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Original Article

Concentration Inequality for m-dependent Random Vectors in Hilbert Spaces

Tran Phuong Thao*, Le Vi

VNU University of Science, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam

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Abstract: Let $\{X_n, n \geq 1\}$ be a sequence of m-dependent random vectors taking values in a real separable Hilbert space. In this work we introduce concentration inequality for the partial sums of $\{X_n, n \geq 1\}$. Then, we give the weak laws of large numbers for weighted sums of $\{X_n, n \geq 1\}$.

Keywords: Hilbert spaces, m-dependent, concentration inequalities.

1. Introduction

One of the main tools in probabilistic analysis is the concentration inequality. Basically, the concentration inequalities are meant to give a sharp prediction of the actual value of a random variable by bounding the error term (from the expected value) with an associated probability. The two simplest concentration inequalities are the Markov's inequality and Chebyshev's inequality. If we want bounds which give us stronger (exponential) convergence, we can use Hoeffding's inequality or McDiarmid's inequality (see [1, 2]). However, such concentration inequalities usually require certain independence assumptions. When the independence assumptions do not hold, it is more difficult to have similar inequalities.

Recently, the subject of dependent random vectors in Hilbert spaces has received a lot of attention. The readers may see some results which have been obtained for negatively associated (NA) random vectors by [3], for negatively quadrant dependent (NQD) random vectors by [4] and for negatively superadditive dependent (NSD) random vectors by [5, 6]. The purpose of this note is to introduce concentration inequalities for sum of m-dependent random vectors in Hilbert spaces. As a consequence,

* Corresponding author.
E-mail address: tranphuongthao@hus.edu.vn

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we have the weak laws of large numbers for weighted sums of m-dependent random vectors in Hilbert spaces.

We start with the definitions of m-dependent random variables.

Definition 1. Let m be a non-negative integer number. A sequence of random varibles $\{X_n, n \geq 1\}$ is said to be m-dependent if for every n and every $j \geq m+1, \{X_{n+m+1}, \dots, X_{n+j}\}$ is independent of $\{X_1, X_2, \dots, X_n\}$. In particular, if $m = 0$, $\{X_n, n \geq 1\}$ is an independent sequence.

Example 1. Let $\{Z_n, n \geq 1\}$ be a sequence of i.i.d. $N(0,1)$ random variables. Then $\{Z_n - Z_{n+1}, n \geq 1\}$ are identically distributed $N(0,2)$ random variables. Let $X_n = Z_n - Z_{n+1}$ then X_n and X_{n+1} contain Z_{n+1} so they are dependent. But we can easily see that for every n, $\{X_1, \dots, X_n\}$ and $\{X_{n+2}, X_{n+3}, \dots\}$ are independent. Therefore, $\{X_n, n \geq 1\}$ is 1-dependent.

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle \cdot, \cdot \rangle$ and let $\{e_j, j \in B\}$ be an orthonormal basis in H.

Definition 2. A sequence $\{X_n, n \geq 1\}$ of H-valued random vectors is said to be m-dependent if for any $j \in B$, the sequence of random variables $\{\langle X_n, e_j \rangle, n \geq 1\}$ is m-dependent.

Example 2. Let $\{Z_n, n \geq 1\}$ be a sequence of i.i.d. $N(0,1)$ random variables. For each $n \geq 1, j \in B$, put $X_n^j = c_{1j} Z_n + \dots + c_{mj} Z_{n+m}$ where $\sum_{i=1}^{m} \sum_{j \in B} c_{ij}^2 < \infty$. We can see that for any $j \in B$, $\{X_n^j, n \geq 1\}$ is m-dependent by definition. We consider $X_n = \sum_{j \in B} X_n^j e_j$, $n \geq 1$, then $\{X_n, n \geq 1\}$ is a sequence of H-valued m-dependent random vectors.

2. The Main Results

In this work we shall always assume that $\{c_n, n \geq 1\}$ is a sequence of positive real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of H-valued m-dependent random vectors with mean 0 such that $\|X_n\| \leq c_n$ for every n, and $S_n = X_1 + X_2 + \dots + X_n$ be the partial sums.

Theorem 1. For all $t > \sqrt{(2m+1)\sum_{i=1}^{n} c_i^2}$ we get

$$P(\|S_n\| \geq t) \leq exp\left\{-\frac{\left(\sqrt{(2m+1)\sum_{i=1}^{n} c_i^2} - t\right)^2}{2\sum_{i=1}^{n}\left(\sum_{j=0}^{m \wedge (n+1)} c_{i+j}\right)^2}\right\}.$$

In particular, if $\{X_n, n \geq 1\}$ is an independent sequence, then for all $t > \sqrt{\sum_{i=1}^{n} c_i^2}$ we have

$$P(\|S_n\| \geq t) \leq exp\left\{-\frac{\left(\sqrt{\sum_{i=1}^{n} c_i^2} - t\right)^2}{2\sum_{i=1}^{n} c_i^2}\right\}.$$

We can use Theorem 1 to prove the following results.

Proposition 1. For all $t > \sqrt{(2m+1)\sum_{i=1}^{n} c_i^2}$ we have

$$P(\|S_n\| \geq t) \leq exp\left\{-\frac{\left(\sqrt{(2m+1)\sum_{i=1}^{n} c_i^2} - t\right)^2}{2(m+1)^2\sum_{i=1}^{n} c_i^2}\right\}.$$

Proposition 2. Let $\{m_n, n \geq 1\}$ be a sequence of positive integer numbers. Let $\{a_{ni}, n \geq 1, 1 \leq i \leq m_n\}$ be an array of positive real numbers such that

$$\sum_{i=1}^{m_n} a_{ni}^2 c_i^2 \to 0 \; as \; n \to \infty,$$

and $U_n = \sum_{i=1}^{m_n} a_{ni} X_i$. We have

$$U_n \overset{P}{\to} 0 \; as \; n \to \infty.$$

3. Proofs of the Main Results

To prove the main results, we need some lemmas. The first lemma is a familiar result in the theory of functional analysis.

Lemma 1. *(Triangle inequality)* Let X, Y be vectors in H. Then,

$$\|X\| - \|Y\| \leq \|X + Y\| \leq \|X\| + \|Y\|.$$

The next lemma is the one used in the proof of the McDiarmiad's inequality (see [5]).

Lemma 2. *(Hoeffding's lemma)* Let X be any real-valued random variable with expected value $E[X] = \eta$, such that $a \leq X \leq b$ almost surely, i.e. with probability one. Then, for all $\lambda \in \mathbb{R}^+$,

$$E\left[e^{\lambda(X - E[X])}\right] \leq exp\left\{\frac{\lambda^2 (b-a)^2}{8}\right\},$$

or equivalently

$$E\left[e^{\lambda X}\right] \leq exp\left\{\lambda\eta + \frac{\lambda^2 (b-a)^2}{8}\right\}.$$

Lemma 3. Let $\{X_n, n \geq 1\}$ be a sequence of H-valued m-dependent random vectors with mean 0 such that $\|X_n\| \leq c_n$ for every n, and $S_n = X_1 + X_2 + \cdots + X_n$. Then,

$$(E\|S_n\|)^2 \leq E[\|S_n\|^2] \leq (2m+1) \sum_{i=1}^{n} c_i^2.$$

Proof. We have

$$\begin{aligned} E[\|S_n\|^2] &= E\left\|\sum_{i=1}^{n} X_i\right\|^2 \\ &= E\sum_{j\in B}\left(\langle \sum_{i=1}^{n} X_i, e_j\rangle\right)^2 \\ &= \sum_{j\in B} E\left(\sum_{i=1}^{n} \langle X_i, e_j\rangle\right)^2 \\ &= \sum_{j\in B}\left(\sum_{i=1}^{n} E\langle X_i, e_j\rangle^2 + \sum_{1\leq i<k\leq n} 2E\big(\langle X_i, e_j\rangle\langle X_k, e_j\rangle\big)\right) \end{aligned}$$

Noting that if $k - i > m$, X_k và X_j are independent, hence

$$\sum_{j\in B} 2E\big(\langle X_i, e_j\rangle\langle X_k, e_j\rangle\big) = 0.$$

If $0 < k - i \leq m$, then

$$\sum_{j\in B} 2E\big(\langle X_i, e_j\rangle\langle X_k, e_j\rangle\big) \leq \sum_{j\in B} E\big(\langle X_i, e_j\rangle^2 + \langle X_k, e_j\rangle^2\big) = E[\|X_i\|^2] + E[\|X_k\|^2] \leq c_i^2 + c_k^2.$$

Therefore

$$\begin{aligned} E[\|S_n\|^2] &\leq \sum_{i=1}^{n} E[\|X_i\|^2] + \sum_{1\leq i<k\leq n, k-i\leq m} (c_i^2 + c_k^2) \\ &\leq (2m+1)\sum_{i=1}^{n} c_i^2 . \end{aligned}$$

We can now prove the main results of this article.

Proof of Theorem 1. Let $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$ and $Y_i = E(\|S_n\||\mathcal{F}_i)$ for all $1 \leq i \leq n$. Then,

$$\begin{cases} Y_n &= E[\|S_n\||\mathcal{F}_n] = \|S_n\|, \\ Y_0 &= E[\|S_n\|] = const. \end{cases}$$

It is easy to see that $(Y_0, Y_1, \dots, Y_n)$ is a martingale. We have

$$\begin{aligned} Y_n - Y_{n-1} &= \|S_n\| - E[\|S_n\||\mathcal{F}_{n-1}] \\ &= \|S_{n-1} + X_n\| - E[\|S_n\||\mathcal{F}_{n-1}]. \end{aligned}$$

From Lemma 1, we deduce that

$$\|S_{n-1}\| - c_n \leq \|S_{n-1}\| - \|X_n\| \leq \|S_{n-1} + X_n\| \leq \|S_n\| + \|X_n\| \leq \|S_{n-1}\| + c_n.$$

Denote

$$\begin{cases} A_{n-1} &= \|S_{n-1}\| - c_n - E[\|S_n\||\mathcal{F}_{n-1}] \\ B_{n-1} &= \|S_{n-1}\| + c_n - E[\|S_n\||\mathcal{F}_{n-1}]' \end{cases}$$

then A_{n-1} and B_{n-1} are $\mathcal{F}_{n-1}$-measured and

$$\begin{cases} A_{n-1} \leq Y_n - Y_{n-1} \leq B_{n-1}, \\ B_{n-1} - A_{n-1} = 2c_n. \end{cases}$$

Apply Lemma 2 for $Y_n - Y_{n-1}$ we get, for all $\lambda > 0$,

$$\begin{aligned} E[exp\{\lambda(Y_n - Y_{n-1})\}] = E\big[E[exp\{\lambda(Y_n - Y_{n-1})\}|\mathcal{F}_{n-1}]\big] &\leq E\left[exp\left\{\frac{\lambda^2(2c_n)^2}{8}\right\}\right] \\ &= exp\left\{\frac{\lambda^2 c_n^2}{2}\right\}. \end{aligned}$$

We write $\|S_n\| = (Y_n - Y_{n-1}) + (Y_{n-1} - Y_{n-2}) + \dots + (Y_1 - Y_0) + Y_0$. We have for $1 \leq i \leq n$,

$$Y_i - Y_{i-1} = E\big[\|S_{i-1} + X_i + \dots + X_{i+m\wedge(n-i)} + X_{i+m\wedge(n-i)+1} + \dots + X_n\||\mathcal{F}_i\big] - Y_{n-1}.$$

From Lemma 1 we deduce

$$\begin{aligned} \|S_{i-1} + X_{i+m\wedge(n-i)+1} + \dots + X_n\| - \sum_{j=0}^{m\wedge(n-i)} c_{i+j} &\leq \|S_{i-1} + X_i + \dots + X_n\| \\ &\leq \|S_{i-1} + X_{i+m\wedge(n-i)+1} + \dots + X_n\| + \sum_{j=0}^{m\wedge(n-i)} c_{i+j} . \end{aligned}$$

Denote

$$\begin{cases} A_{i-1} &= E\big[\|S_{i-1} + X_{i+m\wedge(n-i)+1} + \dots + X_n\||\mathcal{F}_i\big] - \sum_{j=0}^{m\wedge(n-i)} c_{i+j} - Y_{i-1} \\ B_{i-1} &= E\big[\|S_{i-1} + X_{i+m\wedge(n-i)+1} + \dots + X_n\||\mathcal{F}_i\big] + \sum_{j=0}^{m\wedge(n-i)} c_{i+j} - Y_{i-1}. \end{cases}$$

Noting that $X_{i+m\wedge(n-i)+1} + \dots + X_n$ is independent of $\mathcal{F}_i$ and S_{i-1} is $\mathcal{F}_{i-1}$-measured, then $E\big[\|S_{i-1} + X_{i+m\wedge(n-i)+1} + \dots + X_n\||\mathcal{F}_i\big]$ is also $\mathcal{F}_{i-1}$-measured. Therefore A_{n-1} and B_{n-1} are $\mathcal{F}_{i-1}$-measured and

$$\begin{cases} A_{i-1} \leq Y_i - Y_{i-1} \leq B_{i-1}, \\ B_{i-1} - A_{i-1} = 2\sum_{j=0}^{m\wedge(n-i)} c_{i+j} . \end{cases}$$

Apply Lemma 2 for $Y_i - Y_{i-1}$ we get, for all $\lambda > 0$,

$$E[exp\{\lambda(Y_i - Y_{i-1})\}] = E\big[E[exp\{\lambda(Y_i - Y_{i-1})\}|\mathcal{F}_{i-1}]\big] \le E\left[exp\left\{\frac{\lambda^2}{8}\left(2\sum\nolimits_{j=0}^{m\wedge(n-i)} c_{i+j}\right)^2\right\}\right]$$
$$= exp\left\{\frac{\lambda^2}{2}\left(\sum\nolimits_{j=0}^{m\wedge(n-i)} c_{i+j}\right)^2\right\}.$$

Hence for all $\lambda > 0$,

$$\begin{aligned} E\big[e^{\lambda\|S_n\|}\big] &= E\left[E\big[e^{\lambda(Y_n - Y_{n-1})}e^{\lambda Y_{n-1}}|\mathcal{F}_{n-1}\big]\right] \\ &= E\left[e^{\lambda Y_{n-1}}E\big[e^{\lambda(Y_n - Y_{n-1})}|\mathcal{F}_{n-1}\big]\right] \\ &\le E\big[e^{\lambda Y_{n-1}}\big]exp\left\{\frac{\lambda^2 c_n^2}{2}\right\} \\ &\le \cdots \\ &\le exp\left\{\frac{\lambda^2}{2}\sum_{i=0}^{n}\left(\sum_{j=0}^{m\wedge(n-i)} c_{i+j}\right)^2 + \lambda Y_0\right\}. \end{aligned}$$

By Chebysev's inequality,

$$\begin{aligned} P(\|S_n\| \ge t) \le \frac{E\big[e^{\lambda\|S_n\|}\big]}{e^{\lambda t}} &\le exp\left\{\frac{\lambda^2}{2}\sum_{i=0}^{n}\left(\sum_{j=0}^{m\wedge(n-i)} c_{i+j}\right)^2 + \lambda Y_0 - \lambda t\right\} \\ &\le exp\left\{\frac{\lambda^2}{2}\sum_{i=0}^{n}\left(\sum_{j=0}^{m\wedge(n-i)} c_{i+j}\right)^2 + \lambda\left(\sqrt{(2m+1)\sum_{i=1}^{n} c_i^2} - t\right)\right\}, \end{aligned}$$

Where we have used Lemma 3 for the last inequality. The result follows by choosing

$$\lambda = \frac{t - \sqrt{(2m+1)\sum_{i=1}^{n} c_i^2}}{\sum_{i=0}^{n}\left(\sum_{j=0}^{m\wedge(n-i)} c_{i+j}\right)^2} > 0.$$

Proof of Proposition 1. From Theorem 1 we get

$$P(\|S_n\| \ge t) \le exp\left\{-\frac{\left(\sqrt{(2m+1)\sum_{i=1}^{n} c_i^2} - t\right)^2}{2\sum_{i=1}^{n}\left(\sum_{j=0}^{m\wedge(n+i)} c_{i+j}\right)^2}\right\}.$$

By Cauchy-Schwarz's inequality we have

$$\left(\sum_{j=0}^{m\wedge(n+i)} c_{i+j}\right)^2 \le (m \wedge (n-i) + 1)\left(\sum_{j=0}^{m\wedge(n-i)} c_{i+j}^2\right) \le (m+1)\left(\sum_{j=0}^{m\wedge(n+i)} c_{i+j}^2\right).$$

Then

$$\begin{aligned} 2\sum_{i=1}^{n}\left(\sum_{j=0}^{m\wedge(n+i)} c_{i+j}\right)^2 &\le 2(m+1)\sum_{i=1}^{n}\sum_{j=0}^{m\wedge(n+i)} c_{i+j}^2 \\ &\le 2(m+1)^2\sum_{i=1}^{n} c_i^2\,. \end{aligned}$$

We deduce that

$$P(\|S_n\| \geq t) \leq exp\left\{-\frac{\left(\sqrt{(2m+1)\sum_{i=1}^{n} c_i^2} - t\right)^2}{2(m+1)^2 \sum_{i=1}^{n} c_i^2}\right\}.$$

The result follows.

Proof of Proposition 2. It suffices to prove that for all $t > 0$,

$$P(\|U_n\| \geq t) \to 0 \; as \; n \to \infty.$$

Because

$$\sum_{i=1}^{m_n} a_{ni}^2 c_i^2 \to 0 \;\; as \; n \to \infty,$$

there exists a positive integer number n_0 such that $t > \sqrt{(2m+1)\sum_{i=1}^{m_n} a_{ni}^2 c_i^2}$ for all $n > n_0$.

We consider the case when $n > n_0$. From Proposition 1 we have

$$\begin{aligned} P(\|U_n\| \geq t) \quad &\leq exp\left\{-\frac{\left(\sqrt{(2m+1)\sum_{i=1}^{m_n} a_{ni}^2 c_i^2} - t\right)^2}{2(m+1)^2 \sum_{i=1}^{m_n} a_{ni}^2 c_i^2}\right\} \\ &= exp\left\{-\frac{1}{2(m+1)^2}\left(\sqrt{2m+1} - \frac{t}{\sum_{i=1}^{m_n} a_{ni}^2 c_i^2}\right)^2\right\}. \end{aligned}$$

The result follows from the fact that as $n \to \infty$,

$$exp\left\{-\frac{1}{2(m+1)^2}\left(\sqrt{2m+1} - \frac{t}{\sum_{i=1}^{m_n} a_{ni}^2 c_i^2}\right)^2\right\} \to 0.$$

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