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Original Article

Boundary Behavior of General Kobayashi Metrics on h-Extendible Domains

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Abstract: The purpose of this article is to prove the existence of Λ-nontangential limits of the general Kobayashi metrics at h-extendible boundary point. This is a generalization of Yu's result for Λ-nontangential limits.

*Keywords: G*eneral Kobayashi metrics; h-extendible models; Pseudoconvex domains; Finite type.

1. Introduction

Let Ω be a $\mathcal{C}^\infty$- smooth pseudoconvex domain in $\mathbb{C}^{n+1}$ and $\xi_0 \in \partial\Omega$. Let ρ be a local defining function for Ω near ξ_0 and let the multitype $\mathcal{M}(\xi_0) = (1, m_1, \dots, m_n)$ be finite. (For detail definition of multitype, we refer the reader to [1].) Then there are distinguished coordinates $z = (z_0, z')$ with $z' = (z_1, \dots, z_n)$ such that $\xi_0 = 0$ and $\rho(z)$ can be expanded near 0 as follows:

$$\rho(z) = \mathrm{Re}(z_0) + P(z') + R(z),$$

where P is a $(1/m_1, \dots, 1/m_n)$-homogeneous plurisubharmonic polynomial that contains no pluriharmonic terms, R is smooth and satisfies

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$$|R(z)| \leq C\left(|z_0| + \sum_{j=1}^{n} |z_j|^{m_j}\right)^{\gamma},$$

for some constant $\gamma > 1$ and $C > 0$. Here and in what follows, a polynomial P is said to be $(1/m_1, \dots, 1/m_n)$-homogeneous if $P\left(t^{1/m_1} z_1, \dots, t^{1/m_n} z_n\right) = tP(z_1, \dots, z_n)$ for all $t \geq 0$ and $(z_1, \dots, z_n) \in \mathbb{C}^n$.

The domain Ω is called h-*extendible* at p if the model

$$M_P := \{z = (z_0, z') \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Re}(z_0) + P(z') < 0\}$$

is of finite type. Consequently, M_P is *degenerate*, i.e., its boundary contains no nontrivial analytic set passing through the origin, and it is taut (cf. [2, Theorem 3.13]). (For several equivalent conditions to the h-extendibility, we refer the reader to [3].)

For $s, M, N > 0$, let us denote by

$$\Gamma(s; M, N) = \{z \in \Omega : |\mathrm{Im}(z_0)| \leq M|\mathrm{dist}(z, \partial\Omega)|, \sigma(z') \leq N|\mathrm{dist}(z, \partial\Omega)|^s\},$$

where $\sigma(z') := \sum_{j=1}^{n} |z_j|^{m_j}$. Here and in what follows, $\mathrm{dist}(z, \partial\Omega)$ denotes the Euclidean distance from z to the boundary $\partial\Omega$. In addition, $\lesssim$ and $\gtrsim$ denote inequality up to a positive constant. We will also use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

For the sake of simplicity, we define the so-called Λ-cone with vertex at ξ_0 by $\Gamma := \Gamma(1; M, N)$ and denote by $\Gamma^s := \Gamma(s; M, N)$ for some $M, N > 0$. We note that $|z_j|^{m_j} \lesssim |\mathrm{dist}(z, \partial\Omega)|, j = 1, \dots, n$, for $z \in \Gamma$. Recall that a sequence $\{\eta_j\} \subset \Omega$ is said to converge nontangentially to ξ_0 if $|\eta_j - \xi_0| \lesssim \mathrm{dist}(\eta_j, \partial\Omega)$.

Fix a sufficiently small neighborhood U of ξ_0 in $\mathbb{C}^{n+1}$, we may assume that for any point $\eta \in U \cap \Omega$, there exists a positive real number $\epsilon(\eta) > 0$ such that the point $\tilde{\eta} := (\eta_0 + \epsilon(\eta), \eta_1, \dots, \eta_n)$ is in the hypersurface $\{\rho = 0\}$. We note that $\epsilon(\eta) = |\rho(\eta)| \approx \mathrm{dist}(\eta, \partial\Omega)$.

Let us recall the higher order Kobayashi metrics (see [2]). For each integer $k \geq 1$, the k-th order Kobayashi metric is defined by

$$F_\Omega^k(z, X) = \inf\left\{\frac{1}{\lambda} : \lambda > 0, \exists\varphi \in \mathrm{Hol}(\Delta, \Omega),\ \varphi(0) = z, \nu(\varphi) \geq k, \varphi^{(k)}(0) = k!\,\lambda X\right\},$$

where Δ denotes the unit disc in $\mathbb{C}$ and $\nu(\varphi)$ denotes the vanishing order of φ at 0. We note that $F_\Omega := F_\Omega^1$ is just the Kobayashi metric.

Now we define a dilation:

$$\pi_t(z_1, \dots, z_n) = \left(t^{1/m_1} z_1, \dots, t^{1/m_n} z_n\right),\ t \geq 0.$$

Then for any sequence $\{\eta_j\} \subset \Gamma$ converging to the vertex ξ_0, there exists a subsequence $\{\eta_{j_\ell}\} \subset \{\eta_j\}$ such that

$$\lim_{\ell \to \infty} \pi_{1/\epsilon\left(\eta_{j_\ell}\right)}\left({\eta_{j_\ell}}'\right) = \alpha \in \mathbb{C}^n.$$

(Note that $\alpha = 0$ if $\{\eta_{j_\ell}\} \subset \Gamma^s$ for some $s > 1$.) For such $\alpha \in \mathbb{C}^n$, the associated model $M_{P,\alpha}$ is defined as follows:

$$M_{P,\alpha} = \{(z_0, z') \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Re}(z_0) + P(z' + \alpha) - P(\alpha) < 0\}.$$

For simplicity, let us write M_P for $M_{P,0}$.

The Kobayashi metric has been interested in complex analysis of several variables. In particular, the asymptotic boundary behavior of the Kobayashi metric has been a major area of study. In 1975, I. Graham

[4] gave a precise weighted boundary limits of the Kobayashi metric for strongly pseudoconvex domains. There have been many estimates for the metric on several classes of weakly pseudoconvex domains ever since (cf. [4-10]). In particular, the sharp bounds for the metric on pseudoconvex domains of finite type in $\mathbb{C}^2$ [4], smoothly bounded convex domains of finite type in $\mathbb{C}^n (n \geq 2)$ [5] and decoupled domains of finite type [9] are obtained in terms of small/large constants. For general weakly pseudoconvex domains of finite type, there are no sharp bounds known. As a matter of fact, the usual sharp lower estimates for the Kobayashi metric as in [4] do not hold for general domains of finite type.

After that Yu [2] considered the same problem for the general Kobayashi metrics on weakly pseudoconvex domains. Their focus here is again on the precise relationship between the (weighted) boundary limits of the metrics and the Levi invariants of the domain, in the same spirit of Graham's result in [11]. The main difficulty in the case of weakly pseudoconvex domain is that the local Levi geometry of the domain is in general much more complicated and is still not well understood. In particular, there is no universal model for all weakly pseudoconvex domains to compare with. To overcome this difficulty, they first deformed the domain with respect to its multitype and then blow it up to a taut (but unbounded model) domain. The main result of this paper generalizes Graham's result in [11] to a very large class of weakly pseudoconvex domains, called h-extendible domains, which includes almost all the interesting domains mentioned above.

However, in [2] the weighted boundary limits of general Kobayashi metrics are taken over all points in a nontangential cone. The purpose of this paper is to ensure that the Yu's result still holds for Λ-nontangential limits. Namely, we prove the following theorem.

Theorem 1.1. Let Ω be a $\mathcal{C}^\infty$-smooth boundary pseudoconvex domain in $\mathbb{C}^{n+1}$ and $\xi_0 \in \partial\Omega$ such that Ω is h-extendible at ξ_0. Suppose that the multitype of ξ_0 is $(1, m_1, \ldots, m_n)$ with $m_n < +\infty$ and let $\Lambda = (1/m_1, \ldots, 1/m_n)$. Suppose also that the defining function ρ of Ω near 0 has the form

$$\rho(z) = \mathrm{Re}(z_0) + P(z') + R(z),$$

where P is a Λ-homogeneous plurisubharmonic polynomial that contains no pluriharmonic terms, R is smooth and satisfies

$$|R(z)| \leq C\left(|z_0| + \sum_{j=1}^{n} |z_j|^{m_j}\right)^{\gamma},$$

for some constant $\gamma > 1$ and $C > 0$. Let $\{\eta_j\} = \{(\eta_{j0}, \eta_j')\} \subset \Omega \cap U \cap \Gamma$ be a sequence of points converging to 0 such that

$$\lim_{j\to\infty} \pi_{1/\epsilon(\eta_j)}(\eta_j') = \alpha \in \mathbb{C}^n.$$

Then, we have

$$\begin{aligned}\lim_{\Omega\cap U\cap\Gamma\ni\eta_j\to 0} \quad & F^k_{\Omega\cap U}\left(\eta_j, \left(\pi_{\epsilon(\eta_j)}\right)_* X\right) = F^k_{M_{P,\alpha}}((0', -1), X) \\ & = F^k_{M_P}((-1 - P(\alpha), \alpha), X),\ \forall\, X \in \mathbb{C}^{n+1}.\end{aligned}$$

Corollary 1.2. Let $\Omega,\ \xi_0,\ \Gamma^s$ be given as in the Theorem 1.1. If $s > 1$, then we have

$$\lim_{\Omega\cap U\cap\Gamma^s\ni\eta\to 0} F_{\Omega\cap U}(\eta, X)|\rho(\eta)| = F_{M_P}\left((-1, 0'), X_N(0)\right),\ \forall\, X \in \mathbb{C}^{n+1},$$

where $X_N(0)$ is the complex normal component of X at $\xi_0 = 0$.

In order to prove the existence of Λ-nontangential limits of the general Kobayashi metrics at an h-extendible boundary point, we shall blow up the domain Ω by using a rescaling argument. More precisely, we shall construct a sequence of domains $\{\Omega_j\}$ which are the images of $\Omega \cap U$ under a sequence of dilations and translations such that Ω_j converges to M_P as $j \to \infty$. Therefore, the proof of Theorem 1.1 follows from a stability result for general Kobayashi metrics (cf. Theorem 2.3).

2. Stability of the General Kobayashi Metrics

In this section, we shall focus attention on the stability of general Kobayashi metrics. To do this, let us recall that a sequence of domains $\{\Omega_j\}_{j=1}^{\infty}$ in $\mathbb{C}^{n+1}$ is said to converge to $\Omega_\infty \subset \mathbb{C}^{n+1}$ if and only if

For any compact subset $K \subset \Omega_\infty$, there exists $j_0 = j_0(K)$ such that $K \subset \Omega_j$ for all $j \geq j_0$; and

If K is a compact subset which is contained in Ω_j for all sufficiently large j, then $K \subset \Omega_\infty$.

In [12], the first author and Nguyen Quang Dieu proved the following proposition which we shall give a short proof for the reader's convenience.

Proposition 2.1 ([12]). Assume that $\{D_j\}$ is a sequence of domains in $\mathbb{C}^{n+1}$ converging to the model M_P of finite type. Assume also that ω is a domain in $\mathbb{C}^k$ and $\sigma_j: \omega \to D_j$ is a sequence of holomorphic mappings such that $\{\sigma_j(a)\} \Subset M_P$ for some $a \in \omega$. Then $\{\sigma_j\}$ contains a subsequence that converges locally uniformly to a holomorphic map $\sigma: \omega \to M_P$.

In order to give a proof of Proposition 2.1, we need the following lemma.

Lemma 2.2. There exist small neighborhoods U, U' of the origin and $\tau \in (0,1)$ such that one has, for j large enough and for every analytic disc $f: \Delta \to D_j$, that

$f(0) \in U' \Rightarrow f(\Delta_{\tau_0}) \subset U,$

where $\Delta_\tau = \{z \in \mathbb{C}: |z| < \tau\}$.

Proof. We note that there exists a plurisubharmonic peak function for M_P at $(0,0')$ (see [3]). Thus we may find $0 < r < r' < R' < R$, a plurisubharmonic peak function φ on M_P which is continuous on $\overline{M_P}$ such that $\varphi > 0$ on $M_P \cap \{|z| < r\}$ and $\varphi < 0$ on $M_P \cap \{r' < |z| < R'\}$. Let us fix $\epsilon > 0$ small enough. Since the sequence $\{D_j\}$ converges to M_P as $j \to \infty$, one can find $j_0 := j_0(\epsilon) \geq 1$ such that for $j \geq j_0$ we have

$$D_j \subset \Omega_r := M_P^\epsilon \cup \left(\mathbb{C}^{n+1} \setminus \overline{\{|z| < r\}}\right), \tag{1}$$

where $M_P^\epsilon := \{(z, w): \mathrm{Re}(w) + P(z) < \epsilon\}$. By applying to Ω_r and the peak function $\psi(z) := \varphi(z_0 - \epsilon, z')$, it follows that there exist neighborhoods $\widetilde{U}, \widetilde{U}'$ of $(-\epsilon, 0')$ and a constant $\tau \in (0,1)$ such that one has, for every analytic disc $f: \Delta \to \Omega_r$, that

$$f(0) \in \widetilde{U}' \Rightarrow f(\Delta_\tau) \subset \widetilde{U}. \tag{2}$$

Therefore, if we choose $\epsilon > 0$ small enough, then our proof finally follows with $U := \widetilde{U} \cap M_P$ and $U' := \widetilde{U}' \cap M_P$. □

Proof of Proposition 2.1. We first define the following dilation

$$\Delta^\epsilon(z_0, z_1, \ldots, z_n) = \left(\frac{z_0}{\epsilon}, \frac{z_1}{\epsilon^{1/m_1}}, \ldots, \frac{z_n}{\epsilon^{1/m_n}}\right),\ \epsilon > 0. \tag{3}$$

It is note that Lemma 2.2 is still true if Δ is replaced by the unit ball in $\mathbb{C}^k$. Set $A := \overline{\{\sigma_j(a): j \in \mathbb{N}^*\}} \Subset M_P$. Since D_j converges to M_P as $j \to \infty$, there exists an integer $j_0 = j_0(A)$ such that $A \Subset D_j$ for all $j \geq j_0$. Choose a real number $\lambda_0 > 0$ big enough so that $\Delta^{\lambda_0}(A) \subset U'$. Since M_P is invariant under Δ^ϵ for every $\epsilon > 0$, one sees that $\{\Delta^{\lambda_0}(D_j)\}$ converges to $\Delta^{\lambda_0}(M_P) = M_P$. Therefore, it follows that $\Delta^{\lambda_0} \circ \sigma_j(B(a, \tau_0)) \subset U \cap M_P$ for all $j \geq j_0$, where $B(a, \tau_0) := \{z \in \mathbb{C}^{n+1}: |z - a| < \tau_0\}$, and hence

$\sigma_j\big(B(a,\tau_0)\big) \subset \big(\Delta^{\lambda_0}\big)^{-1}(U \cap M_P) \Subset M_P$. For any compact subset K of ω, by using a finite covering of balls of radius τ_0 and continuing the above process, one concludes that there exist a positive number λ_K and an integer j_K such that $\sigma_j(K) \subset \big(\Delta^{\lambda_K}\big)^{-1}(U \cap M_P) \Subset M_P$ for all $j \geq j_K$. Hence, if we denote by

$$L_K := (\Delta^{\lambda_K})^{-1}(U \cap M_P) \bigcup_{j=1}^{j_K-1} \sigma_j(K) \Subset M_P, \tag{4}$$

then $\sigma_j(K) \subset L_K$ for every $j \geq 1$, as desired. In addition, by the Montel theorem and a diagonal process, the sequence $\{\sigma_j\}$ is normal and its limits are holomorphic mappings from ω into the model M_P. This finishes the proof. □

Now let $\{D_j\}$ be a sequence of domains in $\mathbb{C}^{n+1}$ converging to $D_\infty \subset \mathbb{C}^{n+1}$. We consider the following stability problem:

$$\lim_{j\to\infty} F_{D_j}(z,X) = F_{D_\infty}(z,X),\ \forall\ (z,X) \in D \times \mathbb{C}^{n+1}. \tag{5}$$

Some stability results for Kobayashi metric were established [13, 2]. In [13], the stability of the Kobayashi metric is valid only for bounded domains. After, J. Yu [2] generalized their result for unbounded domains D_j that are contained in some fixed taut domain. We note that the tautness condition is not always satisfied. However, thanks to Proposition 2.1we obtain the following theorem without that condition.

Theorem 2.3. Assume that $\{D_j\}$ is a sequence of domains in $\mathbb{C}^{n+1}$ converging to the model M_P of finite type. Then, we have

$$\lim_{j\to\infty} F^k_{D_j}(z,X) = F^k_{M_P}(z,X),\ \forall\ (z,X) \in D \times \mathbb{C}^{n+1},\ k \geq 1.$$

Moreover, the convergence takes place uniformly over compact subsets of $D \times \mathbb{C}^{n+1}$.

Proof. We shall follow the proof of [2, Theorem 2.1] with minor modifications. To do this, let us fix compact subsets $K \Subset M_P$ and $L \Subset \mathbb{C}^{n+1}$. Then it suffices to prove that $F^k_{D_j}(z,X)$ converges to $F^k_{M_P}(z,X)$ uniformly on $K \times L$. Indeed, suppose otherwise.

Then there exist $\epsilon_0 > 0$, a sequence of points $\{z_{j_\ell}\} \subset K$ and a sequence $X_{j_\ell} \subset L$ such that

$$\left|F^k_{D_{j_\ell}}\big(z_{j_\ell}, X_{j_\ell}\big) - F^k_{M_P}\big(z_{j_\ell}, X_{j_\ell}\big)\right| > \epsilon_0,\ \forall\ \ell \geq 1. \tag{6}$$

By the homogeneity of the Kobayashi metrics $F^k(z,X)$ in X, we may assume that $\| X_{j_\ell} \| = 1$ for all $\ell \geq 1$. Moreover, passing to subsequences, we may also assume that $z_{j_\ell} \to z_0 \in K$ and $X_{j_\ell} \to X_0 \in L$ as $\ell \to \infty$. Since M_P is taut, it follows from [11] that $F^k_{M_P}(z,X)$ is continuous on $D \times \mathbb{C}^{n+1}$. Hence, we obtain

$$F^k_{M_P}\big(z_{j_\ell}, X_{j_\ell}\big) \to F^k_{M_P}(z_0, X_0) \tag{7}$$

and thus we have

$$\left|F^k_{D_{j_\ell}}\big(z_{j_\ell}, X_{j_\ell}\big) - F^k_{M_P}(z_0, X_0)\right| > \epsilon_0/2 \tag{8}$$

for ℓ big enough.

By definition, for any $\delta \in (0,1)$ there exists a sequence of analytic discs $\varphi_{j_\ell} \in \mathrm{Hol}\big(\Delta, D_{j_\ell}\big)$ such that $\varphi_{j_\ell}(0) = z_0, \nu\big(\varphi_{j_\ell}\big) = k, \varphi^{(k)}_{j_\ell}(0) = k!\,\lambda_{j_\ell} X_{j_\ell}$, where $\lambda_{j_\ell} > 0$, and

$$F^k_{D_{j_\ell}}\big(z_{j_\ell}, X_{j_\ell}\big) \geq \frac{1}{\lambda_{j_\ell}} - \delta. \tag{9}$$

It follows from Proposition 2.1 that every subsequence of the sequence $\{\varphi_{j_\ell}\}$ has a subsequence converging to some analytic disc $\psi \in \mathrm{Hol}(\Delta, M_P)$ such that $\psi(0) = z_0, \nu(\psi) = k, \psi^{(k)}(0) = k!\,\lambda X_0$, for some $\lambda > 0$. Thus, one obtains that

$$F^k_{M_P}(z_0, X_0) \leq \frac{k!}{\left|\psi^{(k)}(0)\right|} \tag{10}$$

for any such ψ. Therefore, one has

$$\liminf_{\ell\to\infty} F^k_{D_{j_\ell}}\left(z_{j_\ell}, X_{j_\ell}\right) \geq F^k_{M_P}(z_0, X_0) - \delta. \tag{11}$$

On the other hand, as in , by the tautness of M_P, there exists a analytic disc $\varphi \in \mathrm{Hol}(\Delta, M_P)$ such that $\varphi(0) = z_0, \nu(\varphi) = k, \varphi^{(k)}(0) = k!\,\lambda X_0$, where $\lambda = 1/F^k_{M_P}(z_0, X_0)$.

Now for $\delta \in (0,1)$, let us define an analytic disc $\psi^\delta_{j_\ell}: \Delta \to \mathbb{C}^{n+1}$ by settings:

$$\psi^\delta_{j_\ell}(\zeta) := \varphi\big((1-\delta)\zeta\big) + \lambda(1-\delta)^k\zeta^k\left(X_{j_\ell} - X_0\right) + \left(z_{j_\ell} - z_0\right) \text{for all } \zeta \in \Delta. \tag{12}$$

Since $\varphi\left((1-\delta)\overline{\Delta}\right)$ is a compact subset of M_P and $X_{j_\ell} \to X_0$, $z_{j_\ell} \to z_0$ as $\ell \to \infty$, it follows that $\psi^\delta_{j_\ell}(\Delta) \subset D_{j_\ell}$ for all sufficiently large ℓ, that is, $\psi^\delta_{j_\ell} \in \mathrm{Hol}\left(\Delta, D_{j_\ell}\right)$. Moreover, by construction, $\psi^\delta_{j_\ell}(0) = z_{j_\ell}, \nu\left(\psi^\delta_{j_\ell}\right) = k$ and $\left(\psi^\delta_{j_\ell}\right)^{(k)}(0) = k!\,(1-\delta)^k\lambda X_{j_\ell}$. Therefore, again by definition, one has

$$F^k_{D_{j_\ell}}\left(z_{j_\ell}, X_{j_\ell}\right) \leq \frac{1}{(1-\delta)^k\lambda} = \frac{1}{(1-\delta)^k} F^k_{M_P}(z_0, X_0) \tag{13}$$

for all large ℓ. Thus, letting $\delta \to 0^+$, one concludes that

$$\limsup_{\ell\to\infty} F^k_{D_{j_\ell}}\left(z_{j_\ell}, X_{j_\ell}\right) \leq F^k_{M_P}(z_0, X_0). \tag{14}$$

By (11), (14), and (8), we seek a contradiction. Hence, the proof is complete. □

3. Weighted Boundary Limits of the General Kobayashi Metrics

This section is devoted to a proof of Theorem 1.1. First of all, we recall the following definition and proposition (see [12, Section 3] or [2, Section 4]).

Definition 3.1. Let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be a fixed n-tuple of positive numbers and $\mu > 0$. We denote by $\mathcal{O}(\mu, \Lambda)$ the set of smooth functions f defined near the origin of $\mathbb{C}^n$ such that

$$D^\alpha \overline{D}^\beta f(0) = 0 \text{ whenever } \sum_{j=1}^n \left(\alpha_j + \beta_j\right)\lambda_j \leq \mu.$$

If $n = 1$ and $\Lambda = (1)$ then we use $\mathcal{O}(\mu)$ to denote the functions vanishing to order at least μ at the origin.

Proposition 3.1.

If $f \in \mathcal{O}(\mu, \Lambda)$ then $\frac{\partial f}{\partial z_j}$ and $\frac{\partial f}{\partial \bar{z}_j}$ are in $\mathcal{O}\left(\mu - \lambda_j, \Lambda\right)$ for $j = 1, \dots, n$.

Suppose that $f_i,\ 1 \leq i \leq N$, are functions with $f_i \in \mathcal{O}(\mu_i, \Lambda)$. Then

$$\prod_{i=1}^N f_i \in \mathcal{O}(\mu, \Lambda), \text{ where } \mu = \sum_{i=1}^N \mu_i.$$

If $f \in \mathcal{O}(\mu, \Lambda)$, then there are constants $C, \delta > 0$ such that $|f(z)| \leq C\left(\sigma_\Lambda(z)\right)^{\mu+\delta}$ for all z in a small neighborhood of 0.

Now let Ω and ξ_0 be given as in Theorem 1.1. We note that the multitype of ξ_0 is $(1, m_1, \dots, m_n)$ with $m_n < +\infty$. Then by [2, Lemma 4.11] there are local holomorphic coordinates (z_0, z') in which $p = 0$ and Ω can be described a small neighborhood U of $\xi_0 = 0$ as follows:

$$\Omega \cap U = \{(z_0, z') \in U: \mathrm{Re}(z_0) + P(z') + R_1(z) + R_2(\mathrm{Im}\, z_0) + (\mathrm{Im}\, z_0)R(z) < 0\}.$$

Here P is a Λ-homogeneous plurisubharmonic real-valued polynomial containing no pluriharmonic terms, $R_1 \in \mathcal{O}(1, \Lambda), R \in \mathcal{O}(1/2, \Lambda)$, and $R_2 \in \mathcal{O}(2)$.

Recall that for any point $\eta = (\eta_0, \eta')) \in \Omega \cap U$, $\epsilon(\eta)$ denotes a real number such that $\tilde{\eta} := (\eta_0 + \epsilon(\eta), \eta')$ is in the hypersurface $\{\rho = 0\}$. Let $\{\eta_j\} = \{(\eta_{j0}, \eta_j')\} \subset \Omega \cap U \cap \Gamma$ be any sequence of points converging to 0. Let us write $\eta_j = (\eta_{j0}, \eta_{j1}, \dots, \eta_{jn}) \in \mathbb{C} \times \mathbb{C}^n$. Then, by definition, one has $|\eta_{jk}|^{m_j} \lesssim \epsilon(\eta_j)$ for every $1 \le k \le n$. Therefore, after taking a subsequence, we may assume that

$$\lim_{j\to\infty} \pi_{1/\epsilon(\eta_j)}(\eta_j') = \lim_{j\to\infty} \left(\frac{\eta_{j1}}{\epsilon(\eta_j)^{1/m_1}}, \dots, \frac{\eta_{jn}}{\epsilon(\eta_j)^{1/m_n}}\right) = \alpha \in \mathbb{C}^n.$$

Theorem 3.2. Let Ω be a $\mathcal{C}^\infty$-smooth pseudoconvex domain in $\mathbb{C}^n$ and $0 \in \partial\Omega$ such that Ω is h-extendible at 0. Suppose that the multitype of 0 is $(1, m_1, \dots, m_n)$ with $m_n < +\infty$ and let $\Lambda = (1/m_1, \dots, 1/m_n)$. Suppose also that the defining function ρ of Ω near 0 has the form

$$\rho(z_0, z') = \mathrm{Re}(z_0) + P(z') + R_1(z') + R_2(\mathrm{Im}\, z_0) + (\mathrm{Im}\, z_0)R(z),$$

where P is a Λ-homogeneous plurisubharmonic real-valued polynomial containing no pluriharmonic monomials, $R_1 \in \mathcal{O}(1, \Lambda), R \in \mathcal{O}(1/2, \Lambda)$ and $R_2 \in \mathcal{O}(2)$. Let $\{\eta_j\} = \{(\eta_{j0}, \eta_j')\} \subset \Omega \cap U \cap \Gamma$ be a sequence of points converging to 0 such that

$$\lim_{j\to\infty} \pi_{1/\epsilon(\eta_j)}(\alpha_j) = \alpha \in \mathbb{C}^n.$$

Then, we have

$$\begin{aligned}\lim_{\Omega\cap U\cap\Gamma \ni \eta_j \to 0} F^k_{\Omega\cap U}\left(\eta_j, \left(\pi_{\epsilon(\eta_j)}\right)_* X\right) &= F^k_{M_{P,\alpha}}((0', -1), X) \\ &= F^k_{M_P}((-1 - P(\alpha), \alpha), X),\ \forall X \in \mathbb{C}^{n+1}.\end{aligned}$$

Proof. Let $\{\eta_j = (\eta_{j0}, \eta_j')\}$ be a sequence of points converging Λ-nontangentially to the origin in $U \cap \{\rho < 0\} =: U^-$. Then let us consider the associated sequence of points $\tilde{\eta}_j = (\eta_{j0} + \epsilon_j, \eta_j') \in \partial\Omega$, where $\epsilon_j := \epsilon(\eta_j) > 0$. Define sequences of dilations Δ^{ϵ_j} and translations $L_{\tilde{\eta}_j}$ respectively by

$$\Delta^{\epsilon_j}(z_0, z_1, \dots, z_n) = \left(\frac{z_0}{\epsilon_j}, \frac{z_1}{\epsilon_j^{1/m_1}}, \dots, \frac{z_n}{\epsilon_j^{1/m_n}}\right) \tag{15}$$

and

$$L_{\tilde{\eta}_j}(z) = (z_0, z') - \tilde{\eta}_j = (z_0 - \eta_{j0} - \epsilon_j, z' - \eta_j'). \tag{16}$$

By using the change of variables $(\tilde{z}_0, \tilde{z}') := \Delta^{\epsilon_j} \circ L_{\eta_j}(z_0, z')$, i.e.,

$$\begin{cases} z_0 - \eta_{j0} = \epsilon_j \tilde{z}_0 \\ z_k - \eta_{jk} = \epsilon_j^{1/m_k} \tilde{z}_k,\ k = 1, \dots, n, \end{cases} \tag{17}$$

we obtain that $\Delta^{\epsilon_j} \circ L_{\tilde{\eta}_j}(\eta_{j0}, \eta_j') = (-1, 0')$ for every $j \in \mathbb{N}^*$. Furthermore, by using Taylor's theorem, the hypersurface $\Delta^{\epsilon_j} \circ L_{\tilde{\eta}_j}(\{\rho = 0\})$ is defined by an equation of the form

$$
\begin{aligned}
0 &= \epsilon_j^{-1}\rho\left(L_{\tilde{\eta}_j}^{-1}\circ(\Delta^{\epsilon_j})^{-1}(\tilde{z}_0,\tilde{z}')\right)\\
&= \mathrm{Re}(\tilde{z}_0) + R_2{}'(b_j)\mathrm{Im}(\tilde{z}_0) + \mathrm{Im}(\tilde{z}_0)R(\alpha_j) + \epsilon_j^{-1}o(\epsilon_j) + P(\tilde{z}')\\
&+2\mathrm{Re}\sum_{\substack{|p|>0\\ wt(p)\le 1}} \frac{D^pP(\alpha_j)}{p!}\epsilon_j^{wt(p)-1}(\tilde{z}')^p + \sum_{\substack{|p|,|q|>0\\ wt(p+q)<1}} \frac{D^p\overline{D}^qP(\alpha_j)}{p!\,q!}\epsilon_j^{wt(p+q)-1}(\tilde{z}')^p\left(\overline{\tilde{z}'}\right)^q\\
&+2\mathrm{Re}\sum_{\substack{|p|>0\\ wt(p)\le 1}} \frac{D^pR_1(\alpha_j)}{p!}\epsilon_j^{wt(p)-1}(\tilde{z}')^p + \sum_{\substack{|p|,|q|>0\\ wt(p+q)\le 1}} \frac{D^p\overline{D}^qR_1(\alpha)}{p!\,q!}\epsilon_j^{wt(p+q)-1}(\tilde{z}')^p\left(\overline{\tilde{z}'}\right)^q \qquad (18)\\
&+\epsilon_j^{-1}b_j\left(2\mathrm{Re}\sum_{\substack{|p|>0\\ wt(p)\le 1}} \frac{D^pR(\alpha_j)}{p!}\epsilon_j^{wt(p)}(\tilde{z}')^p + \sum_{\substack{|p|,|q|>0\\ wt(p+q)\le 1}} \frac{D^p\overline{D}^qR(\alpha_j)}{p!\,q!}\epsilon_j^{wt(p+q)}(\tilde{z}')^p\left(\overline{\tilde{z}'}\right)^q\right).
\end{aligned}
$$

Here and in what follows, the weight of a multi-index $p = (p_1, \dots, p_n)$ with respect to $\Lambda = (1/m_1, \dots, 1/m_n)$ is defined by

$$wt(p) = \sum_{j=1}^{n} \frac{p_j}{m_j}.$$

Since $\{\eta_j\} \subset \Gamma$ converging to the origin, without loss of generality, we may assume that

$$\lim_{j\to\infty} \pi_{1/\epsilon_j}(\eta_j{}') = \alpha \in \mathbb{C}^n,$$

where $\pi_t(z') = \left(t^{1/m_1}z_1, \dots, t^{1/m_n}z_n\right)$ for $t \ge 0$. Hence, as in the proof of [12, Theorem 1] one has

$$\lim_{j\to\infty} \frac{D^pP(\alpha_j)}{p!}\epsilon_j^{wt(p)-1} = \lim_{j\to\infty} \frac{D^pP\left(\pi_{1/\epsilon_j}(\alpha_j)\right)}{p!} = \frac{D^pP(\alpha)}{p!};$$

$$\lim_{j\to\infty} \frac{D^pR_1(\alpha_j)}{p!}\epsilon_j^{wt(p)-1} = \lim_{j\to\infty} \frac{D^pR(\alpha_j)}{p!}\epsilon_j^{wt(p)} = 0 \text{ whenever } wt(p) \le 1;$$

$$\lim_{j\to\infty} \frac{D^p\overline{D}^qP(\alpha_j)}{p!q!}\epsilon_j^{wt(p+q)-1} = \lim_{j\to\infty} \frac{D^p\overline{D}^qP\left(\pi_{1/\epsilon_j}(\alpha_j)\right)}{p!q!} = \lim_{j\to\infty} \frac{D^p\overline{D}^qP(\alpha)}{p!q!} \text{ whenever } wt(p+q) < 1;$$

$$\lim_{j\to\infty} \frac{D^p\overline{D}^qR_1(\alpha_j)}{p!q!}\epsilon_j^{wt(p+q)-1} = \lim_{j\to\infty} \frac{D^p\overline{D}^qR(\alpha_j)}{p!q!}\epsilon_j^{wt(p+q)} = 0 \text{ whenever } wt(p) + wt(q) \le 1;$$

$$\lim_{j\to\infty} R_2{}'(b_j) = \lim_{j\to\infty} R(\alpha_j) = 0.$$

Hence, after taking some subsequence, we may assume that the sequence of domains $\Omega_j := \Delta^{\epsilon_j} \circ L_{\tilde{\eta}_j}(U^-)$ converges to the following model

$$M_{P,\alpha} := \{(\tilde{z}_0, \tilde{z}') \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Re}(\tilde{z}_0) + P(\tilde{z}' + \alpha) - P(\alpha) < 0\},$$

which is biholomorphically equivalent to the model M_P. Without loss of generality, in what follows we always assume that $\{\Omega_j\}$ converges to M_P.

Since Ω_j converges to M_P as $j \to \infty$, Theorem 2.3 implies that

$$\lim_{j\to\infty} F^k_{\Omega_j}\big((\tilde z_0,\tilde z'),X\big) = F^k_{M_{P,\alpha}}\big((\tilde z_0,\tilde z'),X\big),\ \forall\ \big((\tilde z_0,\tilde z'),X\big)\in M_P\times\mathbb{C}^{n+1}. \tag{19}$$

We note that the convergence takes place uniformly on any compact subsets of $M_P\times\mathbb{C}^{n+1}$. Moreover, since $\Omega_j = T_j(U\cap\Omega)$, where $T_j := \Delta^{\epsilon_j}\circ L_{\tilde\eta_j}$, it follows that

$$\begin{aligned}\lim_{j\to\infty} F^k_{\Omega_j}\big(T_j(\eta_j),X\big) &= \lim_{j\to\infty} F^k_{\Omega\cap U}\left(\eta_j,\big(T_j^{-1}\big)_* X\right) = F^k_{M_{P,\alpha}}\big((-1,0'),X\big)\\ &= F^k_{M_P}\big((-1-P(\alpha),\alpha),X\big),\ \forall\ X\in\mathbb{C}^{n+1}.\end{aligned} \tag{20}$$

Hence, the proof follows easily from the fact that $\big(T_j^{-1}\big)_* = \big(\pi_{\epsilon(\eta)}\big)_*$. □

Remark 3.1. In [2], by virtue of the nontangential convergence of the sequence $\{\eta_j\}$, it was proved that all Ω_j are contained in a fixed taut domain. Hence, the the nontangential limits of Kobayashi metric are obtained (see [2, Theorem 5.2])

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let ρ be a local defining function for Ω on a neighborhood U of ξ_0 as in the hypothesis, i.e., $\xi_0 = 0$ and

$$\rho(\tilde z) = \mathrm{Re}(\tilde z_0) + P(\tilde z') + R(\tilde z), \tag{21}$$

where P is a $(1/m_1,\dots,1/m_n)$-homogeneous plurisubharmonic polynomial that contains no pluriharmonic terms, R is smooth and satisfies

$$|R(\tilde z)| \le C\left(|\tilde z_0| + \sum_{j=1}^n |\tilde z_j|^{m_j}\right)^{\gamma}, \tag{22}$$

for some constant $\gamma > 1$ and $C > 0$. Moreover, by [2, Lemma 4.11], we may assume that, shrinking U if necessary, there exists a biholomorphic map $(z_0,z') = \Phi(\tilde z_0,\tilde z')$, defined on U by

$$\begin{cases} z' = \tilde z';\\ z_0 = \tilde z_0 + b_1(\tilde z')z_0 + b_2(\tilde z')w^2 + b_3(\tilde z'),\end{cases} \tag{23}$$

where b_1, b_2, b_3 are smooth functions of $\tilde z'$ satisfying that $b_j(\tilde z') = O(|z|^2)$, $j = 1,2,3$, such that

$$\rho\big(\Phi^{-1}(z_0,z')\big) = \mathrm{Re}(z_0) + P(z') + R_1(z') + R_2(\mathrm{Im}\, z_0) + (\mathrm{Im}\, z_0)R(z'), \tag{24}$$

where P is a Λ-homogeneous plurisubharmonic real-valued polynomial containing no polynomials, $R_1\in\mathcal{O}(1,\Lambda)$, $R\in\mathcal{O}(1/2,\Lambda)$ and $R_2\in\mathcal{O}(2)$.

Now let us denote by $D := \Phi(U\cap\Omega)$ and then we apply Theorem 3.2 to obtain that

$$\begin{aligned}\lim_{D\cap\Gamma\ni\eta_j\to 0} F^k_D\left(\eta_j,\big(\pi_{\epsilon(\eta_j)}\big)_* X\right) &= F^k_{M_{P,\alpha}}\big((-1,0'),X\big)\\ &= F^k_{M_P}\big((-1-P(\alpha),\alpha),X\big),\ \forall\ X\in\mathbb{C}^{n+1}.\end{aligned} \tag{25}$$

Let $\tilde X_j := \big(\pi_{\epsilon(\eta_j)}\big)_* X$ and $Y_j = (\pi^{-1}_{\epsilon(\eta_j)})_*\circ\Phi_{*,\eta_j}\tilde X_j$. A computation shows that $\lim_{j\to\infty}\Phi_{*,\eta_j} = \mathrm{Id}$ and hence

$$\lim_{j\to\infty} Y_j = \lim_{j\to\infty}\big(\pi^{-1}_{\epsilon(\eta_j)}\big)_*\circ\Phi_{*,\eta_j}\big(\pi_{\epsilon(\eta_j)}\big)_* X = X. \tag{26}$$

Therefore, by the invariance of the metric one has

$$\begin{aligned}&\lim_{\Omega\cap\Gamma\ni\eta_j\to 0} F^k_D\left(\eta_j,\big(\pi_{\epsilon(\eta_j)}\big)_* X\right) = \lim_{\Omega\cap\Gamma\ni\eta_j\to 0} F^k_D(\eta_j,\tilde X_j)\\ &= \lim_{j\to\infty} F^k_{\Omega_j}\big((-1,0'),Y_j\big) = F^k_{M_P}\big((-1,0'),X\big).\end{aligned} \tag{27}$$

□

Proof of Corollary 1.2. For $s > 1$, one has

$$\lim_{\Omega\cap U\cap\Gamma^s\ni\eta\to 0}\pi_{1/\epsilon(\eta)}(\alpha) = 0. \tag{28}$$

Therefore, according to the proof of Theorem 1.1, we conclude that

$$\lim_{\Omega\cap U\cap\Gamma^s\ni\eta\to 0} F^k_{\Omega\cap U}\left(\eta, \left(\pi_{\epsilon(\eta)}\right)_* Y\right) = F^k_{M_P}\big((-1,0'), Y\big),\ \forall\, Y \in \mathbb{C}^{n+1}. \tag{29}$$

For $X = (X_0, X')$, set $Y := \epsilon(\eta)X = |\rho(\eta)|$ and notice that

$$\lim_{\eta\to 0}\left(\pi_{\epsilon(\eta)}\right)_* Y = \lim_{\eta\to 0}\left(\pi_{\epsilon(\eta)}\right)_* \epsilon(\eta)X = X_0 = X_N(\xi_0). \tag{30}$$

Hence, the proof follows.

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