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Original Article

Solvability and Stability of Switched Discrete-time Singular Systems with the Same Switching Rules in Coefficient Matrices

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Abstract: The aim of this work is to study the problem of solvability and stability for switched discrete-time linear singular (SDLS) systems with the same switching rules in coefficient matrices under Lipschitz perturbation. Firstly, we prove the unique existence of the solution, as well as describe the manifold solution. Secondly, by utilizing a Lyapunov function, we derive certain conditions that guarantee the stability of these systems. Finally, we illustrate obtained results through an example.

Keywords: SDLS systems, index, solvability, stability, Lipschitz perturbation.

1. Introduction

In this work, we investigate stability of switched discrete-time linear singular (SDLS) systems with the same switching rules in coefficient matrices under Lipschitz perturbation of the form:

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + f_{\sigma(k)}\big(x(k)\big), \tag{1}$$

where $\sigma: \mathbb{N} \cup \{0\} \to \{1,2,\dots,N\} =: \underline{N}$, is a switching signal taking values in the finite set $\underline{N}, E_i, A_i \in \mathbb{R}^{n\times n}$ and $f_i: \mathbb{R}^n \to \mathbb{R}^n, i \in \underline{N}$, are perturbations, $x(k) \in \mathbb{R}^n$ is state vector at time $k \in \mathbb{N}$. Suppose that the matrices E_i are singular for all $i \in \underline{N}$. For notational convenience, we put $\sigma(-1) = 1$.

There are already quite a few works devoted to the stability of SDLS systems, due to their importance in both theoretical and practical aspects ([1-3],...). Recently, in [4, 5], the authors have studied solvability and stability of SDLS systems which have no perturbations. In [6], the authors study sovability and stability for SDLS systems with the different switching rules in matrices E and A under Lipschitz pertubation f. However, to the best of our knowledge, there are still no results about sovability

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and stability for SDLS systems with the same switching rules in matrices E and A under Lipschitz pertubation f of Eq. (1). Thus, in this work we have attempted to fill this gap.

The paper is organized as follows. In Section 2, we summarize some preliminary results of SDLS systems of index-1. In Section 3, we study solvability of SDLS systems under Lipschitz perturbations. Section 4 deals with stability of these systems.

2. Preliminaries

Consider the homogeneous SDLS systems

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \tag{2}$$

where $\sigma: \mathbb{N} \cup \{0\} \to \underline{N}$ denotes the switching signal that determines which of the $n \in \mathbb{N}$ modes is active at time k. Suppose that the matrices E_i are singular for all $i \in \underline{N}$. Assume that the system (2) is of index-1 ([4, 7]), i.e., the following hypotheses are fulfilled:

(a) $\operatorname{rank} E_i = r < n, \forall i \in \underline{N}$,

(b) $S_i \cap \ker E_j = \{0\}, \forall i, j \in \underline{N}$, where $S_i = A_i^{-1}(\operatorname{Im} E_i) = \{\xi \in \mathbb{R}^n : A_i\xi \in \operatorname{Im} E_i\}$.

It is proved that from hypothesis (b) we have

$$S_i \oplus \ker E_j = \mathbb{R}^n, \forall i, j \in \underline{N}$$

(see, e.g. [4]). Let the matrix $V_i := \{s_i^1, \dots, s_i^r, h_i^{r+1}, \dots, h_i^n\}$, whose columns form bases of S_i and ker E_i, respectively, and $Q = \operatorname{diag}(O_r, I_{n-r}), P = I_n - Q$. Here O_r is the $r \times r$ zero matrix and I_{n-r} stands for the $(n-r) \times (n-r)$ identity matrix. Then the matrix $Q_i := V_i Q V_i^{-1}$ defines a projection onto ker E_i along S_i (i.e., $Q_i^2 = Q_i$ and $\operatorname{Im} Q_i = \ker E_i$), and $P_i := I_n - Q_i = V_i P V_i^{-1}$ is the projection onto S_i along ker E_i. Further we define the so-called connecting operators $Q_{ij} := V_j Q V_i^{-1}$.

The following theorem from [4] presents a characterization of the system (2).

Theorem 2.1. ([4]). For switched discrete-time linear singular homogeneous system of index-1 (2), the following assertions hold:

(a) $G_{ij} = E_i + A_i Q_{ij}$ is non-singular ;

(b) $E_i P_i = E_i$;

(c) $P_i = G_{ij}^{-1} E_i$;

(d) $V_i^{-1} G_{ij}^{-1} A_i V_j Q = Q$.

for all $i, j \in \underline{N}$.

3. Solvability

Consider switched discrete-time singular system with perturbations (1). Let us associate system (1) with the initial condition

$$P_{\sigma(k_0-1)}x(k_0) = P_{\sigma(k_0-1)}\gamma \tag{3}$$

where γ is a given vector in $\mathbb{R}^n$ and k_0 is a fixed nonnegative integer.

Theorem 3.1. Let $f_{\sigma(k)}(x)$ be a Lipschitz continuous function with a sufficient small Lipschitz coefficient, i.e.,

$$\|f_i(x) - f_i(\tilde{x})\| \le L_i \parallel x - \tilde{x} \parallel, \forall x, \tilde{x} \in \mathbb{R}^n, i \in \underline{N}, \tag{4}$$

and

$$\omega_i := L_i \max\left\{\left\|Q_{ij} G_{ij}^{-1}\right\| : j \in \underline{N}\right\} < 1, \forall i \in \underline{N}. \tag{5}$$

Then the IVP (1), (3) has a unique solution.

Proof.

Multiplying on both sides of equation (1) from the left by $P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}$ and $Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}$, respectively and observing that

$$G^{-1}_{\sigma(k)\sigma(k-1)}E_{\sigma(k)} = P_{\sigma(k)},\ P_{\sigma(k)}Q_{\sigma(k)} = Q_{\sigma(k)}P_{\sigma(k)} = O$$

we get

$$P_{\sigma(k)}x(k+1) = P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}x(k) + P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}f_{\sigma(k)}(x(k)), \quad (6)$$

$$Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}x(k) = -Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}f_{\sigma(k)}(x(k)). \quad (7)$$

Let $u(k) = P_{\sigma(k-1)}x(k), v(k) = Q_{\sigma(k-1)}x(k), (k \in \mathbb{N})$ we get

$$\begin{aligned}
&P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}v(k)\\
&= P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}Q_{\sigma(k-1)}x(k)\\
&= P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}Q_{\sigma(k)\sigma(k-1)}V_{\sigma(k)}QV^{-1}_{\sigma(k-1)}x(k)\\
&= P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}\left(G_{\sigma(k)\sigma(k-1)} - E_{\sigma(k)}\right)V_{\sigma(k)}QV^{-1}_{\sigma(k-1)}x(k)\\
&= \left(P_{\sigma(k)} - P_{\sigma(k)}P_{\sigma(k)}\right)V_{\sigma(k)}QV^{-1}_{\sigma(k-1)}x(k)\\
&= 0
\end{aligned}$$

and from (6)

$$u(k+1) = P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}u(k) + P_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}f_{\sigma(k)}(u(k)+v(k)) \quad (8)$$

By item (d) of Theorem 2.1,

$$G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}Q_{\sigma(k)\sigma(k-1)} = V_{\sigma(k)}QV^{-1}_{\sigma(k)} = Q_{\sigma(k)}$$

In addition, we use the fact that $Q_j = Q_{ij} \cdot Q_{ji}, Q_i \cdot Q_{ji} = Q_{ji}$, the left side of (7) can be expressed as

$$\begin{aligned}
&Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}x(k) = Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}(u(k)+v(k))\\
&= Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}u(k) + Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}Q_{\sigma(k)\sigma(k-1)}Q_{\sigma(k-1)\sigma(k)}x(k)\\
&= Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}u(k) + Q_{\sigma(k-1)\sigma(k)}x(k)
\end{aligned}$$

Hence, it follows from (7) that

$$Q_{\sigma(k-1)\sigma(k)}x(k) = -Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}A_{\sigma(k)}u(k) - Q_{\sigma(k)}G^{-1}_{\sigma(k)\sigma(k-1)}f_{\sigma(k)}(x(k))$$

Now acting $Q_{\sigma(k)\sigma(k-1)}$ on both sides of the last relation we get

$$v(k) = Q_{\sigma(k-1)}x(k) = -Q_{\sigma(k)\sigma(k-1)}G^{-1}_{\sigma(k)\sigma(k-1)}\left[f_{\sigma(k)}(u(k)+v(k)) + A_{\sigma(k)}u(k)\right]. \quad (9)$$

By equation (8), suppose that $u := u(k)(k \geq k_0)$ is known, where

$$u(k_0) = P_{\sigma(k_0-1)}x(k_0) = P_{\sigma(k_0-1)}\gamma$$

is given. We consider an operator $T_{ij}: \operatorname{Im} Q_{ij} \to \operatorname{Im} Q_{ij}$ defined by

$$T_{ij}(v) := -Q_{ij}G^{-1}_{ij}[f_i(u+v) + A_i u]$$

Since

$$\begin{aligned}
\|T_{ij}(v) - T_{ij}(\tilde{v})\| &= \| Q_{ij}G^{-1}_{ij}[f_i(u+v) - f_i(u+\tilde{v}) \|\\
&\leq \|Q_{ij}G^{-1}_{ij}\| \|f_i(u+v) - f_i(u+\tilde{v})\|\\
&\leq \|Q_{ij}G^{-1}_{ij}\| L_i \| v - \tilde{v} \| \leq \omega_i \| v - \tilde{v} \| < \| v - \tilde{v} \|,
\end{aligned}$$

the operator T_{ij} is contractive. Therefore equation (9) has a unique solution given by a mapping $g_{\sigma(k)}: \operatorname{Im} P_{\sigma(k-1)} \to \operatorname{Im} Q_{\sigma(k-1)}, g_{\sigma(k)}(u(k)) = v(k)$. Moreover, it is easy to show that $g_{\sigma(k)}$ is a Lipschitz continuous mapping having the Lipschitz constant

$$K_{\sigma(k)} := \omega_{\sigma(k)}\left(L_{\sigma(k)} + \|A_{\sigma(k)}\|\right)L^{-1}_{\sigma(k)}\left(1 - \omega_{\sigma(k)}\right)^{-1} \quad (10)$$

Thus, the IVP (1), (3) has a unique solution given by

$$x(k) = u(k) + g_{\sigma(k)}(u(k)), \tag{11}$$

with $u(k_0) = P_{\sigma(k_0-1)}\gamma$. The proof is complete.

In what follows without loss of generality, assume that $f_i(0) = 0, \forall i \in \underline{N}$. This implies that $g_{\sigma(k)}(0) = 0$ and equation (1) possesses a trivial solution $x(k) \equiv 0$. It follows from (11) that each solution $x(k)$ of the IVP (1), (3) satisfies $x(k) = P_{\sigma(k-1)}x(k) + g_{\sigma(k)}\big(P_{\sigma(k-1)}x(k)\big)$ or equivalently,

$$Q_{\sigma(k-1)}x(k) = -Q_{\sigma(k)\sigma(k-1)}G^{-1}_{\sigma(k)\sigma(k-1)}\big[f_{\sigma(k)}x(k)\big) + A_{\sigma(k)}P_{\sigma(k-1)}x(k)\big].$$

For $i \in \underline{N}$, we set

$$\Delta_i := \big\{x \in \mathbb{R}^n : Q_j x = -Q_{ij}G_{ij}^{-1}\big(f_i(x) + A_iP_jx\big), \text{ for some } j \in \underline{N}\big\}. \tag{12}$$

If $x = x(k)$ is any solution of the IVP (1), (3), then obviously $x(k) \in \Delta_{\sigma(k)} (k \geq k_0)$. Conversely, for each $\theta \in \Delta_i$, there exists a solution of (1) passing θ. Indeed, let σ be a switching signal satisfying $\sigma(k) = i$ and $x(m, k; \theta)(m \geq k)$ be a solution of (1) satisfying the initial condition $P_{\sigma(k-1)}x(k) = P_{\sigma(k-1)}\theta$. Clearly,

$$\begin{aligned} x(k, k; \theta) &= P_{\sigma(k-1)}x(k) + g_{\sigma(k)}\big(P_{\sigma(k-1)}x(k)\big) \\ &= P_{\sigma(k-1)}\theta + Q_{\sigma(k-1)}\theta = \theta. \end{aligned}$$

We will prove that the set Δ_i does not depend on the choice of projections in the following proposition.

Proposition 3.2. Let $i, j \in \underline{N}$ and the solution manifold Δ_i be defined in (12). Then, the following hold:

(a) $\Delta_i = \Omega_i := \{x \in \mathbb{R}^n : f_i(x) + A_ix \in \operatorname{Im} E_i\}$.

(b) $\Delta_i \cap \ker E_j = \{0\}$.

Proof. (a) Let $x \in \Delta_i$, then exists $j \in \underline{N}$ such that

$$Q_jx = -Q_{ij}G_{ij}^{-1}\big(f_i(x) + A_iP_jx\big),$$

hence

$$x = P_jx + Q_jx = -Q_{ij}G_{ij}^{-1}f_i(x) + \big(I - Q_{ij}G_{ij}^{-1}A_i\big)P_jx.$$

From the last relation, we get

$$f_i(x) + A_ix = \big(I - A_iQ_{ij}G_{ij}^{-1}\big)f_i(x) + A_i\big(I - Q_{ij}G_{ij}^{-1}A_i\big)P_jx.$$

Observing that

$$A_i\big(I - Q_{ij}G_{ij}^{-1}A_i\big)P_jx = \big(I - A_iQ_{ij}G_{ij}^{-1}\big)A_iP_jx$$

we find

$$f_i(x) + A_i(x) = \big(I - A_iQ_{ij}G_{ij}^{-1}\big)\big(f_i(x) + A_iP_j(x)\big)$$

Since

$$A_iQ_{ij}G_{ij}^{-1} = \big(G_{ij} - E_i\big)G_{ij}^{-1} = I - E_iG_{ij}^{-1}$$

it implies that

$$f_i(x) + A_i(x) = E_iG_{ij}^{-1}\big(f_i(x) + A_iP_j(x)\big) \in \operatorname{Im} E_i.$$

hence $x \in \Omega_i$.

Conversely, let $x \in \mathbb{R}^n$ such that $f_i(x) + A_ix \in \operatorname{Im} E_i$, i.e, there exists $\xi \in \mathbb{R}^n$, satisfied $f_i(x) + A_ix = E_i\xi$. We have to prove that

$$Q_jx = -Q_{ij}G_{ij}^{-1}\big(f_i(x) + A_iP_jx\big),$$

or equivalently,

$$x = -Q_{ij}G_{ij}^{-1}(f_i(x) + A_ix) + Q_{ij}G_{ij}^{-1}A_iQ_jx + P_jx$$

Denoting the right-hand side of the last relation by ω_i and note that

$$Q_{ij}G_{ij}^{-1}(f_i(x) + A_ix) = Q_{ij}G_{ij}^{-1}E_i\xi = Q_{ij}P_i\xi = V_jQV_i^{-1}V_iPV_i^{-1}\xi = 0$$

Using Theorem 2.1 we get

$$\begin{aligned}\omega_i &= Q_{ij}G_{ij}^{-1}A_iQ_jx + P_jx\\ &= Q_{ij}G_{ij}^{-1}A_iV_jQV_i^{-1}V_iQV_j^{-1}x + P_jx\\ &= Q_{ij}G_{ij}^{-1}\big(G_{ij} - E_i\big)V_iQV_j^{-1}x + P_jx\\ &= Q_{ij}V_iQV_j^{-1}x - Q_{ij}G_{ij}^{-1}E_iV_iQV_j^{-1}x + P_jx\\ &= V_jQV_i^{-1}V_iQV_j^{-1}x - V_jQV_i^{-1}P_iV_iQV_j^{-1}x + P_jx\\ &= Q_jx - V_jQPQV_j^{-1}x + P_jx\\ &= Q_jx + P_jx = x.\end{aligned}$$

Thus, $x \in \Delta_i$ and the item (a) of Lemma 3.2 is proved.

(b) Let $x \in \Delta_i \cap \ker E_j$. Then $P_jx = 0$ and $x \in \Delta_i$, hence $x = P_jx + g_i\big(P_jx\big) = 0$. The proof of Lemma 3.2 is complete.

Since $G_{\sigma(k_0-1)\sigma(k_0)}^{-1}E_{\sigma(k_0-1)} = P_{\sigma(k_0-1)}$, it is easy to see that the initial condition (3) is equivalent to the condition

$$E_{\sigma(k_0-1)}x(k_0) = E_{\sigma(k_0-1)}\gamma,\ (k_0 \geq 0) \tag{13}$$

which is independent of the choice of projections. Thus both initial conditions (3) and (13) are equivalent for all $k_0 \in \mathbb{N}$. The unique solution of the IVP (1), (3) or (1), (13) will be denoted by $x(k) = x(k, k_0; \gamma)$.

4. Stability

In this section, the notions of stability system are introduced and the necessary and sufficient conditions for stability of SDLS systems are established.

Definition 4.1. The system (1) is said to be

i) Stable if for each $\epsilon > 0$, any $k_0 \geq 0$ and for all switching signals there exists a $\delta = \delta(\epsilon, k_0) \in (0, \epsilon]$ such that $\|P_{\sigma(k_0-1)}\gamma\| < \delta$ implies $\|x(k, k_0; \gamma)\| < \epsilon$ for all $k \geq k_0$, uniformly stable if it is stable and δ does not depend on k_0;

ii) Asymptotically stable if it is stable and for any $k_0 \geq 0$ and for all switching signals there exists a $\delta = \delta(k_0) > 0$ such that the inequality $\|P_{\sigma(k_0-1)}\gamma\| < \delta$ implies $\|x(k, k_0; \gamma)\| \to 0$ as $k \to +\infty$.

Remark 4.2. In the above definition, if replacing the initial condition $P_{\sigma(k_0-1)}\gamma$ by $E_{\sigma(k_0-1)}\gamma$ then we get notions of E-stability, E-asymptotical stability (respectively). However, since the relation $G_{ij}^{-1}E_i = P_i$ and $E_iP_i = E_i$ for all $i, j \in \underline{N}$, it is easy to show that they are equivalent to above notions (respectively).

Denote by $\mathcal{K}$ the class of all increasing functions ψ from $[0, \infty)$ into itself such that $\psi(0) = 0, \psi(x) > 0$ for $x \neq 0$ and $\lim_{x\to 0^+}\psi(x) = 0$.

Lemma 4.3. The system (1) is stable if and only if there exists a function $\psi \in \mathcal{K}$, such that for each nonnegative integer k_0 and for all switching signals, there holds the inequality

$$\| x(k) \| \leq \psi(\|x(k_0)\|),\ \forall k \geq k_0 \tag{14}$$

Proof. Assume that for all switching signals and for each nonnegative integer k_0, there exists a function $\psi \in \mathcal{K}$ satisfying condition (14). Since ψ is increasing and continuous at 0, for each positive ϵ there exists $\delta = \delta(\epsilon) \in (0, \epsilon]$ such that $\psi(\delta) < \epsilon$. Let $K := \max_{i\in N} K_i$, where K_i is given by (10). If $x(k)$ is an arbitrary solution of (1) satisfying

$\|P_{\sigma(k_0-1)}x(k_0)\| < \delta_1 := \dfrac{\delta}{K+1}$ then

$$\|x(k_0)\| = \left\|P_{\sigma(k_0-1)}x(k_0) + g_{\sigma(k_0)}\left(P_{\sigma(k_0-1)}x(k_0)\right)\right\| \\ \leq \|P_{\sigma(k_0-1)}x(k_0)\|\left(1 + K_{\sigma(k_0)}\right) \leq \|P_{\sigma(k_0-1)}x(k_0)\|(1+K) < \delta. \quad (15)$$

This implies that

$$\| x(k) \| \leq \psi(\|x(k_0)\|) \leq \psi(\delta) < \epsilon,\ \forall k \geq k_0, \forall \sigma$$

which implies that the system (1) is stable.

Conversely, suppose that the trivial solution of (1) is stable, i.e., for each positive ϵ there exists a $\delta = \delta(\epsilon) \in (0, \epsilon]$, such that if $x(k)$ is any solution of (1) satisfying the inequality $\|P_{\sigma(k_0-1)}x(k_0)\| < \delta$ for all switching signals then $\| x(k) \| < \epsilon$ for all $k \geq k_0$. Denote by $\alpha(\epsilon)$ the supremum of such $\delta(\epsilon)$. Clearly, if $\|P_{\sigma(k_0-1)}x(k_0)\| < \alpha(\epsilon)$ for some k_0 and for all σ, then $\| x(k) \| < \epsilon$ for all $k \geq k_0$. Further, the function $\alpha(\epsilon)$ is positive and increasing and moreover, $\alpha(\epsilon) \leq \epsilon$. Putting $\beta(\epsilon) := \frac{\epsilon\alpha(\epsilon)}{(\epsilon+1)H}$ for $\epsilon \geq 0$, where $H := \max\{\|P_i\| : i \in \underline{N}\}$. It is easy to see that $0 < \beta(\epsilon) < \frac{\alpha(\epsilon)}{H} \leq \frac{\epsilon}{H}$, β is strictly increasing and continuous at 0. Then there exists the strictly increasing inverse of β from $\operatorname{Im}\beta$ to $[0, \infty)$ which can be expanded to $\psi \in \mathcal{K}$. Let $x(k)$ be a solution of (1) and k_0 be a fixed nonnegative integer. Set $\epsilon_k := \| x(k) \|$ and consider two possibilities. If $\| x(k) \| = 0$ then $\| x(k) \| = 0 \leq \psi(\|x(k_0)\|)$ since ψ is nonnegative. Now suppose that $\epsilon_k := \| x(k) \| > 0$. If $\|x(k_0)\| < \beta(\epsilon_k)$ then

$$\|P_{\sigma(k_0-1)}x(k_0)\| \leq H\beta(\epsilon_k) < \alpha(\epsilon_k).$$

This implies that $\| x(k) \| < \epsilon_k = \| x(k) \|, \forall k \geq k_0$, which is contradiction. Therefore $\|x(k_0)\| \geq \beta(\epsilon_k)$ which is equivalent to

$$\| x(k) \| = \epsilon_k \leq \beta^{-1}(\|x(k_0)\|) = \psi(\|x(k_0)\|).$$

The proof is complete.

Remark 4.4. The above lemma is developed and modified from Lemma 3.3 in [7]. Here, ψ is a function of $\|x(k_0)\|$ which doesn't depend of the choice of projections and $\psi \in \mathcal{K}$ containing the class of all continuous and strictly increasing functions $\hat{\psi}$ from $[0, \infty)$ into itself, such that $\hat{\psi}(0) = 0$. Moreover, to prove the converse, we have constructed the function ψ which is different from Lemma 3.3 in [7].

Theorem 4.5. The existence of the Lyapunov functions $V_\sigma : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_+$being continuous in the second variable at $\gamma = 0$ and the functions $a, \psi_k \in \mathcal{K}$, such that

i) $a(\| y \|) \leq V_\sigma(k, y) \leq \psi_k(\| y \|), \forall k \geq 0, \forall y \in \Delta_{\sigma(k)}, \forall \sigma$,

ii) $\Delta V_\sigma(k, y(k)) := V_\sigma(k+1, y(k+1)) - V_\sigma(k, y(k)) \leq 0, \forall k \geq 0, \forall \sigma$, for any solution $y(k)$ of (1) corresponding σ, is a necessary and sufficient condition for the stability of the SDLS system (1).

Proof.

Necessity. Suppose that the system (1) is stable. For each k_0, then according to Lemma 4.3, there exist functions $\psi_{k_0} \in \mathcal{K}(k_0 \geq 0)$, such that for any solution $x(k)$ of (1),

$$\| x(k) \| \leq \psi_{k_0}(\|x(k_0)\|), \forall k \geq k_0, \forall \sigma \quad (16)$$

We define the Lyapunov function

$$V_\sigma(k_0, \gamma) := \sup_{m \in \mathbb{N}} \|x_\sigma(k_0 + m, k_0; \gamma)\|, \text{ for each } \gamma \in \mathbb{R}^n, k_0 \in \mathbb{N} \quad (17)$$

where $x_\sigma(k_0 + m, k_0; \gamma)$ is the unique solution of (1) corresponding to switching signal σ satisfying the initial condition $P_{\sigma(k_0-1)}x_\sigma(k_0) = P_{\sigma(k_0-1)}\gamma$. Inequality (16) ensures the correctness of definition (17). By (15), we have

$$\|x_\sigma(k_0)\| \leq (K+1)\|P_{\sigma(k_0-1)}x_\sigma(k_0)\| = (K+1)\|P_{\sigma(k_0-1)}\gamma\| \leq (K+1)H \parallel \gamma \parallel,$$

where the constants K, H are given Lemma 4.3. Define $\hat{\psi}_{k_0}(t) := \psi_{k_0}[(K+1)Ht]$ for $t \geq 0$. Then we imply that

$$V_\sigma(k_0,\gamma) \leq \psi_{k_0}(\|x_\sigma(k_0)\|) \leq \psi_{k_0}((K+1)H \parallel \gamma \parallel) = \hat{\psi}_{k_0}(\parallel \gamma \parallel), \forall k_0 \geq 0, \forall \gamma \in \mathbb{R}^n, \forall \sigma$$

This implies that $V_\sigma(k_0,0) = 0$ and the continuity of the function V w.r.t the second variable at $\gamma = 0$. For each $y \in \Delta_{\sigma(k_0)}$, by (3.11), we have

$$V_\sigma(k_0,y) = \sup_{l\in\mathbb{N}} \|x_\sigma(k_0+l,k_0;y)\| \geq \|x_\sigma(k_0,k_0;y)\| = \parallel y \parallel := a(\parallel y \parallel) \quad (18)$$

On the other hand, for each $k_0 \geq 0$ due to the unique solvability of (1)-(3), it is easy to see that

$$\{x_\sigma(k_0+l,k_0;y(k_0)): l \geq 0\} = \{y(k_0+l): l \geq 0)\}$$
$$\supset \{y(k_0+l): l \geq 1)\} \supset \{x_\sigma(k_0+1+l,k_0+1;y(k_0+1)): l \geq 0\}, \quad (19)$$

where $\sigma_y(k)$ is the switching signal corresponding $y(k)$. Thus

$$V_\sigma(k+1,y(k+1)) = \sup_{l\geq 0} \|x_\sigma(k+1+l,k+1;y(k+1))\|$$
$$\leq \sup_{l\geq 0} \|x_\sigma(k+l,k;y(k))\| = V_\sigma(k,y(k))$$

which implies $\Delta V_\sigma(k,y(k)) \leq 0$. The necessity part is proved.

Sufficiency. We argue by contradiction by assuming that the system (1) is not stable, i.e., there exist a positive ϵ_0, a nonnegative integer k_0 and a switching signal σ, such that for all $\delta \in (0,\epsilon_0]$, there exists a solution $x_\sigma(k)$ of (1) satisfying the inequalities $\|P_{\sigma(k_0-1)}x_\sigma(k_0)\| < \delta$ and $\|x_\sigma(k_1)\| \geq \epsilon_0$ for some $k_1 \geq k_0$.

Since $V_\sigma(k_0,0) = 0$ and $V_\sigma(k_0,\gamma)$ is continuous at $\gamma = 0$, there exists a $\delta_0' = \delta_0'(\epsilon,k_0) > 0$, such that for all $\xi \in \mathbb{R}^n, \parallel \xi \parallel < \delta_0'$ and for all σ we have $V_\sigma(k_0,\xi) < \epsilon_1 := a(\epsilon_0)$. Choosing $\delta_0 \leq \left\{\frac{\delta_0'}{K+1},\epsilon_0\right\}$ we can find solution $x_\sigma(k)$ of (1) satisfying $\|P_{\sigma(k_0-1)}x_\sigma(k_0)\| < \delta_0$, however $\|x_\sigma(k_1)\| \geq \epsilon_0$ for some $k_1 \geq k_0$. Since $\|P_{\sigma(k_0-1)}x_\sigma(k_0)\| < \delta_0 \leq \frac{\delta_0'}{K+1}$, $\|x_\sigma(k_0)\| < \delta_0'$ and one gets $V_\sigma(k_0,x_\sigma(k_0)) < \epsilon_1$. On the other hand, using the properties of the function V, we find

$$V_\sigma(k_0,x_\sigma(k_0)) \geq V_\sigma(k_1,x_\sigma(k_1)) \geq a(\|x_\sigma(k_1)\|) \geq a(\epsilon_0) = \epsilon_1,$$

which leads to a contradiction. The proof of Theorem 4.5 is complete.

If the system (1) is uniformly stable, then the function ψ_k in the above theorem can be chosen independently on k. Therefore, a similar argument as in the above proof leads to the next result.

Theorem 4.6. The system (1) is uniformly stable if and only if there exist two functions $a, b \in \mathcal{K}$ and the Lyapunov function $V_\sigma: \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_+$, such that

i) $a(\parallel y \parallel) \leq V_\sigma(k,y) \leq b(\parallel y \parallel), \forall k \geq 0, \forall y \in \Delta_{\sigma(k)}, \forall \sigma$,

ii) $\Delta V_\sigma(k,y(k)) := V_\sigma(k+1,y(k+1)) - V_\sigma(k,y(k)) \leq 0, \forall k \geq 0, \forall \sigma$, for any solution $y(k)$ of (1) corresponding σ.

Next, we present sufficient conditions for asymptotical stability of the system (1).

Theorem 4.7. Suppose that there exist the functions $a, c, \psi_k \in \mathcal{K}$ and the Lyapunov function $V_\sigma: \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, such that

i) $a(\parallel y \parallel) \leq V_\sigma(k,y) \leq \psi_k(\parallel y \parallel), \forall k \geq 0, \forall y \in \Delta_{\sigma(k)}, \forall \sigma$,

ii) $\Delta V_\sigma(k,y(k)) := V_\sigma(k+1,y(k+1)) - V_\sigma(k,y(k)) \leq -c(\parallel y(k) \parallel), \forall k \geq 0, \forall \sigma$, for any solution $y(k)$ of (1) corresponding σ.

Then the system (1) is asymptotically stable.

Proof.

From Theorem 4.5, we have the system (1) is stable. By item (ii), $\{V_\sigma(k,y(k))\}$ is a decreasing sequence and is below bounded by 0. Therefore, there exists the limit $\lim_{k\to\infty} V_\sigma(k,y(k))$. This implies that

$$\lim_{k\to\infty}[V_\sigma(k+1,y(k+1)) - V_\sigma(k,y(k))] = 0$$

and hence $\lim_{k\to\infty} c(\| y(k) \|) = 0$. Since $c \in \mathcal{K}$, it implies that $\lim_{k\to\infty} \| y(k) \| = 0$. Indeed, assume that $\lim_{k\to\infty} \| y(k) \| \neq 0$. Then for some $\epsilon > 0$, there exists a sequence $\{k_m\} \subset \mathbb{N}$ such that $k_m \to \infty$ and $\|y(k_m)\| > \epsilon$. This implies that $c(\|y(k_m)\|) \geq c(\epsilon) > 0$ which is a contradiction. The proof is complete.

Example 4.8. In this example we will use the Euclidean norms of vectors and matrices. Consider the SDLS (1) with switching signal $\sigma: \mathbb{N} \cup \{0\} \to \{1,2,\dots,N\} = \underline{N}$ and

$$E_i = (i+1)\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ A_i = \begin{pmatrix} i+1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$f_i(x) = \frac{\sin(x_1)}{i+1}(0,0,1)^T;\ x = (x_1,x_2,x_3)^T \in \mathbb{R}^3,\ i \in \underline{N}.$$

In this case, $\ker E_i = \text{span}\{(0,0,1)^T\}$ and $S_i = \text{span}\{(1,-1,0)^T, (1,i,0)^T\}$. Clearly, $S_i \cap \ker E_i = \{0\}$ and $\text{rank}\, E_i = 2 < 3$, hence the SDLS (1) is of index-1.

We have $V_i = \begin{pmatrix} 1 & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \forall i,j \in \underline{N};\ Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, therefore we calculated $V_i^{-1} = \frac{1}{i+1}\begin{pmatrix} i & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & i+1 \end{pmatrix}$, $Q_i = Q;\ P_i = I_n - Q_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

A simple calculation shows that $Q_{ij} = V_j Q V_i^{-1} = Q, \forall i,j \in \underline{N}$ and

$$G_{ij} = E_i + A_i Q_{ij} = \begin{pmatrix} i+1 & i+1 & 0 \\ i+1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ G_{ij}^{-1} = \frac{1}{i+1}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & i+1 \end{pmatrix}$$

Further, the function $f_i(x)$ is Lipschitz with the Lipschitz coefficient $L_i = \frac{1}{i+1}$. Moreover, $f_i(0) = 0$ and $\omega_i = L_i \max\|Q_{ij}G_{ij}^{-1}: j \in \underline{N}\| = \frac{1}{(i+1)} < 1, \forall i \in \underline{N}$. According to Theorem 3.1, the SDLS (1), (3) has unique solution.

From the definition of Δ_i, we have $x \in \Delta_i$ if only if

$$Q_j x = -Q_{ij}G_{ij}^{-1}\big(f_i(x) + A_i P_j x\big)$$

This relation leads to $x_3 = -\frac{\sin x_1}{(i+1)}$. Thus,

$$\Delta_i = \Omega_i = \left\{x = (x_1,x_2,x_3)^T : x_3 = -\frac{\sin x_1}{(i+1)}\right\}, \forall i \in \underline{N}$$

Consider a function $V_\sigma(k,\gamma) := 3\|P_{\sigma(k-1)}\gamma\|$ for all $\gamma \in \mathbb{R}^3$. We get for each $y \in \Delta_i$,

$$\| y \| = \sqrt{y_1^2 + y_2^2 + y_3^2} = \sqrt{y_1^2 + y_2^2 + \frac{\sin^2 y_1}{(i+1)^2}} \leq \sqrt{2y_1^2 + y_2^2} \leq 3\sqrt{y_1^2 + y_2^2} = 3\|P_{\sigma(k-1)}y\|$$

Moreover, $V_\sigma(k,y) = 3\|P_{\sigma(k-1)}y\| \leq 3 \| y \|$. Thus, item (i) of Theorem 4.6 is satisfied.

We suppose that $y(k)$ is a solution of (1) and putting $y(k) = u(k) + v(k)$, where $u(k) = P_{\sigma(k-1)}y(k);\ v(k) = Q_{\sigma(k-1)}y(k)$, we have

$$\Delta V_\sigma(k, y(k)) = V(k+1, y(k+1)) - V(k, y(k))$$
$$= 3\left(\|P_{\sigma(k-1)} y(k+1)\| - \|P_{\sigma(k-1)} y(k)\|\right) = 3(\| u(k+1) \| - \| u(k) \|)$$

Using Equation (8) we find

$$u(k+1) = P_j G_{ij}^{-1} A_i u(k) + P_j G_{ij}^{-1} f_i(x(k)) = \frac{1}{(i+1)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} u(k)$$

hence, $\| u(k+1) \| \leq \frac{2}{(i+1)} \| u(k) \|$ and leading to $\| u(k+1) \| - \| u(k) \| \leq 0$. According to Theorem 4.6, the the SDLS system (1) is uniformly stable.

We illustrate the solution of this SDLS system for the case $N = 2$ with the specific switching rule $\sigma(k) = (k \text{mod} 2) + 1$. Choose initial value $x(0) = (2,1,3)^T$. Here, we consider a simple switching signals that is sequentially switched: if k even, take the equation system $E_1 x(k+1) = A_1 x(k) + f_1(x(k))$, otherwise consider $E_2 x(k+1) = A_2 x(k) + f_2(x(k))$. At each system, at step k, we found $x_1(k)$ and $x_2(k)$ is found based on $x_1(k-1), x_2(k-1)$, while $x_3(k)$ is determined by $x_1(k)$. Illustrating the solution of the system for 20 steps, we see that after 8 steps the solution converges to 0, see Figure 1. Algorithm 1 provides an algpseudocode for a figure which simulates the stable solution for this problem. It is seen that this algorithm depends on N and the switching signal σ.

Algorithim 1

Initiate $x(0)$
for $k = 0$ **to** 20
  If k is even **then**
    Solve the system $E_1 x(k+1) = A_1 x(k) + f_1(x(k))$
  else if k is odd **then**
    Solve the system $E_2 x(k+1) = A_2 x(k) + f_2(x(k))$
  end if
end for

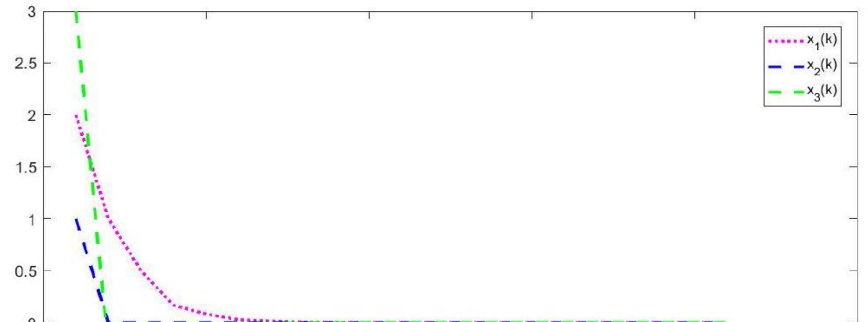

Figure 1. Simulation of the stable solution $X(x_1, x_2, x(3))$ with $N = 2$ and $\sigma(k) = (k \text{ mod} 2) + 1$.

5. Conclusion

In this work, we have studied SDLS systems with the same switching rules in matrices E and A under Lipschitz pertubation f of the form (1). We derive solvability for these equations. The stability of SDLS systems is investigated by using methods of the Lyapunov functions and the solution evaluation.

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