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Original Article

Phenomenological Analogy between Gross-Pitaevskii Theory for Bose-Einstein Condensate and Newton Equation for Classical Mechanics

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Abstract: In this work we found an interesting analogy between Gross-Pitaevskii theory for Bose-Einstein condensate at zero temperature and Newton equation of classical particle in the classical physics. Although this analogy is a pure phenomenology, it brings a new perspective to physics in general, in particular to quantum physics, as well as to many-body physics.

Keywords: Bose-Einstein condensates, Gross-Pitaevskii equation, Newton equation, Phenomenological analogy.

1. Introduction

It is well-known that a Bose-Einstein condensate (BEC) is formed as the temperature of the bosonic system lowered to the critical temperature [1]. Below the critical temperature, the greater number of condensate atoms are, the lower temperature is [2]. Theoretically, all of atoms are condensed at zero temperature. This implies that a phase transition from the normal (non-condensate) phase to the condensate phase takes place at a critical temperature. The condensation phase transition has been observed in experiments [3] after 70 years since the prediction of Einstein [4].

In the condensed phase, state of all of atoms are described as a whole by a wave function, which is the solution of the Gross-Pitaevskii (GP) equation [1, 2]. In case of the stationary potential, the GP equation is the time-independent equation, which is a nonlinear Schrodinger equation. Mathematically, the wave function is a function of the coordinates, whereas it expresses statistically the probability of a finding particle in its own space. It is well-known that the time-independent GP equation has no

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analytical solution. Except some special cases, the solution of time-independent GP equation only can be solved numerically.

In non-relative classical mechanics, the state of a particle is described by the equation of motion, which shows the state of the particle in space-time. Particularly, the equation of motion expresses coordinates of particle as a function of time. The most important thing in establishing the equation of motion is to find the acceleration of the particle. As known that the acceleration can be found by solving second Newton equation [5].

Mathematically, both time-independent GP and the second Newton equation are of the second-order differential equations. In this contribution we introduce a phenomenological analogy between these equations, which will bring a useful insight.

2. Interface of a Two Component Bose-Einstein Condensates Confined Between Two Parallel Plates

In this Section we will explore some analogies between GP theory for BEC and Newton equation for classical mechnics. To begin with, we first recall relevent knownledge about Newton equations. It is well-known that a particle of mass *m* will be accelarated when there is a force $\vec{F}$ acting on. The Newton equation shows the relation between the acting force and accelation $\vec{a}$ via the vector equation [5]

$$m\vec{a} = \vec{F}. \tag{1}$$

The instantaneous velocity and accelation is defined as the first- and second-order derivative of the coordinate with respect to time, respectively

$$\vec{v} = \frac{d\vec{r}}{dt},\ \vec{a} = \frac{d^2\vec{r}}{dt^2}, \tag{2}$$

in which $\vec{r} = (x, y, z).$ For a simplest case, a particle moves in one-dimensional projectile, a combination of Eqs. (1) and (2) allows us to rewrite Newton equation along one axis, let's say *z:*

$$m\frac{d^2z}{dt^2} = F_z, \tag{3}$$

with F_z being the projection of the force into *z*-axis. Assuming that at the initial state, the particle is moving through the position z_0 and velocity v_0. Double integrals Eq. (3) over time one can find the time-dependence of the coordinate in form

$$z = z(z_0, v_0, t). \tag{4}$$

Eq. (4) is sometime called the equation of motion.

We now consider the BEC, which is freely moving in (*x*,*y*)-derections, i.e the translational symmetry is satisfied along O*x* and O*y* axes. Along O*z*-axis, the atoms can move in semi-infinite space because a hard-wall is located at $z = 0$ and perpendicular to O*z*-axis. Experimentally, a hard-wall can be created the magnetic force [6], both magnetic and optical force [7] or the optical lattice [7, 8]. Theoretically, at zero temperature all of bosonic atoms with mass of *m* occupy the same state, which is ground state. The system is considered as a whole and described by the Lagrangian density without external potential [1]:

$$L = \psi^*\left(-i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\right)\psi - \mu|\psi|^2 + \frac{g}{2}|\psi|^4, \tag{5}$$

where $\psi = \psi(\vec{r}, t)$ is the wave function - a function of both the coordinate and time, μ is the chemical potential, $\hbar$ is reduced Planck constant. The coupling constant can be expressed as $g = 4\pi\hbar^2 a_s / m$ with a_s being *s*-wave scattering length. Here we consider the case with repulsive interaction, i.e. $a_s > 0$. From Lagrangian density (5) one has the time-independent GP equation:

$$\frac{\hbar^2}{2m}\frac{d^2\Psi(z)}{dz^2} = -\mu\Psi(z) + g\Psi(z)^3, \tag{6}$$

where the wave function of the ground state is $\Psi(z) = \psi(z,t)\exp(-\frac{i}{\hbar}\mu t)$ [8]. We now employ the idea of Joseph *et. al*. [9,10], in which the phenomenological analogy was applied for a two-component Bose-Einstein condensate within framwork of the density functional theory, to examine the phenomenological analogy between (3) and (6) in the dynamic point of view thoughout this work. It is easily seen that these equations have the same mathematical form, this allows us to interpret the GP equation (6) as the Newton equation (3), which describes a particle with "mass"of $\hbar^2 / 2m$ moving under of acting force:

$$F_z = -\mu\Psi(z) + g\Psi(z)^2. \tag{7}$$

The acting force in (7) consists of two components, namely, the first term represents a resistant force and the remaining is the pulling force. In this regard, the motion of this particle is determined by the "coordinate" $\Psi(z)$ in "time" z.

For more detail, we should transform the GP equation (6) into the dimensionless form. To do so, we introduce the reduced wave function and dimensionless coordinate:

$$\tilde{z} = \frac{z}{\xi}, \tilde{\psi} = \frac{\Psi}{\sqrt{n_0}},$$

in which n_0 is the particle density in bulk, the healing length is defined as $\xi = \hbar / \sqrt{2mgn_0}$. In mean-field theory the chemical potential is $\mu = gn_0$. Consequently, GP equation (6) has the dimensionless form:

$$\frac{d^2\tilde{\psi}}{d\tilde{z}^2} = -\tilde{\psi} + \tilde{\psi}^3. \tag{8}$$

Phenomenologically, the dimensionless GP equation (8) described motion of particle with "mass" of $\tilde{m} = 1$ along $\tilde{\psi}$ -"axis" in "time" $\tilde{z}$. With the presence of the hard-wall at origin $\tilde{z} = 0$, the Dirichlet boundary condition is invoked:

$$\tilde{\psi}(0) = 0, \tilde{\psi}(\infty) = 1. \tag{9}$$

The well-known solution of Eq. (8) with constraint condition (9) is [1, 2]:

$$\tilde{\psi}(z) = \tanh\left(\frac{z}{\xi\sqrt{2}}\right). \tag{10}$$

The "equation of motion" (10) shows the "time" $\tilde{z}$ -dependent of the "coordinate" $\tilde{\psi}$ of the particle with unity "mass" by dimensionless acting force:

$$\tilde{F} = -\tilde{\psi} + \tilde{\psi}^3. \tag{11}$$

In dynamic point of view, which is equivelent to Eq. (4) of the dynamic mechanics. Figure 1 shows graphically the evoluation of (10), in which the solid and dashed lines correspond to "coordinate" $\tilde{\psi}$ and "velocity" $d\tilde{\psi}/d\tilde{z}$ versus "time" $\tilde{z}$. In dynamic point of view, one can examine as folows:

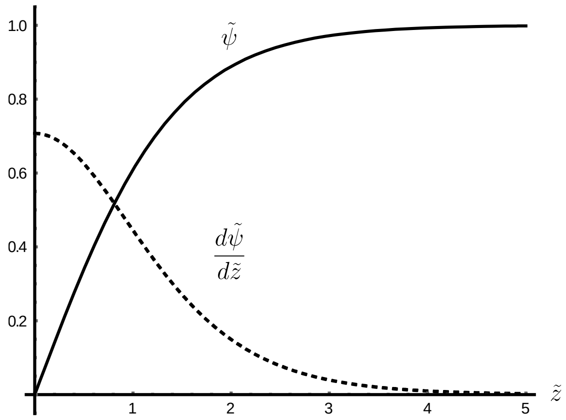

Figure 1. The evolution of the "coordinate" $\tilde{\psi}$ and "velocity" $d\tilde{\psi}/d\tilde{z}$ versus "time" $\tilde{z}$.

- At the initial state, the particle passes the "coordinate" $\tilde{\psi} = 0$ and "velocity" $d\tilde{\psi}/d\tilde{z} = 0.707.$
- The wave function is normalized to the bulk density therefore the reduced wave function is always smaller than unity. This implies that the resultant force (11) is always negative, which plays the role of the resistant force. As a consequence, the motion is the slowing down, the "velocity" decreases and the "coordinate" approaches unity. When "time" $\tilde{z}$ is large enough, the "velocity" vanishes and the particle stops.

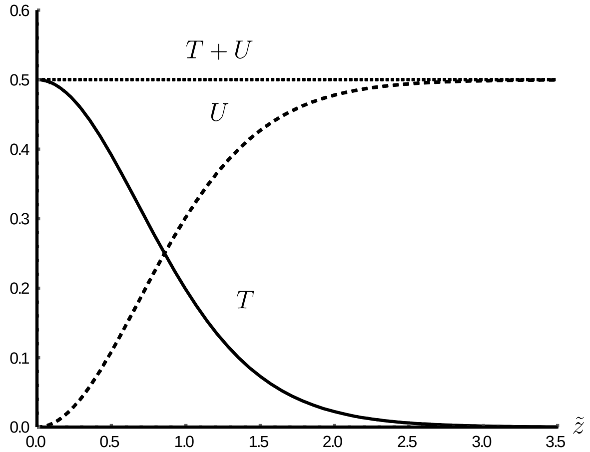

Figure 2. The evolution of the kinetic energy and potential as functions of "time" $\tilde{z}$.

Now we investigate in energy point of view. Multiplying Eq. (8) by $d\tilde{\psi}/d\tilde{z}$ and then integrating over $\tilde{z}$ from $\tilde{z}$ to infinity one has

$$\left(\frac{d\tilde{\psi}}{d\tilde{z}}\right)^2 + \tilde{\psi}^2 - \frac{\tilde{\psi}^4}{2} = \frac{1}{2}. \tag{12}$$

Let K and U respectively be the kinetic energy and potential of the particle, Eq. (12) can be rewritten in form

$$K + U = \frac{1}{2}, \tag{13}$$

in which the kinetic energy is

$$K = \left(\frac{d\tilde{\psi}}{d\tilde{z}}\right)^2,$$

and the potential has the form

$$U = \tilde{\psi}^2 - \frac{\tilde{\psi}^4}{2}. \tag{14}$$

Eq. (12) or (13) is called "constant of motion", which shows that the energy of the particle is conserved. The behaviors of the kinetic energy and potential are sketched in Fig. 2 respectively by solid and dashed lines. In good agreement with Fig. 1, as the "time" $\tilde{z}$ increases, the kinetic energy decreases whereas the potential increases. Furthermore, the total energy of particle is unchanged and always takes value 1/2 (dotted line). Phenomenologically, like in mechanics, one can think of the trasformation energy from the kinetic energy to potential. When the "time" $\tilde{z}$ is large enough, the kinetic energy vanishes and total energy is in form of the potential. This potential is said to create the surface energy of Bose gas [5].

4. Conclusion

In forgoing section, the phenomenological analogies between GP theory and Newton equation for the nonrelative classical mechanics have been considered. In GP theory, the state of BEC is described as a whole by the wave function of the ground state. The absence of external field, the wave function is real and dependent of the coordinate. In these analogies, the wave function and coordinate take the part of the coordinate and time in Newton equation for the classical mechanics.

Although this analogy is purely mathematical and phenomenological, it provides a new insight into the physics of condensed matter, in particularly in quantum matter. It can be applied for the multi-component Bose-Einstein mixtures.

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