], Bukovšek and Šmigoc presented some fundamental properties of symmetric non-negative matrices over the semiring of real numbers $\mathbb{R}$ and considered symmetric non-negative matrices trifactorization. In [7], Tung and Hoang introduced the asymptotic spectrum of non-negative matrices and some characteristic properties regarding the asymptotic non-negative rank of matrices with entries in the semiring of non-negative real numbers $\mathbb{R}^{+}$.

Regarding the rank of matrices over the positive cones of a given semiring, according to [8], every positive cone (ordering) *P* of a commutative semiring *S* always contains the positive subsemiring $P(S)$ of *S*. Therefore, every matrix A with entries in $P(S)$ can be factored into the product of two matrices over semirings $P(S), P$ and *S*. For example, if matrix *A* has *m* rows and *n* columns then it can be factored as $A = I_m A = A I_n$, where A, I_m, I_n are matrices with entries in $P(S) \subset P \subset S$. Let k be the smallest non-negative integer such that there exist matrices *B* and *C* with entries in the positive cone *P*, and $A = BC$, where *k* is the number of columns in matrix *B*. Then, it can be easily verified that $rank_N(A) \leq k \leq rank(A)$, where $rank_N(A)$ is the non-negative rank of matrix A defined in [3, Definition 7.5.1], and $rank(A)$ is the factor rank of matrix *A* over the simering *S*. In this work, we provide some fundamental properties of the PC rank of matrices over positive cones of commutative semirings. We also give sufficient conditions for the equality of the PC rank, factor rank, and non-negative rank of matrices. Moreover, we describe the structure of idempotent matrices over positive cones of commutative semirings whose PC rank and factor rank coincide.

2. Preliminaries

A semiring [9] is a nonempty set *S* equipped two binary operations, addition $(+)$ and multiplication $(.)$, such that:

- $(S,+)$ is a commutative monoid with identity element 0_S;
- $(S,.)$ is a monoid with identity element 1_S;
- Multiplication distributes on both sides with addition;
- $0_S.m = m.0_S = 0_S, \forall m \in S$.

If $(S,.)$ is a commutative monoid then *S* is a *commutative semiring*. The semiring *S* is called *zerosumfree* if $a + b = 0 \Rightarrow a = b = 0, \forall a, b \in S$.

The subset *R* of *S* is called a *subsemiring* of the semiring *S* if $\{0,1\} \subset R$ and it is closed under the addition and multiplication on *R*.

For two commutative semirings *R* and *S*, a mapping $\varphi : S \to R$ is called a *homomorphism* if it satisfies the following conditions:

i) $\varphi(0_S) = 0_R$, $\varphi(1_S) = 1_R$;

ii) $\varphi(a+b) = \varphi(a) + \varphi(b), \forall a, b \in S$;

iii) $\varphi(ab) = \varphi(a)\varphi(b), \forall a, b \in S$.

If there is no risk of confusion, we can write 0 instead of $0_S, 0_R$ and 1 instead of $1_S, 1_R$.

Recall in [4] and [8] that let S be a commutative semiring and P is a subsemiring of S containing the set S^2, where $S^2 = \{s^2 \mid s \in S\}$. We define $V(P) = \{x \in P \mid \exists y \in P : x + y = 0\}$. If $1 \notin V(P)$ then the subsemiring P is called a *prepositive cone* of the semiring S. Moreover, if the prepositive cone P satisfies the following conditions, it is called a *positive cone* of the semiring S:

i) If $a \in S$ and $a \notin P$ then there exists an element $b \in P$ such that $a + b = 0$;

ii) For any $a, b \in S$, if $ab \in V(P)$ then $a \in P$ or $b \in P$.

Notice that, if S is a commutative semiring, then the set $P(S)$ consisting of finite sums of elements in S^2 is a subsemiring of S. Furthermore, if P is a positive cone of S one can have $P(S) \subset P$ (since P is a subsemiring of S and $S^2 \subset P$). In the general case, the subsemiring $P(S)$ may not be a positive cone of the semiring S. According to [8, Theorem 3.3], a commutative semiring S has at least one positive cone if and only if $1 + x \neq 0, \forall x \in P(S)$. In that case, we denote $PC(S)$ as the set of positive cones of the semiring S. It is easy to verify that if S is a zero-sumfree semiring, one obtains $PC(S) = \{S\}$.

Example 2.1. Let $\mathbb{R}$ be the semiring of real numbers and $\mathbb{B}$ be the Boolean semiring, we have the set $S = \mathbb{R} \times \mathbb{B}$ forming a commutative semiring with the operations of addition $(\oplus)$ and multiplication $(\odot)$ defined as follows: For any $(a,b),(s,t) \in S$, $(a,b) \oplus (s,t) = (a+s, b+t)$ and $(a,b) \odot (s,t) = (as, bt)$. Here, the additive monoid $(S, \oplus)$ has the unit element $(0,0)$, and the multiplicative monoid $(S, \odot)$ has the unit element $(1,1)$. We have, $V(S) = \{(x,0) \mid x \in \mathbb{R}\}$, this implies that $(1,1) \notin V(S)$. Thus $P = S$ is a positive cone of the semiring S. It is easy to verify that $P(S) = \mathbb{R}^+ \times \mathbb{B}$, where $\mathbb{R}^+$ is a subsemiring of $\mathbb{R}$ consisting of non-negative real numbers. Since $V(P(S)) = \{(0,0)\}$ hence $(1,1) \notin V(P(S))$, and so, $P(S)$ is a prepositive cone of the semiring S. However, $P(S)$ is not a positive cone of the semiring S since the element $(-1,1)$ does not belong to $P(S)$ and $(-1,1) + (x,y) = (x-1,1) \neq (0,0), \forall (x,y) \in P(S)$.

Example 2.2. For $S = \mathbb{R}^+[x]$, a semiring consisting of polynomials in the variable x with coefficients in $\mathbb{R}^+$, we have $P(S) = \left\{ \sum_{i \in J} (p_i(x))^2 \mid p_i(x) \in S, 0 < |J| < +\infty \right\}$, where $|J|$ is the number of elements in a certain set J. We have $V(P(S)) = \{0\}$ and $1 \notin V(P(S))$, this follows that $P(S)$ is a prepositive cone of the semiring S. On the other hand, the polynomial $p(x) = x + 1 \in S$ but $p(x) \notin P(S)$, satisfies the conditon $p(x) + q(x) \neq 0, \forall q(x) \in P(S)$. Therefore, $P(S)$ is not a positive cone of the semiring S.

Proposition 2.3 [8, Lemma 3.5]. Let P be a subset of the commutative semiring S, and $a, b \in S$.

i) If P is a prepositive cone of S, then $a \in V(P)$ and $a + b \in V(P)$ imply $b \in V(P)$.

ii) If P is a positive cone of S, then $rm \in V(P)$ for all $m \in V(P), r \in S$.

Note that from Proposition 2.3, we have $ar+bs\in V(P)$ for all $a,b\in V(P)$ and $r,s\in S$. Let S be a commutative semiring, we denote by $M_{m\times n}(S)$ the set of $m\times n$ matrices over semiring S, and $M_n(S)$ is the set of square matrices of order n. The sum of all entries on the main diagonal of matrix $A=(a_{ij})\in M_n(S)$ is called *trace* of A, denoted by $Tr(A)$. It is easy to verify that for any matrices $A,B\in M_n(S)$, we have $Tr(AB)=Tr(BA)$. A matrix $E\in M_n(S)$ is called *idempotent* if $E=E^2$. According to [10], two square matrices $E\in M_n(S), F\in M_m(S)$ are said to be *equivalent* over the semiring S if there exist matrices $A\in M_{n\times m}(S), B\in M_{m\times n}(S)$ such that $E=AB$ and $F=BA$, and denoted by $E\cong_S F$. Note that, for P being a subsemiring of the semiring S, and $E\in M_n(P), F\in M_m(P)$ being idempotent matrices over P, if $E\cong_P F$ then $E\cong_S F$ but the converse is generally not true. The *factor rank* of a matrix $A\in M_{m\times n}(S)$ is the smallest non-negative integer k such that there exist matrices $M\in M_{m\times k}(S)$ and $N\in M_{k\times n}(S)$ satisfying $A=MN$. The factor rank of the matrix A is denoted by $rank(A)$. Note that if $A\in M_{m\times p}(S), B\in M_{p\times n}(S)$ then $rank(AB)\le \min\{rank(A), rank(B)\}$. If a matrix A belongs to the set $M_{m\times n}(P(S))$, then A is called a *non-negative matrix* over the semiring S. According to [3, Definition 7.5.1], *non-negative rank* of a non-negative matrix $A\in M_{m\times n}(P(S))$, denoted by $rank_N(A)$, is the smallest non-negative integer q such that there exist matrices $B\in M_{m\times q}(P(S))$ and $C\in M_{q\times n}(P(S))$ satisfying $A=BC$. According to [3, Proposition 7.5.2], let S be a commutative semiring, and a matrix $A\in M_{m\times n}(P(S))$, we always have $rank(A)\le rank_N(A)\le \min\{m,n\}$. Notice that if $S=\mathbb{R}$, then $P(S)=P(\mathbb{R})=\mathbb{R}^+$ the semiring of nonnegative real numbers. In [1], Cohen and Rothblum provided a non-negative matrix $A=\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&1\end{pmatrix}$ over subsemiring $P(\mathbb{R})$ satisfying the inequality $rank(A)<rank_N(A)$. Furthermore, by [1, Theorem 4.1], for any non-negative matrix A over subsemiring $P(\mathbb{R})$ of $\mathbb{R}$, if $rank(A)\le 2$ then $rank(A)=rank_N(A)$.

3. Main Results

In this section, we investigate some fundamental properties regarding the rank of matrices with entries in a positive cone of a given commutative semiring. Moreover, we compare them with factor rank, non-negative rank, and consider some cases where these rank functions coincide.

Definition 3.1. Let S be a commutative semiring with $PC(S)\neq\varnothing$ and $P\in PC(S)$. The *PC rank* of a matrix $A\in M_{m\times n}(P)$ is the smallest non-negative integer k such that there exist matrices

$B \in M_{m\times k}(P)$ and $C \in M_{k\times n}(P)$ satisfying $A = BC$. The PC rank of matrix $A \in M_{m\times n}(P)$ is denoted by $rank_{PC}(A)$.

Remark 3.2. If P is a positive cone of the commutative semiring S, then for any $A \in M_{m\times n}(P)$, we always have $A = AI_n = I_m A$. Since entries of matrices I_m, I_n belong to *P*, the PC rank of matrix A always exists and satisfies the condition $rank_{PC}(A) \le \min\{m,n\}$. In addition, since $P(S) \subset P \subset S$, hence, for any matrix $A \in M_{m\times n}(P(S))$, we get the inequality $rank(A) \le rank_{PC}(A) \le rank_N(A)$. The following examples provide cases where the equal sign "=" does not hold.

Example 3.3. Consider the commutative semiring $S = \mathbb{R}$ with $P(S) = \mathbb{R}^+$ and $V(P(S)) = \{0\}$. Therefore, $1 \notin V(P(S))$. For any $x \in S$ and $x \notin P(S)$, there always exists an element $-x \in P(S)$ such that $x + (-x) = 0$. Furthermore, for any $x, y \in S$ such that $x.y \in V(P(S)) = \{0\}$, we have $x = 0 \in P(S)$ or $y = 0 \in P(S)$. Thus $P = P(S)$ is a positive cone of *S*. We have the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in M_4(P(S))$. According to [1], $rank(A) = 3 < 4 = rank_N(A)$. Moreover, since $P = P(S)$ hence $rank_N(A) = rank_{PC}(A)$, and so $rank(A) < rank_{PC}(A) = rank_N(A)$.

Lemma 3.4. Let S, R be commutative semirings. If there exists a homomorphism $\varphi : S \to R$, then for any matrix $A = (a_{ij}) \in M_{m\times n}(P(S))$, we have the matrix $\varphi(A)$ defined by $\varphi(A) = (\varphi(a_{ij})) \in M_{m\times n}(P(R))$, satisfying the inequality $rank_N(\varphi(A)) \le rank_N(A)$.

Proof. For any matrix $A = (a_{ij}) \in M_{m\times n}(P(S))$, we have $a_{ij} \in P(S), \forall i = 1,...,m; j = 1,...,n$. Since φ is a homomorphism hence $\varphi(a_{ij}) \in P(R), \forall i = 1,...,m; j = 1,...,n$, this follows that $\varphi(A) = (\varphi(a_{ij})) \in M_{m\times n}(P(R))$. Suppose that $rank_N(A) = k$, then there exist matrices $B \in M_{m\times k}(P(S)), C \in M_{k\times n}(P(S))$ such that $A = BC$, and so $\varphi(A) = \varphi(B).\varphi(C)$ (because φ is a homomorphism). This implies that $rank_N(\varphi(A)) \le k = rank_N(A)$. □

Example 3.5. Consider the commutative semiring $S = \mathbb{R} \times \mathbb{B}$ introduced in section 2 and the matrix $A = \begin{pmatrix} (1,0) & (0,0) & (0,0) & (1,0) \\ (1,0) & (1,0) & (0,0) & (0,0) \\ (0,0) & (1,0) & (1,0) & (0,0) \\ (0,0) & (0,0) & (1,0) & (1,0) \end{pmatrix}$, we observe that entries of *A* belong to $P(S) = \mathbb{R}^+ \times \mathbb{B}$.

Then, $P = S$ is a positive cone of the semiring *S*. Furthermore, we have:

$$A=\begin{pmatrix}(1,0) & (0,0) & (0,0)\\ (1,0) & (1,0) & (0,0)\\ (0,0) & (1,0) & (1,0)\\ (0,0) & (0,0) & (1,0)\end{pmatrix}\begin{pmatrix}(1,1) & (0,0) & (0,0) & (1,0)\\ (0,0) & (1,1) & (0,0) & (-1,0)\\ (0,0) & (0,0) & (1,1) & (1,0)\end{pmatrix}$$, this follows that:

$$rank(A)=rank_{PC}(A)\le 3.$$

Now, consider the mapping $\varphi : S\to\mathbb{R}$ defined by $\varphi((x,y))=x,\forall(x,y)\in S$. It is easy to verify that φ is a homomorphism and $\varphi(A)=\begin{pmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\in M_4(P(\mathbb{R}))=M_4(\mathbb{R}^+)$. Applying Lemma 3.4, we obtain the inequality $rank_N(\varphi(A))\le rank_N(A)$. By Example 3.3, we have $rank_N(\varphi(A))=4$, and so $rank_N(A)=4$. Thus $rank_{PC}(A)\le 3<4=rank_N(A)$.

Next, we provide some basic properties regarding the PC rank of matrices over the positive cones of a given commutative semiring. The following proposition provides some basic properties of the PC rank, and the proof is similar to the one introduced in [11].

Proposition 3.6. Let P is a positive cone of the commutative semiring S and matrices $A\in M_{m\times n}(P)$, $B\in M_{n\times p}(P)$, $C\in M_{m\times q}(P)$, $D\in M_{m\times n}(P)$, $E\in M_{k\times t}(P)$. Then, the following statements hold:

i) $rank_{PC}(AB)\le\min\{rank_{PC}(A),rank_{PC}(B)\}$.

ii) $\max\{rank_{PC}(A),rank_{PC}(C)\}\le rank_{PC}\begin{pmatrix}A & C\end{pmatrix}\le rank_{PC}(A)+rank_{PC}(C)$.

iii) $\max\{rank_{PC}(A),rank_{PC}(E)\}\le rank_{PC}\begin{pmatrix}A&0\\0&E\end{pmatrix}\le rank_{PC}(A)+rank_{PC}(E)$.

iv) $rank_{PC}(A+D)\le\min\{rank_{PC}(A)+rank_{PC}(D),m,n\}$.

Where $\begin{pmatrix}A & C\end{pmatrix}$ and $\begin{pmatrix}A&0\\0&E\end{pmatrix}$ are block matrices formed by matrices A,C,E.

Proof.

Suppose that $rank_{PC}(A)=r, rank_{PC}(B)=s, rank_{PC}(C)=u$, then there exist matrices $M\in M_{m\times r}(P)$, $N\in M_{r\times n}(P)$, $H\in M_{n\times s}(P)$, $G\in M_{s\times p}(P)$, $X\in M_{m\times u}(P)$ and $Y\in M_{u\times q}(P)$ such that $A=MN$, $B=HG$ and $C=XY$.

i) Since $AB=MNHG$, hence, $rank_{PC}(AB)\le\min\{r,s\}=\min\{rank_{PC}(A),rank_{PC}(B)\}$.

ii) We have $\begin{pmatrix}A & C\end{pmatrix}=\begin{pmatrix}MN & XY\end{pmatrix}=\begin{pmatrix}M & X\end{pmatrix}\begin{pmatrix}N&0\\0&Y\end{pmatrix}$, this implies that $rank_{PC}\begin{pmatrix}A & C\end{pmatrix}\le r+u=rank_{PC}(A)+rank_{PC}(C)$.

Suppose that $rank_{PC}\begin{pmatrix}A & C\end{pmatrix}=v$, then there exist matrices $M'\in M_{m\times v}(P), N'\in M_{v\times(n+q)}(P)$ such that $\begin{pmatrix}A & C\end{pmatrix}=M'N'$. Set $N'=\begin{pmatrix}H' & G'\end{pmatrix}$, where $H'\in M_{v\times n}(P), G'\in M_{v\times q}(P)$. We have $\begin{pmatrix}A & C\end{pmatrix}=\begin{pmatrix}M'H' & M'G'\end{pmatrix}$, hence, $A=M'H', C=M'G'$. Thus $rank_{PC}(A)\le v$ and $rank_{PC}(C)\le v$, this implies that $\max\{rank_{PC}(A), rank_{PC}(C)\}\le v=rank_{PC}\begin{pmatrix}A & C\end{pmatrix}$.

iii) and *iv)* follow from *i)* and *ii)*.

Definition 3.7. Let P be a positive cone of the commutative semiring *S*. A square matrix $E\in M_n(P)$ is called *PC - full* if $rank_{PC}(E)=n$.

Remark 3.8. If P is a positive cone of the commutative semiring *S*, then every identity matrix I_n is PC - full. Indeed, since *S* is a commutative semiring, hence, *P* is also a commutative semiring. For any matrices $A, B\in M_k(P)$ such that $AB=I_k$, then by [12], we have $BA=I_k$. Applying [10, Mệnh đề 3.12] to the semiring *P* with PC rank, we obtain the following result: For any matrices $G\in M_{n\times m}(P), H\in M_{m\times n}(P)$ such that $GH=I_n$, it follows that $m\ge n$. Thus $rank_{PC}(I_n)=n$. Similarly, applying [10, Lemma 3.12] to the commutative semiring *S*, we also have $rank(I_n)=n$.

Note that, for any matrix $A\in M_{m\times n}(P)$, where P is a positive cone of the commutative semiring *S*, if $rank_{PC}(A)=k$ then there exist matrices $B\in M_{m\times k}(P)$ and $C\in M_{k\times n}(P)$ such that $A=BC=BI_kC$. This gives us a hint on how to compute the PC rank of matrices through the PC rank of PC - full idempotent matrices, as showed in the following proposition.

Proposition 3.9. Let P be a positive cone of the commutative semiring *S* and a matrix $A\in M_{m\times n}(P)$. Then, $rank_{PC}(A)=\min\{rank_{PC}(E)\mid A=BEC, E\in \mathbb{EM}_{fpc}(P)\}$, where B, C are matrices with entries in *P*, $\mathbb{EM}_{fpc}(P)$ is the set of PCs - full idempotent matrices over *P*.

Proof. We have $A=I_mI_mA=AI_nI_n$, this implies that the set $\{rank_{PC}(E)\mid A=BEC, E\in \mathbb{EM}_{fpc}(P)\}$ is a nonempty set, where B, C are matrices with entries in *P*. Furthermore, if $rank_{PC}(A)=k$ then there exist matrices $B\in M_{m\times k}(P)$ and $C\in M_{k\times n}(P)$ such that $A=BC=BI_kC$, since $rank_{PC}(I_k)=k$ (by Remark 3.8), hence, we obtain the inequality $rank_{PC}(A)\ge \min\{rank_{PC}(E)\mid A=BEC, E\in \mathbb{EM}_{fpc}(P)\}$. Conversely, suppose that $\min\{rank_{PC}(E)\mid A=BEC, E\in \mathbb{EM}_{fpc}(P)\}=k$, then there exists a matrix $E\in \mathbb{EM}_{fpc}(P)$ with order *k*, and matrices $B\in M_{m\times k}(P), C\in M_{k\times n}(P)$ such that $A=BEC$ and $rank_{PC}(E)=k$. Applying Proposition 3.6, we have $rank_{PC}(A)=rank_{PC}(BEC)\le rank_{PC}(E)=k$. So, the proposition has been proven. □

Next, we investigate some conditions under which the PC rank of matrices coincides with the non-negative rank or the factor rank.

Proposition 3.10. Let P be a positive cone of the commutative semiring *S*. Then,

i) If $A\in M_n(P)$ is an invertible matrix over the semiring *S* then $rank_{PC}(A)=rank(A)=n$.

ii) If $A\in M_n(P(S))$ is an invertible matrix over the semiring S then $rank_N(A)=rank_{PC}(A)=rank(A)=n$.

Proof.

i) If $A \in M_n(P)$ with $rank(A) = k \le n$, then there exist matrices $B \in M_{n\times k}(S)$ và $C \in M_{k\times n}(S)$ such that $A = BC$. Because A is an invertible matrix over the semiring *S*, hence, $I_n = A^{-1}BC$. By Remark 3.8, we have $n = rank(I_n) \le k$, and so $rank(A) = n$. Moreover, $rank(A) \le rank_{PC}(A) \le n.$ Thus $rank_{PC}(A) = rank(A) = n$.

ii) If an invertible matrix $A \in M_n(P(S))$ with $rank(A) = k \le n$, similarly, we can show that $rank(A) = n$. Furthermore, $n = rank(A) \le rank_{PC}(A) \le rank_N(A) \le n$, this follows that $rank_N(A) = rank_{PC}(A) = rank(A) = n$. □

Remark 3.11. Let P be a positive cone of the commutative semiring *S*, if $A \in M_n(P(S))$ is an invertible matrix over the semiring *S*, then A^{-1} may not belong to the set $M_n(P(S))$. Moreover, if $A \in M_n(P)$ is an invertible matrix over the semiring *S*, then A^{-1} may not belong to the set $M_n(P)$. For example, consider the semiring $S = \mathbb{R}$ consisting of real numbers, we have $A = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} \in M_2(P(\mathbb{R}))$ is an invertible matrix over $\mathbb{R}$, and $A^{-1} = \begin{pmatrix} 1 & -\frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix} \notin M_2(P(\mathbb{R}))$. It is easy to verify that $P(\mathbb{R}) = \mathbb{R}^+$, and $P = P(\mathbb{R})$ is a positive cone of *S*. Then, $A^{-1} \notin M_2(P)$. However, consider the matrix $B = \begin{pmatrix} 0 & 4 \\ 9 & 0 \end{pmatrix} \in M_2(P(\mathbb{R}))$, we have $B^{-1} = \begin{pmatrix} 0 & \frac{1}{9} \\ \frac{1}{4} & 0 \end{pmatrix} \in M_2(P(\mathbb{R}))$.

Proposition 3.12. Let P be a positive cone of the commutative semiring *S*, and *Q* be a subsemiring of *S* such that $P \subset Q$. Then, the following statements are equivalent:

i) If any square matrix $A \in M(P)$ is invertible over *Q*, then $A^{-1} \in M(P)$.

ii) $P = Q$.

Proof.

$ii) \Rightarrow i)$: Obviously.

$i) \Rightarrow ii)$: If $P \subset Q$ and $P \ne Q$, then there exists an element $a \in Q \setminus P$. Since P is a positive cone of the semiring *S*, hence, there exists an element $b \in P$ such that $a + b = 0$. We have $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in M_2(P)$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, this implies that $A^{-1} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in M_2(Q)$. So, the matrix A is invertible over *Q*. Therefore, $A^{-1} \in M_2(P)$, and so $a \in P$, this is contradictory to $a \notin P$. Thus $P = Q$. □

The following Corollary can be readily deduced from the Proposition 3.12.

Corollary 3.13. Let P is a positive cone of the commutative semiring *S*. Then, the following statements hold:

i) If $P(S)$ is a positive cone of the semiring *S*, and if any square matrix *A* in $M(P(S))$, which is invertible over the semiring *P*, always has its inverse A^{-1} also in $M(P(S))$, then, $rank_N(B) = rank_{PC}(B), \forall B \in M(P(S))$.

ii) If any square matrix *A* in $M(P)$, which is invertible over the semiring *S*, always has its inverse A^{-1} also in $M(P)$, then, $rank(B) = rank_{PC}(B), \forall B \in M(P)$.

iii) If $P(S)$ is a positive cone of the semiring *S*, and if any square matrix *A* in $M(P(S))$, which is invertible over the semiring *S*, always has its inverse A^{-1} also in $M(P(S))$, then, $rank_N(B) = rank_{PC}(B) = rank(B), \forall B \in M(P(S))$.

Next, we examine the cases where the PC rank and the factor rank of idempotent matrices are equal over commutative semirings.

Theorem 3.14. Let P is a positive cone of the commutative semiring *S*, and $E \in M_n(V(P))$ be an idempotent matrix. Then, $rank(E) = rank_{PC}(E)$.

Proof. Suppose that $rank(E) = k$, then there exist matrices $B \in M_{n\times k}(S)$ and $C \in M_{k\times n}(S)$ such that $E = BC$. Let $F = CEB \in M_k(S)$, we have $F^2 = (CEB)(CEB) = CE^3B = CEB = F$, this follows that F is an idempotent matrix. Furthermore, since $E \in M_n(V(P))$, hence, by Proposition 2.3, we have $CE \in M_{k\times n}(V(P))$, $EB \in M_{n\times k}(V(P))$, and so $F = (CE)(EB) \in M_k(V(P))$, this implies that $BF \in M_{n\times k}(V(P))$, $FC \in M_{k\times n}(V(P))$. Moreover, we can decompose the matrix E into $E = E^5 = B(CBCB)(CBCB)C = (BF)(FC)$, hence, $rank_{PC}(E) \le k = rank(E)$. Therefore, $rank(E) = rank_{PC}(E)$. □

Corollary 3.15. Let P is a positive cone of the commutative semiring *S*, and $E \in M_n(P)$ be an idempotent matrix with $Tr(E) \in V(P)$. Then, the following statements hold:

i) If $rank(E) = 1$ then $rank_{PC}(E) = 1$.

ii) If $rank(E) = 2$, then there exists an idempotent matrix $F \in M_2(P)$ that is PC - full and $E \cong_S F$, such that $rank(E) = rank(F) = rank_{PC}(F) = 2$.

Proof.

i) If $rank(E) = 1$, then there exist matrices $B = (b_{i1}) \in M_{n\times 1}(S)$ and $C = (c_{1j}) \in M_{1\times n}(S)$ such that $E = BC$. Then, $Tr(E) = \sum_{l=1}^{n} c_{1l} b_{l1} \in V(P)$, this follows that $F = CB \in M_1(V(P))$. Hence, $BF \in M_{n\times 1}(V(P))$ and $FC \in M_{1\times n}(V(P))$, and so $E = E^3 = (BF)(FC) \in M_n(V(P))$. By Theorem 3.14, we obtain $rank_{PC}(E) = rank(E) = 1$.

ii) If $rank(E)=2$, then there exist matrices $B=(b_{ij})\in M_{n\times 2}(S)$ and $C=(c_{ij})\in M_{2\times n}(S)$ such that $E=BC$. Let $H=CB=\begin{pmatrix} q_1 & p_1 \\ p_2 & q_2 \end{pmatrix}\in M_2(S)$, we have $q_1+q_2=Tr(CB)=Tr(BC)=Tr(E)\in V(P)$ and $H^2=\begin{pmatrix} q_1^2+p_1p_2 & p_1(q_1+q_2) \\ p_2(q_1+q_2) & q_2^2+p_1p_2 \end{pmatrix}=CEB$. Setting $F=H^2$, we have $F^2=(CEB)(CEB)=CE^3B=CEB=F$, hence, F is an idempotent matrix and

$$F=F^2=\begin{pmatrix} (q_1^2+p_1p_2)^2+p_1p_2(q_1+q_2)^2 & (q_1+q_2)\left[p_1(q_1^2+p_1p_2)+p_1(q_2^2+p_1p_2)\right] \\ (q_1+q_2)\left[p_2(q_1^2+p_1p_2)+p_2(q_2^2+p_1p_2)\right] & (q_2^2+p_1p_2)^2+p_1p_2(q_1+q_2)^2 \end{pmatrix}.$$

Because $(q_1^2+p_1p_2)^2, (q_2^2+p_1p_2)^2\in P(S)\subset P$ and $q_1+q_2\in V(P)$, applying Proposition 2.3, we have the elements $p_1p_2(q_1+q_2)^2$, $(q_1+q_2)\left[p_1(q_1^2+p_1p_2)+p_1(q_2^2+p_1p_2)\right]$ and $(q_1+q_2)\left[p_2(q_1^2+p_1p_2)+p_2(q_2^2+p_1p_2)\right]$ belonging to $V(P)\subset P$. Thus the idempotent matrix $F\in M_2(P)$. Moreover, since $E=E^3=BFC$, hence, $2=rank(E)\le rank(F)\le rank_{PC}(F)\le 2$, this follows that $rank_{PC}(F)=rank(F)=rank(E)=2$, and so F is a PC - full matrix. In additon, because $(EB)C=E^2=E$ and $C(EB)=F$, hence, $E\cong_S F$. □

Remark 3.16. If the matrix $F=H^2=\begin{pmatrix} q_1^2+p_1p_2 & p_1(q_1+q_2) \\ p_2(q_1+q_2) & q_2^2+p_1p_2 \end{pmatrix}$, as determined in Corollary 3.15, has all its entries on the main diagonal belonging to $V(P)$, then $F\in M_2(V(P))$. This follows that $E=BFC=(BF)(FC)\in M_n(V(P))$. By Theorem 3.14, we obtain the equality $rank(E)=rank_{PC}(E)=rank(F)=rank_{PC}(F)=2$.

Example 3.17. The semiring $S=\mathbb{R}\times\mathbb{B}$ has $V(S)=\{(x,0)\mid x\in\mathbb{R}\}$, hence, $(1,1)\notin V(S)$. We have $P=S$ as a positive cone of the semiring *S*. Then, $E=\begin{pmatrix} \left(\frac{1}{2},0\right) & (-2,0) \\ \left(-\frac{1}{8},0\right) & \left(\frac{1}{2},0\right) \end{pmatrix}, G=\begin{pmatrix} \left(\frac{1}{2},0\right) & (1,0) & (0,0) \\ \left(\frac{1}{4},0\right) & \left(\frac{1}{2},0\right) & (0,0) \\ (0,0) & (0,0) & (1,0) \end{pmatrix}$ are idempotent matrices over the positive cone *P*. Furthermore, $Tr(E)=(1,0)\in V(P)$, $Tr(G)=(2,0)\in V(P)$ and $rank(E)=1$, $rank(G)=2$.

Lemma 3.18. Let *S* be a commutative semiring, and $E=(e_{ij})\in M_n(S)$ be an square matrix. Then for any $k\in\{1,2,...,n\}$, the entries of the matrix $B=E^k$ are determined by the formula

$$b_{ij}=\begin{cases}\sum\limits_{l_1,l_2,...,l_{k-1}\in\{1,2,...,n\}} e_{il_1}e_{l_1l_2}...e_{l_{k-2}l_{k-1}}e_{l_{k-1}j},\ k>1\\ e_{ij},\ k=1\end{cases},\forall i,j\in\{1,2,...,n\}, \text{ where } B=(b_{ij}).$$

Proof. If $k=1$ then $B=E$, hence, $b_{ij}=e_{ij},\forall i,j\in\{1,2,...,n\}$. If $k=2$ then $B=E^2$, hence, $b_{ij}=\sum\limits_{l\in\{1,2,...,n\}}e_{il}e_{lj},\forall i,j\in\{1,2,...,n\}$. Suppose that the Lemma is true with $k=q,\ (2\le q<n)$. Let $C=E^q$, the entries of the matrix $B=E^{q+1}=CE$ are determined by $b_{ij}=\sum\limits_{t\in\{1,2,...,n\}}c_{it}e_{tj}$, where $C=(c_{ij})$ and $B=(b_{ij})$. Since $c_{it}=\sum\limits_{l_1,l_2,...,l_{q-1}\in\{1,2,...,n\}}e_{il_1}e_{l_1l_2}...e_{l_{q-2}l_{q-1}}e_{l_{q-1}t}$, hence, $b_{ij}=\sum\limits_{l_1,l_2,...,l_{q-1},t\in\{1,2,...,n\}}e_{il_1}e_{l_1l_2}...e_{l_{q-2}l_{q-1}}e_{l_{q-1}t}e_{tj}$. the Lemma has been proven. □

Lemma 3.19. Let *P* be a positive cone of the commutative semiring *S*, and $A=(a_{ij})\in M_n(P)$ be an idempotent matrix. If $a_{ii}=0,\forall i\in\{1,...,n\}$ then $A=M_n(V(P))$.

Proof. If $n=1$ then $A=(0)\in M_1(V(P))$. If $n>1$, then for each $k\in\{1,2,...,n\}$, by Lemma 3.18, the entries of the main diagonal of the matrix $B=A^k$ are determined by the formula

$$b_{ii}=\begin{cases}\sum\limits_{l_1,l_2,...,l_{k-1}\in\{1,2,...,n\}} a_{il_1}a_{l_1l_2}...a_{l_{k-2}l_{k-1}}a_{l_{k-1}i},\ k>1\\ 0,\ k=1\end{cases},\ \forall i\in\{1,2,...,n\}, \text{ where } B=(b_{ij}).$$

Since $A=(a_{ij})$ is an idempotent matrix with $a_{ii}=0,\forall i\in\{1,...,n\}$, hence, $A=A^k=B$, and so $b_{ii}=\sum\limits_{l_1,l_2,...,l_{k-1}\in\{1,2,...,n\}}a_{il_1}a_{l_1l_2}...a_{l_{k-2}l_{k-1}}a_{l_{k-1}i}=0,\forall i\in\{1,2,...,n\},k\in\{2,3,...,n\}$. This implies that $a_{il_1}a_{l_1l_2}...a_{l_{k-2}l_{k-1}}a_{l_{k-1}i}\in V(P),\forall i\in\{1,2,...,n\}$ (because $A=(a_{ij})\in M_n(P)$). Now let $E=A^n$, then the entries of the matrix *E* are detemined by the following formula:

$$e_{ij}=\begin{cases}\sum\limits_{l_1,l_2,...,l_{n-1}\in\{1,2,...,n\}} a_{il_1}a_{l_1l_2}...a_{l_{n-2}l_{n-1}}a_{l_{n-1}j},\ i\neq j\\ 0,\ i=j\end{cases},\forall i,j\in\{1,2,...,n\}.$$

Let $l_0=i,\ l_n=j$, then for any $l_1,l_2,...,l_{n-1}\in\{1,2,...,n\}$, since $l_0,l_1,...,l_n\in\{1,2,...,n\}$, hence, there exist at least two elements $r,t\in\{0,1,...,n\}$ such that $r<t$ and $l_r=l_t$. This follows that $a_{l_rl_{r+1}}...a_{l_{t-1}l_t}\in V(P)$, and so $a_{il_1}a_{l_1l_2}...a_{l_{n-2}l_{n-1}}a_{l_{n-1}j}=a_{il_1}a_{l_1l_2}...a_{l_rl_{r+1}}...a_{l_{t-1}l_t}...a_{l_{n-2}l_{n-1}}a_{l_{n-1}j}\in V(P)$ (by Proposition 2.3), therefore, $e_{ij}\in V(P),\forall i,j\in\{1,2,...,n\}$. Thus $E=A^n=A\in M_n(V(P))$. □

Corollary 3.20. Let S be a commutative ring, and *Q* be a positive cone of the commutative semiring *R*. Then, the following statements hold:

i) The semiring $P=S\times Q$ is a positive cone of the semiring $S\times R$.

ii) If $E \in M_n(P)$ is an idempotent matrix with entries on the main diagonal belonging to set $S \times \{0_R\}$, then $rank(E) = rank_{PC}(E)$.

Proof.

i) Since Q is a positive cone of the commutative semiring R, hence, $P = S \times Q$ is a subsemiring of the semiring $S \times R$. It easy to see that $V(P) = S \times V(Q)$ (because S is a commutative ring). We have $(1_S, 1_R) \notin V(P)$ (since Q is a positive cone of R hence $1_R \notin V(Q)$). This follows that P is a prepositive cone of the semiring $S \times R$. For any element $(x, y) \in S \times R$ and $(x, y) \notin P = S \times Q$, we have $y \notin Q$. Since Q is a positive cone of R, hence, there exists an element $b \in Q$ such that $y + b = 0_R$, therefore, the element $(-x, b) \in P$ satisfies $(x, y) + (-x, b) = (0_S, 0_R)$. Furthermore, if there exist elements $(x, y), (u, v) \in S \times R$ such that $(x, y)(u, v) = (xu, yv) \in S \times V(Q)$, then $yv \in V(Q)$, and so, $y \in Q$ or $v \in Q$. This implies that $(x, y) \in S \times Q = P$ or $(u, v) \in S \times Q = P$. Thus P is a positive cone of the semiring $S \times R$.

ii) If the idempotent matrix $E = (e_{ij}) \in M_n(P)$ has entries on the main diagonal belonging to the set $S \times \{0_R\}$, then let $e_{ij} = (s_{ij}, q_{ij}), \forall i, j \in \{1, ..., n\}$, we have $e_{ii} = (s_{ii}, 0_R), \forall i \in \{1, ..., n\}$. Consider the homomorphism $\varphi : S \times R \to R$ with $\varphi(s, q) = q, \forall (s, q) \in S \times R$. Then, the matrix $\varphi(E) = (q_{ij}) \in M_n(Q)$ is an idempotent matrix with entries on the main diagonal equal to 0_R. By Lemma 3.19, we have $\varphi(E) \in M_n(V(Q))$, this follows that $e_{ij} = (s_{ij}, q_{ij}) \in S \times V(Q) = V(P), \forall i, j \in \{1, ..., n\}$, and so $E \in M_n(V(P))$. Applying Theorem 3.14, we obtain $rank(E) = rank_{PC}(E)$. □

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