VNU Journal of Science: Mathematics – Physics, Vol. 40, No. 4 (2024) 91-98

VNU Journal of Science: Mathematics - Physics

Journal homepage: https://js.vnu.edu.vn/MaP

Original Article

On the Cofiniteness of Certain Local Cohomology Modules for a Pair of Ideals

Nguyen Van Hoang*

University of Transport and Communications, 3 Cau Giay, Lang Thuong, Dong Da, Hanoi, Vietnam

Received 31 July 2024
Revised 24 October 2024; Accepted 10 December 2024

Abstract: Let m, n be non-negative integers and N be an R-module such that $Ext_R^j(R/I, N)$ is finitely generated for all $j \leq m + n$ and $H_{I,J}^i(N) \in S_n(I,J)$ for all $i \leq m$, where $S_n(I,J)$ is a class of modules. We hence prove that (1) if $n = 1$ then $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \leq m$; (2) if $n \geq 2$ and $Ext_R^{\ell}(R/I, H_{I,J}^{t+i-\ell}(N))$ is finitely generated for all $1 \leq t \leq n-1, \ell \leq t-1, i \leq m$, then $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \leq m$. These extend the results of Khazaei-Sazeedeh [10, Thm 2.10, Thm 2.11] for local cohomology modules for a pair of ideals. Finally, we prove that $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \geq 0$ whenever I is principal, N is an R-module satisfying $Ext_R^j(R/I, N)$ is finitely generated for all $j \geq 0$, and $H_{I,J}^0(N)$ is in dimension < 2. This extends a theorem [13, Thm 1] of Kawasaki.

Keywords: cofinite module, local cohomology, local cohomology for a pair of ideals, in dimension < 2 module.
MSC2020-Mathematics Subject Classification System: 13D45, 14B15, 13E05.

1. Introduction

Throughout this note the ring R is commutative Noetherian. Let j be a non-negative integer, I, J ideals of R, and N an R-module. The j^{th} local cohomology functor $H_{I,J}^j(-)$ w.r.t a pair of ideals (I,J) was defined by Takahashi et al. [1] as the j^{th} right derived functor of (I,J)-torsion functor $\Gamma_{I,J}(-)$. They called $H_{I,J}^j(N)$ the j^{th} local cohomology module of N w.r.t a pair of ideals (I,J). It is clear that $H_{I,0}^j(N)$ is just the ordinary local cohomology module $H_I^j(N)$ of N w.r.t the ideal I.

* Corresponding author.
E-mail address: hoangnv@utc.edu.vn

https//doi.org/10.25073/2588-1124/vnumap.4960

In [2], Grothendieck conjectured that $(0:_{H_I^j(N)} I)$ is finitely generated for all $j \geq 0$ and all finitely generated modules N. In [3], Hartshorne provided a counterexample to this conjecture. He also introduced an R-module K to be I-cofinite if $Supp_R(K) \subseteq V(I)$ and $Ext_R^j(R/I, K)$ is finitely generated for all j and he asked a question: *For which rings R and ideals I are the modules $H_I^j(N)$ is I-cofinite for all finitely generated modules N?* Hartshorne proved that if N is a finitely generated R-module where R is a complete regular local ring, then $H_I^j(N)$ is I-cofinite in the case I is a prime ideal with $\dim_R(R/I) = 1$ (see [3, Coro 7.7]). This problem was studied more extensively by the numerous mathematicians (see papers [4-10]). The aim of this note is to investigate a question similar to the one above for the theory of local cohomology w.r.t a pair of ideals. Before stating main results, we recall the notion of (I,J)-cofinite module which is introduced by Tehranian-and Talemi (see [11, Def 2.1]) as follows: An R-module K is called *(I,J)-cofinite* if $Supp_R(K) \subseteq W(I,J)$ and $Ext_R^j(R/I, K)$ is finitely generated for all j, where $W(I,J) = \{p \in Spec\, R \mid I^n \subseteq p + J$ for some $n \in N_0\}$ (see [1, Def 3.1]). We then introduce a class $S_n(I,J)$ of modules (see Definition 2.1). The first main result in this note is the following theorem on the cofiniteness of module $H_{I,J}^i(N)$ w.r.t a pair of ideals (I,J).

Theorem 1.1. *Let m be an integer with $m \geq 0$, and N an R-module. If $Ext_R^j(R/I,N)$ is finitely generated for all $j \leq m+1$ and $H_{I,J}^i(N) \in S_1(I,J)$ for all $i \leq m$, then $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \leq m$.*

This result covers [10, Thm 2.10] of Khazaei-Sazeedeh for local cohomology modules (see Corollary 2.3). Theorem 1.1 also has a consequence on the co-finiteness of $H_{I,J}^i$(N) when these modules are in dimension < 2 (see Corollary 2.5), in which an R-module K is called in dimension < 2 if there exists a finitely generated R-submodule T of K such that $dimSupp_R(K/T) \leq 1$ (see [12, Def 2.1] of Asadollahi-Naghipour). We next prove the co-finiteness of $H_{I,J}^i$(N) concerning the condition that $H_{I,J}^i$(N) belongs to a class module $S_n(I,J)$. The following theorem is the second main result in this note.

Theorem 1.2. *Let n be an integer with $n \geq 2$. Let m be a non-negative integer such that $Ext_R^j(R/I,N)$ is finitely generated for all $j \leq m+n$ and $H_{I,J}^i(N) \in S_n(I,J)$ for all $i \leq m$. If $Ext_R^\ell(R/I, H_{I,J}^{t+i-\ell}(N))$ is finitely generated for all $1 \leq t \leq n-1, 0 \leq \ell \leq t-1$ and $i \leq m$, then $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \leq m$.*

By replacing $J = 0$ in Theorem 1.2, we obtain again a result of Khazaei-Sazeedeh in [10, Thm 2.11]. The last of this section, we use Theorem 1.1 and Corollary 2.5 to study the co-finiteness of $H_{I,J}^i(N)$ when the ideal I is principal. Kawasaki proved that if I is a principal ideal then local cohomology module $H_I^i(M)$ is I-cofinite for all $i \geq 0$ when M is a finitely generated R-module (see [13, Thm 1]). We next extend the result of Kawasaki in [13] to the case of local cohomology module $H_{I,J}^i(N)$ where I is a principal ideal and the module N is not necessary finitely generated. The following theorem is the third main result in this note.

Theorem 1.3. *Let I and J be ideals of R such that I is a principal ideal. Let N be an R-module such that $Ext_R^j(R/I,N)$ is finitely generated for all $j \geq 0$. Assume that the R-module $H_{I,J}^0(N)$ is in dimension < 2. Then the R-module $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \geq 0$.*

This note is organized into two sections. In Section 2, we first introduce a condition $P_n(I,J)$ and a class $S_n(I,J)$ of modules. We second prove three main theorems and their consequences.

2. Main Results

We first recall a class $S_n(I,J)$ of modules which is introduced in [14].

Definition 2.1. (see [14, Def 2.3]) Let n be a non-negative integer, I,J ideals of R and K an R-module.

We say that an R-module K satisfies the condition $P_n(I,J)$ if the following statement holds: *Suppose that $Ext_R^j(R/I,K)$ is finitely generated for all $j \leq n$ and $Supp_R(K) \subseteq W(I,J)$. Then the module K is (I,J)-cofinite.*

We define a class of R-modules as follows:

$$S_n(I,J) = \{K \in Mod - R \mid K \text{ satisfies the condition } P_n(I,J)\},$$

where $Mod - R$ is the category of modules over the ring R.

Remark 2.2. Firstly, we observe that $S_0(I,J) \subseteq S_1(I,J) \subseteq S_2(I,J) \subseteq \ldots$. Secondly, we recall that an R-module K satisfies the condition $P_n(I)$ if K is I-cofinite whenever $Ext_R^j(R/I,K)$ is finitely generated for all $j \leq n$ and $Supp_R(K) \subseteq V(I)$ (see [10, Def 2.1]). Hence, if we replace J by 0 in $P_n(I,J)$, we obtain that $P_n(I,0) = P_n(I)$. Moreover, we also have

$$S_n(I,0) = S_n(I) = \{K \in Mod - R \mid K \text{ satisfies the condition } P_n(I)\}$$

and the notion (I,J)-cofinite is an extension of the notion I-cofinite because the notion $(I,0)$-cofinite coincides exactly with the notion I-cofinite.

We next prove Theorem 1.1 on the co-finiteness of $H_{I,J}^i(N)$ with respect to a pair of ideals (I,J) concerning the condition $S_1(I,J)$.

Proof of Theorem 1.1. We process by induction on $m \geq 0$. Assume that $m = 0$. Then the R-module $Ext_R^j(R/I,N)$ is finitely generated for $j = 0,1$ by the hypothesis. Therefore, the isomorphism $Hom_R(R/I,\Gamma_{I,J}(N)) \cong Hom_R(R/I,N)$ and the exact sequence

$$Hom_R(R/I,N/\Gamma_{I,J}(N)) = 0 \to Ext_R^1(R/I,\Gamma_{I,J}(N)) \to Ext_R^1(R/I,N)$$

imply that $Ext_R^j(R/I,\Gamma_{I,J}(N))$ is finite for all $j \leq 1$. Moreover, since $\Gamma_{I,J}(N) \in S_1(I,J)$ by the hypothesis for $m = 0$, we obtain that $H_{I,J}^0(N)$ is (I,J)-cofinite as $H_{I,J}^0(N) \cong \Gamma_{I,J}(N)$.

We now assume that $m > 0$ and the result have been proved for all values less than m. Consider the short exact sequence

$$0 \to \overline{N} \to E \to P \to 0, \quad (\dagger)$$

where $\overline{N} = N/\Gamma_{I,J}(N), P = E/\overline{N}$, and $E = E_R(\overline{N})$ the injective envelope of R-module $\overline{N}$. Note that $\Gamma_{I,J}(E) = 0$ since $\Gamma_{I,J}(\overline{N}) = 0$. By the case $m = 0$ we obtain that $\Gamma_{I,J}(N)$ is (I,J)-cofinite. Thus $Ext_R^j(R/I,\overline{N})$ is finitely generated for all $j \leq m + 1$ by the following exact sequences

$$Ext_R^j(R/I,N) \to Ext_R^j(R/I,\overline{N}) \to Ext_R^{j+1}(R/I,\Gamma_{I,J}(N))$$

and the hypothesis of N. On the other hand, by the exact sequence ($\dagger$), we have isomorphisms $Ext_R^j(R/I,P) \cong Ext_R^{j+1}(R/I,\overline{N})$ for all $j \geq 0$. Therefore the R-module $Ext_R^j(R/I,P)$ is finitely generated for all $j \leq m$. Also, by the sequence ($\dagger$) and by the fact $\Gamma_{I,J}(E) = 0$, we have isomorphisms $H_{I,J}^i(P) \cong H_{I,J}^{i+1}(\overline{N})$ for all $i \geq 0$; moreover, we also get $H_{I,J}^{i+1}(\overline{N}) \cong H_{I,J}^{i+1}(N)$ for all $i \geq 0$. Therefore, we obtain that $H_{I,J}^i(P) \cong H_{I,J}^{i+1}(N) \in S_1(I,J)$ for all $i \leq m-1$ by the hypothesis. Finally, by applying the inductive assumption for the R-module P, we obtain that $H_{I,J}^i(P)$ is (I,J)-cofinite for all $i \leq m-1$, and so $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \leq m$, as required. □

By replacing $J = 0$ in Theorem 1.1 we get the following consequence.

Corollary 2.3. (see [10, Thm 2.10]) *Let m be a non-negative integer and N an R-module. If $Ext_R^i(R/I,N)$ is finitely generated for all $i \leq m+1$ and $H_I^i(N) \in S_1(I)$ for all $i \leq m$, then $H_I^i(N)$ is I-cofinite for all $i \leq m$.*

Remark 2.4. Before considering more consequences of Theorem 1.1, we need to recall that an R-module K is called *in dimension* < 2 if there exists a finitely generated submodule T of K such that $dimSupp_R(K/T) < 2$ (see [12, Def 2.1]). The following is a result on the co-finiteness of $H_{I,J}^i(N)$.

Corollary 2.5. (covers [9, Thm 1.1]) *Let m be a non-negative integer and N an R-module. If $Ext_R^j(R/I,N)$ is finitely generated for all $j \leq m+1$ and $H_{I,J}^i(N)$ is in dimension < 2 for all $i \leq m$, then $H_{I,J}^i(N)$ is (I,J)-cofinite for all $i \leq m$.*

Proof. By Theorem 1.1, we need only to show $H_{I,J}^i(N) \in S_1(I,J)$ for all $i \leq m$ provided that $Ext_R^j(R/I,N)$ is finitely generated for all $j \leq m+1$ and $H_{I,J}^i(N)$ is in dimension < 2 for all $i \leq m$.

Fix an integer $i \in \{0,1,\dots,m\}$. Set $K = H_{I,J}^i(N)$. Hence, K is in dimension < 2 by the hypothesis. Note that $Supp_R(K) \subseteq W(I,J)$ by [1, Prop 1.7 and Coro 1.13]. We now prove that the R-module K satisfies the condition $P_1(I,J)$. To do this, we assume that the R-modules $Hom_R(R/I,K)$ and $Ext_R^1(R/I,K)$ are finitely generated, we claim that $Ext_R^j(R/I,K)$ is finitely generated for all j. By the hypothesis of K, there exists a finitely generated submodule T of K such that $dimSupp_R(K/T) \leq 1$ and $Supp_R(K/T) \subseteq W(I,J)$. We have the following exact sequence

$$0 \to Hom_R(R/I,T) \to Hom_R(R/I,K) \to Hom_R(R/I,K/T)$$
$$\to Ext_R^1(R/I,T) \to Ext_R^1(R/I,K) \to Ext_R^1(R/I,K/T)$$
$$\to Ext_R^2(R/I,T) \to Ext_R^2(R/I,K) \to Ext_R^2(R/I,K/T) \to \dots$$

It follows that $Hom_R(R/I,K/T)$ and $Ext_R^1(R/I,K/T)$ are finitely generated. Keep in mind that $dimSupp_R(K/T) \leq 1$. Thus, we get by [15, Thm 2.5] that $Ext_R^j(R/I,K/T)$ is finitely generated for all j. Therefore, by the above exact sequence again, we obtain that $Ext_R^j(R/I,K)$ is finitely generated for all j, and the claim is proved. That means the R-module $H_{I,J}^i(N)$ satisfies the condition $P_1(I,J)$ for any $i \in \{0,1,\dots,m\}$. Hence $H_{I,J}^i(N) \in S_1(I,J)$ for all $0 \leq i \leq m$, as required. □

We next prove Theorem 1.2 on the co-finiteness of $H_{I,J}^i(N)$ with respect to a pair of ideals (I,J) in the case where the module $H_{I,J}^i(N)$ belongs to a class $S_n(I,J)$ for some integer $n \geq 2$.

Proof of Theorem 1.2. We prove by induction on $m \geq 0$. Suppose $m = 0$. We have by the hypothesis that $Ext_R^j(R/I,N)$ is finitely generated for all $j \leq n+0$ (*), $Ext_R^\ell(R/I,H_{I,J}^{t+0-\ell}(N))$ is finitely generated for all $1 \leq t \leq n-1, 0 \leq \ell \leq t-1$ and $i = 0$, and $\Gamma_{I,J}(N) = H_{I,J}^0(N) \in S_n(I,J)$. Hence, in order to show that the module $H_{I,J}^0(N)$ is (I,J)-cofinite, we need only to prove that $Ext_R^v(R/I,H_{I,J}^0(N))$ is finitely generated for all $v \leq n$. We now have the following exact sequences

$$0 \to \Gamma_{I,J}(N) \to N \to \overline{N} = N/\Gamma_{I,J}(N) \to 0 \qquad (1)$$
$$0 \to \overline{N} \to E_0 \to N_1 = E_0/\overline{N} \to 0 \qquad (1')$$

where E_0 is the injective envelope of R-module $\overline{N}$. By the sequence (1), we get the following exact sequence

$$Ext_R^{v-1}(R/I,\overline{N}) \to Ext_R^v(R/I,\Gamma_{I,J}(N)) \to Ext_R^v(R/I,N).$$

Hence, from (*), in order to show the finiteness of $Ext_R^v(R/I,\Gamma_{I,J}(N))$ for all $v \leq n$, we need to Claim that $Ext_R^{v-1}(R/I,\overline{N})$ is finitely generated for all $v \leq n$.

The case $v - 1 = 0$ is clear since

$$Hom_R(R/I,\overline{N}) \cong (0:_{\overline{N}} I) \subseteq \Gamma_I(\overline{N}) \subseteq \Gamma_{I,J}(\overline{N}) = 0$$

by [1, Corollary 1.13]. For the case $v - 1 = 1$, by the sequence (1'), we get that

$$Ext_R^1(R/I,\overline{N}) \cong Hom_R(R/I,N_1)$$
$$\cong Hom_R(R/I,\Gamma_{I,J}(N_1))$$
$$\cong Hom_R(R/I,H_{I,J}^1(\overline{N}))$$

since $\Gamma_{I,J}(E_0) = 0$. Remind that

$$Hom_R\left(R/I,H_{I,J}^1(\overline{N})\right) \cong Ext_R^0(R/I,H_{I,J}^{1+0-0}(N))$$

is finitely generated by the hypothesis (for $t = 1, \ell = 0, i = 0$). Hence $Ext_R^1(R/I,\overline{N})$ is finitely generated.

We next prove the finiteness of $Ext_R^{v-1}(R/I,\overline{N})$ in the case $1 < v - 1 \le n - 1$ (**). Fix an integer $v - 1 \in \{2,3,\ldots, n - 1\}$. For any integer $u > 1$, we have the following exact sequences

$$0 \to \Gamma_{I,J}(N_{u-1}) \to N_{u-1} \to \overline{N_{u-1}} \to 0, \qquad \text{(u)}$$
$$0 \to \overline{N_{u-1}} \to E_{u-1} \to N_u \to 0 \qquad \text{(u')}$$

where $\overline{N_{u-1}} = N_{u-1}/\Gamma_{I,J}(N_{u-1})$, $E_{u-1} = E_R(\overline{N_{u-1}})$ the injective envelope of $\overline{N_{u-1}}$ and $N_u = E_{u-1}/\overline{N_{u-1}}$. By the sequence (1'), we get an isomorphism

$$Ext_R^{v-2}(R/I,N_1) \cong Ext_R^{v-1}(R/I,\overline{N}).$$

Moreover, by the sequence (u) with $u = 2$, we obtain the following exact sequence

$$Ext_R^{v-2}(R/I,\ \Gamma_{I,J}(N_1)) \to Ext_R^{v-2}(R/I,N_1) \to Ext_R^{v-2}(R/I,\overline{N_1})\ .$$

Therefore, it is enough to show that the modules $Ext_R^{v-2}(R/I,\Gamma_{I,J}(N_1))$ and $Ext_R^{v-2}(R/I,\overline{N_1})$ are finitely generated. Note that $\Gamma_{I,J}(N_1) \cong H_{I,J}^1(\overline{N}) \cong H_{I,J}^1(N)$. Thus, the R-module

$$Ext_R^{v-2}(R/I,\Gamma_{I,J}(N_1)) \cong Ext_R^{v-2}(R/I,H_{I,J}^1(N))$$

is finitely generated by the hypothesis (for $t = v - 1, \ell = v - 2, i = 0$). By the sequence (u') with u = 2, we have an isomorphism

$$Ext_R^{v-2}(R/I,\overline{N_1}) \cong Ext_R^{v-3}(R/I,N_2)$$

By the sequence (u) with u = 3, we have the following exact sequence

$$Ext_R^{v-3}(R/I,\Gamma_{I,J}(N_2)) \to Ext_R^{v-3}(R/I,N_2) \to Ext_R^{v-3}(R/I,\overline{N_2})$$

Thus, it suffices to show that $Ext_R^{v-3}(R/I,\Gamma_{I,J}(N_2))$ and $Ext_R^{v-3}(R/I,\overline{N_2})$ are finitely generated. We have the following isomorphisms

$$Ext_R^{v-3}(R/I,\Gamma_{I,J}(N_2)) \cong Ext_R^{v-3}(R/I,H_{I,J}^1(\overline{N_1}))$$
$$\cong Ext_R^{v-3}(R/I,H_{I,J}^1(N_1))$$
$$\cong Ext_R^{v-3}(R/I,H_{I,J}^2(\overline{N}))$$
$$\cong Ext_R^{v-3}(R/I,H_{I,J}^2(N))$$

and the last module is finitely generated by the hypothesis (for $t = v - 1, \ell = v - 3, i = 0$). Continuing the same arguments as above, after finitely many steps, we need only to show that R-the modules $Ext_R^1(R/I,\Gamma_{I,J}(N_{v-2}))$ and $Ext_R^1(R/I,\overline{N_{v-2}})$ are finitely generated. We also have the following isomorphisms

$$Ext_R^1(R/I,\Gamma_{I,J}(N_{v-2})) \cong Ext_R^1(R/I,H_{I,J}^1(\overline{N_{v-3}})) \cong Ext_R^1(R/I,H_{I,J}^1(N_{v-3}))$$
$$\cong Ext_R^1(R/I,H_{I,J}^2(\overline{N_{v-4}})) \cong Ext_R^1(R/I,H_{I,J}^2(N_{v-4}))$$
$$\ldots$$
$$\cong Ext_R^1(R/I,H_{I,J}^{v-3}(\overline{N_1})) \cong Ext_R^1(R/I,H_{I,J}^{v-3}(N_1))$$
$$\cong Ext_R^1(R/I,H_{I,J}^{v-2}(\overline{N})) \cong Ext_R^1(R/I,H_{I,J}^{v-2}(N))$$

and the last module is finitely generated by the hypothesis (for $t = v - 1, \ell = 1, i = 0$). Moreover, by the sequences (u) and (u') for $u = v - 1, v - 2, \ldots, 1$ we obtain the following isomorphisms

$$Ext_R^1(R/I,\overline{N_{v-2}}) \cong Hom_R(R/I,N_{v-1}) \cong Hom_R(R/I,\Gamma_{I,J}(N_{v-1}))$$
$$\cong Hom_R(R/I,H_{I,J}^1(\overline{N_{v-2}})) \cong Hom_R(R/I,H_{I,J}^1(N_{v-2}))$$

$$\cong Hom_R(R/I, H^2_{I,J}(\overline{N_{v-3}})) \cong Hom_R(R/I, H^2_{I,J}(N_{v-3}))$$
$$\ldots$$
$$\cong Hom_R(R/I, H^{v-2}_{I,J}(\overline{N_1})) \cong Hom_R(R/I, H^{v-2}_{I,J}(N_1))$$
$$\cong Hom_R(R/I, H^{v-1}_{I,J}(\overline{N})) \cong Hom_R(R/I, H^{v-1}_{I,J}(N)).$$

Note that the last module in the above isomorphisms is finitely generated by the hypothesis (for $t = v-1, \ell = 0, i = 0$). Hence $Ext^1_R(R/I, \overline{N_{v-2}})$ is finitely generated. Hence the statement (**) is proved, and so that the Claim also is proved. Thus, we have proved the theorem in the case $m = 0$.

Assume that $m > 0$ and the theorem is true for all values $< m$. By the inductive assumption, $H^0_{I,J}(N) = \Gamma_{I,J}(N)$ is (I,J)-cofinite, and so the module $Ext^j_R(R/I, \overline{N})$ is finitely generated for all $j \leq m+n$ by the sequence (1) and by the hypothesis. Hence, we obtain by (1), (1') and the hypothesis that $Ext^j_R(R/I, N_1) \cong Ext^{j+1}_R(R/I, \overline{N})$ is finitely generated for all $j \leq n+m-1$. Moreover, we have by (1') and (1) that

$$H^i_{I,J}(N_1) \cong H^{i+1}_{I,J}(\overline{N}) \cong H^{i+1}_{I,J}(N) \in S_n(I,J)$$

for all $i \leq m-1$ by the hypothesis. On the other hand, by (1) and (1'), we have

$$Ext^\ell_R(R/I, H^{t+i-\ell}_{I,J}(N_1)) \cong Ext^\ell_R(R/I, H^{t+i+1-\ell}_{I,J}(\overline{N}))$$
$$\cong Ext^\ell_R(R/I, H^{t+i+1-\ell}_{I,J}(N))$$

for all $1 \leq t \leq n-1, 0 \leq \ell \leq t-1$ and $i \leq m-1$. Moreover, we get by the hypothesis of N that the last module in the above isomorphism is finitely generated for all $1 \leq t \leq n-1, 0 \leq \ell \leq t-1$ and $i \leq m-1$. Hence the R-module $Ext^\ell_R(R/I, H^{t+i-\ell}_{I,J}(N_1))$ is finitely generated for all $1 \leq t \leq n-1, 0 \leq \ell \leq t-1$ and $i \leq m-1$. Therefore, by the inductive assumption for R-module N_1, we obtain that the module $H^i_{IJ}(N_1)$ is (I,J)-cofinite for all $i \leq m-1$. Hence, $H^i_{I,J}(N) \cong H^{i-1}_{I,J}(N_1)$ is (I,J)-cofinite for all $i \leq m$, and so the proof of our theorem is completed. □

Note that Theorem 1.2 is an extension of a theorem of Khazaei-Sazeedeh in [10, Thm 2.11]. Moreover, by replacing $n = 2$ in Theorem 1.2 we get an immediate result as the following consequence.

Corollary 2.6. *Let m be a non-negative integer such that $Ext^j_R(R/I, N)$ is finitely generated for all $j \leq m+2$ and $H^i_{I,J}(N) \in S_2(I,J)$ for all $i \leq m$. If $Hom_R(R/I, H^{1+i}_{I,J}(N))$ is finitely generated for all $i \leq m$, then $H^i_{I,J}(N)$ is (I,J)-cofinite for all $i \leq m$.*

For the last of this section, before proving Theorem 1.3 we need to recall the following lemma on the class $S_n(I,J)$ and the modules $H^i_{I,J}(N)$.

Lemma 2.7. (see [14, Thm 3.1]) *Let n be a non-negative integer. Let N be an R-module such that $Ext^j_R(R/I, N)$ is finitely generated for all j. Let t be a non-negative integer such that $H^i_{I,J}(N) = 0$ for all $i \neq t, t+1$. Then $H^{t+1}_{I,J}(N) \in S_n(I,J)$ if and only if $H^t_{I,J}(N) \in S_{n+2}(I,J)$.*

We now are ready to prove the last theorem in this note.

Proof of Theorem 1.3. By the hypothesis we have that $Ext^j_R(R/I, N)$ is finitely generated for all j and $H^0_{I,J}(N)$ is in dimension < 2. From this we obtain by [16, Thm 1.1, (i)] that $H^0_{I,J}(N)$ is an (I,J)-cofinite module over R. It yields that $Ext^j_R(R/I, H^0_{I,J}(N))$ is finitely generated for all $j \geq 0$. Hence $H^0_{I,J}(N)$ belongs to the class of modules $S_2(I,J)$.

Note that since I is a principal ideal, there exists an element $a \in I$ such that $I = (a)$. Hence $H^i_{I,J}(N) \cong H^i(C^\bullet_{\underline{a},J} \otimes_R N)$ for all $i \geq 0$, where $C^\bullet_{\underline{a},J} = (0 \to R \to R_{a,J} \to 0)$ is a complex (see [1, Def 2.1, Def 2.2 and Thm 2.4]). Thus, we obtain that $H^i_{I,J}(N) = 0$ for all $i \neq 0, 1$. Hence, the conditions in the hypothesis of Lemma 2.7 are satisfied for number $t = 0$. Keep in mind that $H^0_{I,J}(N) \in S_2(I,J)$ by the above paragraph, and hence the module $H^1_{I,J}(N)$ belongs to the class of modules $S_0(I,J)$ by again Lemma 2.7.

On the other hand, by again the hypothesis that $Ext^j_R(R/I,N)$ is finitely generated for all $j \geq 0$ and $H^0_{I,J}(N)$ is in dimension < 2, we obtain by [16, Thm 1.1, (ii)] that $Hom_R(R/I, H^1_{I,J}(N))$ is finitely generated. Therefore, since $H^1_{I,J}(N) \in S_0(I,J)$, we get by the definition of class $S_0(I,J)$ that the R-module $Ext^j_R(R/I, H^1_{I,J}(N))$ is finitely generated for all $j \geq 0$, that is, the R-module $H^1_{I,J}(N)$ is (I,J)-cofinite, as required. □

By replacing $J = 0$ in Theorem 1.3, we obtain the following corollary on the cofiniteness of local cohomology modules in [14, Thm 3.4].

Corollary 2.8. *Let I be a principal ideal of R and N an R-module such that $Ext^j_R(R/I,N)$ is finitely generated for all $j \geq 0$. Assume that the module $H^0_I(N)$ is in dimension < 2. Then the R-module $H^i_I(N)$ is $I-$cofinite for all $i \geq 0$.*

As a consequence of Corollary 2.8 we obtain a theorem of K. I. Kawasaki in the paper [13] as the following result.

Corollary 2.9. (see [13, Thm 1]) *Let I be a principal ideal of R and N a finitely generated R-module. Then $H^i_I(N)$ is $I-$cofinite for all $i \geq 0$.*

Proof. Since N is finitely generated, we obtain that the R-module $Ext^j_R(R/I,N)$ is finitely generated for all $j \geq 0$, and $H^0_I(N)$ is in dimension < 2. Therefore, the conclusion of the corollary follows from Corollary 2.8. □

Finally, we give an example on a non-finitely generated R-module N satisfying the assumption of Theorem 1.3.

Example 2.10. Let $R = k[X]$ be the ring of polynomials in one variable X with coefficients in field k. Let $I = (X)$ be a pricipal ideal of R. Note that since R is a PID, any divisible module is injective. The injective hull of R is the fraction field $K = k(X)$. Since K/R is divisible, it is injective. Hence, an injective resolution of R-module R is given by $0 \to R \to K \to K/R \to 0$. We apply functor $\Gamma_I(-)$ and calculate the local cohomology as the cohomology of the complex $0 \to \Gamma_I(K) \to \Gamma_I(K/R) \to 0$. We then obtain that $H^0_I(R) = 0$ and $H^j_I(R) = 0$ for all $j > 1$. We also have

$$H^1_I(R) = \Gamma_I(K/R) = R_X/R = k[X,X^{-1}]/k[X].$$

We set $N = H^1_I(R)$. We obtain by [17, Exercise 4.2.4 (i)] that the R-module N is not finitely generated. Moreover, since I is a principal ideal, we get that $N = H^1_I(R)$ is I-cofinite by [13, Thm 1]. Hence $Ext^j_R(R/I,N)$ is finitely generated for all $j \geq 0$. On the other hand,

$$Supp_R(N) = Supp_R(H^1_I(R)) \subseteq Supp_R(R) \cap V(I) = V(I) \subseteq Max(R).$$

It yields that $dimSupp_R(N) < 2$. Thus, N is an in dimension < 2 module. We then have $H^0_{I,J}(N) = \Gamma_{I,J}(N)$ is in dimension < 2 for any ideal J of R (since the class of in dimension < 2 module is a Serre subcategory (cf. [18, Section 4]) and $\Gamma_{I,J}(N)$ is a submodule of N). Therefore, we have shown that N is a non-finitely generated R-module satisfying the hypothesis of Theorem 1.3.

Acknowledgements

The author would like to thank anonymous referees and the editor for many valuable comments and suggestions, which helped us to improve this work.

References

[1] R. Takahashi, Y. Yoshinok, Yoshizawa, Local Cohomology Based on A Nonclosed Support Defined by A Pair of Ideals, J. Pure and Appl, Algebra, Vol. 213, No. 4, 2009, pp. 582-600, https://doi.org/10.1016/j.jpaa.2008.09.008.

[2] A. Grothendieck, M. Raynaud, Local Cohomology of Coherent Sheaves and Global and Local Lefschetz Theorems (SGA2), (in French), in Algebraic geometry seminar of Bois Marie, 1962, Recomposed and annotated edition in Advanced Studies in Pure Mathematics 2, North-Holland Publishing Company - Amsterdam, 1968, New Updated Edition by Yves Laszlo of the Book: Documents Mathematics, Paris, Mathematical Society of France, Paris, Vol. 4, 2005, https://doi.org/10.48550/arXiv.math/0511279.

[3] R. Hartshorne, Affine Duality and Cofiniteness, Invent. Math. Vol. 9, 1970, pp. 145-164, https://doi.org/10.1007/BF01404554.

[4] K. I. Yoshida, Cofiniteness of Local Cohomology Modules for Ideals of Dimension One, Nagoya Math. J, Vol. 147, 1997, pp. 179-191, https://doi.org/10.1017/S0027763000006371.

[5] D. Delfino, T. Marley, Cofinite Modules and Local Cohomology, J. Pure Appl. Alg. Vol. 121, No. 1, 1997, pp. 45-52, https://doi.org/10.1016/S0022-4049(96)00044-8.

[6] K. Bahmanpour, R. Naghipour, Cofiniteness of Local Cohomology Modules for Ideals of Small Dimension, J. Alg, Vol. 321, No. 7, 2009, pp. 1997-2011, https://doi.org/10.1016/j.jalgebra.2008.12.020.

[7] M. Aghapournahr, K. Bahmanpour, Cofiniteness of Weakly Laskerian Local Cohomology Modules, Bull. Math. Soc. Sci. Math. Roumanie Tome Vol. 57, No. 4, 2014, pp. 347-356, http://www.jstor.org/stable/43678855 (accessed on: June 24th, 2024).

[8] K. Bahmanpour, R. Naghipour, M. Sedghi, Modules Cofinite and Weakly Cofinite with Respect to an Ideal, J. Alg and Its Appl., Vol. 17, No. 3, 2018, pp. 1850056, https://doi.org/10.1142/S0219498818500561.

[9] N. V. Hoang, N. T. Ngoan, On the Cofiniteness of Local Cohomology Modules in Dimension < 2, Hokkaido Math. J, Vol. 52, No. 1, 2023, pp. 65-73, https://doi.org/10.14492/hokmj/2020-428.

[10] M. Khazaei, R. Sazeedeh, A Criterion for Cofiniteness of Modules, Rend. Sem. Mat. Univ. Padova, Vol. 151, 2024, pp. 201-211, https://doi.org/10.4171/rsmup/128.

[11] A. Tehranian, A. P. E. Talemi, Cofiniteness of Local Cohomology Based on A Nonclosed Support Defined by A Pair of Ideals, Bull. Iranian Math. Soc. Vol. 36, No. 02, 2010, pp. 145-155, http://bims.iranjournals.ir/article_16_5c327b3a154ef3140eee1b10988a0689.pdf (accessed on: June 21st, 2024).

[12] D. Asadollahi, R. Naghipour, Faltings' Local-global Principle for the Finiteness of Local Cohomology Modules, Comm. Alg. Vol. 43, No. 3, 2015, pp. 953-958, https://doi.org/10.1080/00927872.2013.849261.

[13] K. I. Kawasaki, Cofiniteness of Local Cohomology Modules for Principal Ideals, Bull. London Math. Soc, Vol. 30, No. 3, 1998, pp. 241-246, https://doi.org/10.1112/S0024609397004347.

[14] N. V. Hoang, A Note on the Cofiniteness of Local Cohomology Modules for A Pair of Ideals, TNU Journal of Science and Technology, Vol. 229, No. 6, 2024, pp. 75-81, https://doi.org/10.34238/tnu-jst.9517.

[15] M. Aghpournahr, Cofiniteness of Local Cohomology Modules for A Pair of Ideals for Small Dimensions, J. Alg and Its App, Vol. 17, No. 2, 2018, pp. 1850020, https://doi.org/10.1142/S0219498818500202.

[16] N. T. Ngoan, N. V. Hoang, N. H. Hoang, On the Cofiniteness of In Dimension < 2 Local Cohomology Modules for A Pair of Ideals, East-West J. of Math, Vol. 24, No. 2, 2023, pp. 118-127, http://eastwestmath.org/index.php/ewm/article/view/490/392 (accessed on: June 21st, 2024).

[17] M. P. Brodmann, R. Y. Sharp, Local Cohomology: An Algebraic Introduction with Geometry Applications, Second Edition, Cambridge University Press, 2013.

[18] L. Melkersson, Modules Cofinite with Respect to an Ideal, J. Alg, Vol. 285, No. 2, 2005, pp. 649-668, https://doi.org/10.1016/j.jalgebra.2004.08.037.