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Original Article

Topological Solitons in High-order Nonlinear Material with Moiré Photonic Lattices

Nguyen Thanh Luan[1], Ngo Trong Dat[1], Nguyen Luong Thien[2], Duong Chinh Cuong[3], Nguyen Duc Anh Quan[4], Do Duc Tho[4], Nguyen Viet Hung[5,*]

[1]Hanoi University of Science and Technology, 1 Dai Co Viet, Hanoi, Vietnam
[2]Space Technology Institute, 18 Hoang Quoc Viet, Hanoi, Vietnam
[3]Phenikaa University, To Huu, Hanoi, Vietnam
[4]Faculty of Engineering Physics, Hanoi University of Science and Technology, 1 Dai Co Viet, Hanoi, Vietnam
[5]Faculty of Electronic Materials and Devices, Hanoi University of Science and Technology, 1 Dai Co Viet, Hanoi, Vietnam

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Abstract: Moiré photonic lattices provide tunable geometric configurations that enable the formation and control of topological solitons. These solitons depend on the interplay between the underlying lattice geometry and high-order nonlinearities such as third and fifth-order effects. In this work, we employ Moiré lattices generated in a high-order nonlinear material to investigate the existence of topological solitons under diverse geometries, which are controlled by the twisting angle of sublattices. The formation of solitons in both commensurate and incommensurate Moiré lattice configurations allows us to explore deeper into the impact of geometric transitions on soliton stability and localization. The findings have potential applications in advanced photonic systems, including topological photonics and all-optical switching, where soliton stability and control are significant factors that can be optimized to enhance performance and functionality.

Keywords: Topological solitons, Moiré photonic lattices, high-order nonlinear materials, photonic waveguides.

* Corresponding author.
E-mail address: hung.nguyenviet1@hust.edu.vn

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1. Introduction

Moiré photonic lattices have attracted significantly extensive interests in recent years, particularly in the context of the search for topological solitons in high-order nonlinear materials. This is due to the unique structural dynamics and enhanced stability of such lattices. Moiré patterns are created by superimposing two periodic structures with a slight rotational misalignment [1]. The concept of Moiré patterns has been known and utilized in various fields such as art, textile industry, architecture, and twistronics for many years [2-4]. In 2018, researchers discovered that stacking two layers of graphene with a "magical" offset angle of 1.1 degrees could induce superconductivity [5]. Additionally, these patterns have been used to manipulate cold atoms based on graphene-based systems [6, 7]. The potential of Moiré lattices to investigate a wide range of unique physical phenomena has recently prompted a surge of interest in the fields of optics, photonics, and condensed matter physics. They are among the tools utilized for the control and manipulation of light propagation, including delocalization-localization of light, magic-angle lasers, and topological defects [8-10] and can be enabled to enhance spatial dispersion and manipulate light on metasurface optics [11, 12]. The mutual rotation of two identical sublattices allows the generation of commensurable and incommensurable Moiré patterns with tunable amplitudes and twist angles. This tunability is crucial for studying the localization and delocalization of light, as well as for investigating the physics of flat-band structure [13].

In the fields of mathematics and physics, a soliton is defined as a nonlinear, self-reinforcing, localized wave packet that maintains its shape while propagating at a constant velocity [14]. This stability is noteworthy, as solitons are capable of re-establishing their form even after colliding with other solitons [15]. Solitons provide stable solutions to a wide class of weakly nonlinear dispersive partial differential equations, which describe various physical systems. A topological soliton, also known as a topological defect, is a solution to a set of partial differential equations that is stable against decay to the trivial solution. Vortex solitons are a specific type of topological soliton characterized by a phase singularity, which means they have a point where the phase of the wave function is undefined. This results in a "vortex" structure, where the wave function circulates around the singularity. Vortex solitons have been studied for their potential applications in various fields such as optical tweezers that trap particles [16], enlarging the capacity of optical communication [17] and high-order quantum entanglement [18]. Their unique properties and stability make them valuable for understanding complex physical phenomena and developing advanced technological applications. In nonlinear optics, fundamental solitons in the media with saturable [19] and vortex solitons with cubic (Kerr) [20] on Moiré lattices have been studied. The nonlinear cubic Schrödinger equation with external lattices corresponds to numerous optical materials such as potassium niobate ($KNbO_3$) [21] or lithium niobate ($LiNbO_3$) [22]. For carbon disulfide (CS_2) material, which elucidates high-order nonlinearities where the competition between cubic and quintic nonlinearities leads to unique soliton dynamics [23].

Both third-order and fifth-order nonlinear optical media can support solitons, but the third-order Kerr term will exhibit catastrophic self-focusing if the beam intensity exceeds a threshold [24]. In the presence of fifth-order nonlinearity, the system becomes more stable because the self-defocusing effects of the fifth-order term counteract the self-focusing. The solitons in these competing cubic-quintic nonlinear systems are more robust and resilient to external perturbations, contributing to their increased stability.

In this work, we investigated the formation and stability of topological solitons in a high-order nonlinear material within Moiré photonic lattices. By using square operator method (SOM), we explored the different configurations that were controlled by the twisting angles of sublattices and how they influence the existence of solitons and behavior of soliton characteristics. This method allowed us to examine the impact of both commensurate and incommensurate Moiré lattice structures on soliton formation. We then systematically analyzed the stability of these solitons by doing direct numerical simulation using split-step Fourier method.

2. Model and Method

In the paraxial approximation [24], the propagation of laser beams in a nonlinear optical medium is governed by the nonlinear Schrodinger equation (NLSE), written for the complex envelope *A(x,y,z)* of the electric field as the following:

$$2ik\frac{\partial A}{\partial z}+\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+2k^2\big(\delta n+n_{\mathrm{NL}}(|A|^2)\big)A=0, \tag{1}$$

where $k=\omega/c$ is wavevector, δn is the variation of linear refractive index and $n_{\mathrm{NL}}(|A|^2)$ is a contribution of nonlinear refractive index. The latter term is a function of the beam's intensity. To account for the higher-order nonlinear material, we will consider carbon disulfide as a specific case in this work as its nonlinear optical properties were reported in [23]. The total refractive index of CS_2 can be expressed as $n=\{1+Re[\chi^{(1)}+3\chi^{(3)}|A|^2+10\,\chi^{(5)}|A|^4]\}^{1/2}$, with $Re[\chi^{(3)}]\approx 2.8\times 10^{-21}(m^2/V^2)$, $Re[\chi^{(5)}]\approx -1.2\times 10^{-39}(m^4/V^4)$ at wavelength 920 nm. By substituting this into (1), we obtain following equation:

$$2ik\frac{\partial A}{\partial z}+\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\frac{\omega^2}{c^2}\left[\delta n\,A+3\chi^{(3)}A|A|^2+10\chi^{(5)}A|A|^4\right]=0. \tag{2}$$

After normalizing, the equation (2) turns into:

$$i\frac{\partial\psi}{\partial Z}+\frac{1}{2}\left(\frac{\partial^2\psi}{\partial X^2}+\frac{\partial^2\psi}{\partial Y^2}\right)+I(r)\psi+|\psi|^2\psi-(\beta-i\gamma)|\psi|^4\psi=0. \tag{3}$$

Here $X=x/w_0, Y=y/w_0$, and $Z=z/l$, with w_0 being the initial beam waist and $l=n_0\omega w_0^2/c$ a characteristic length. The rescaling intensity is $I_r=2c^2/3\omega^2w^2\chi^{(3)}$ and the normalized field equals to $\psi=A/\sqrt{I_r}$. The coefficients β and γ nonlinearities can be derived from the optical properties of CS_2, satisfying the relation $(\beta-i\gamma)=-\frac{20}{9}\frac{c^2}{\omega^2w_0^2}\frac{\chi^{(5)}}{[\chi^{(3)}]^2}$. After doing the calculations, we get the values $\beta=0.028$ and $\gamma=0$. Therefore, this conservative case will ensure the existence of stable solitons. We then apply equation (3) with Moiré photonic lattices and the final model to search for soliton solutions is presented as:

$$i\frac{\partial\psi}{\partial Z}+\frac{1}{2}\nabla_\perp^2\psi+I(r)\psi+|\psi|^2\psi-0.028|\psi|^4\psi=0, \tag{4}$$

where ψ is the dimensionless light field amplitude, Z is the propagation distance, $r=(X,Y)$ are the transverse coordinates, $\nabla=(\partial_X,\partial_Y)$ is the gradient operator and $I(r)=|I_1V(R(\theta)r)+I_2V(r)|^2$ is the function describing the Moiré lattices created by the superposition of two square sublattices with I_1, I_2 are the amplitudes of both square sublattices, $V(r)$ is the potential of the sublattices [6]. In Fig. 1 there are shown some configurations of Moiré lattices according to commensurate and incommensurate cases.

In order to obtain the soliton solutions of Eq. (4), we use SOM. Finding soliton solutions for complex function *U(r)* and propagation constant μ in formula $\psi(r,z)=U(r)e^{i\mu z}$.

$$-\mu U+\frac{1}{2}(U_{xx}+U_{yy})+I(x,y)U+|U|^2U-0.028|U|^4U=0 \tag{5}$$

Starting with the operator $\mathcal{L}_0$:

$$\mathcal{L}_0U=-\mu U+\frac{1}{2}\Delta U+I(X,Y)U+\gamma|U|^2U-\beta|U|^4U \tag{6}$$

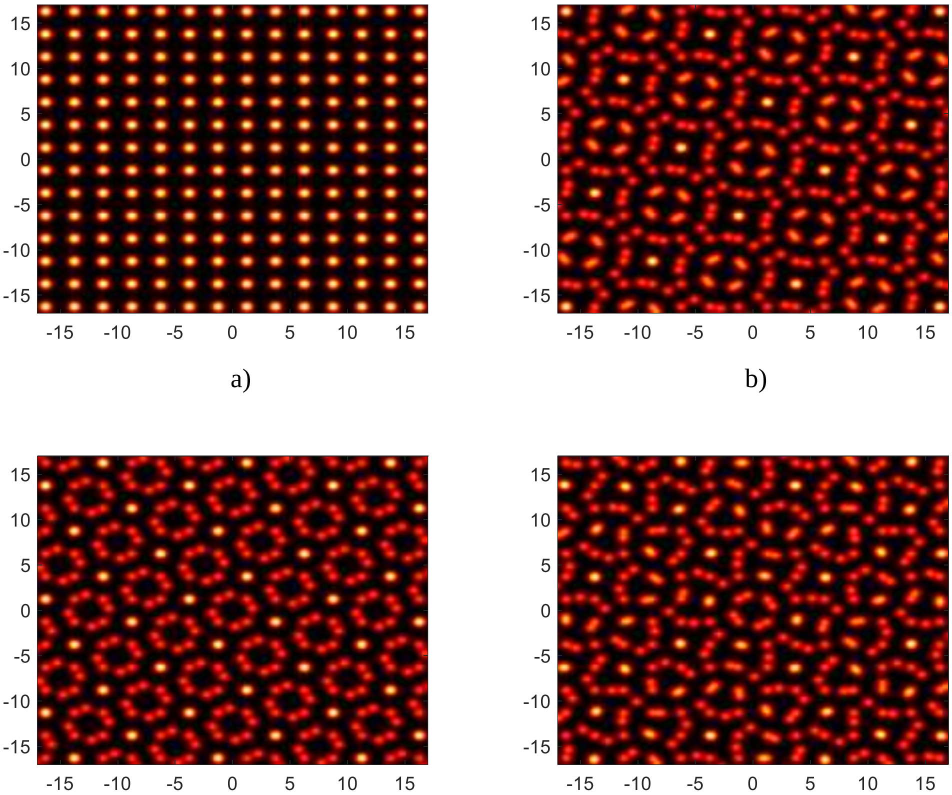
Here, μ represents the eigenvalue or propagation constant. By decomposing $\mathcal{L}_0$ into its real and imaginary components and applying the Fourier transform, we derive the sub-operators T_1 and T_2:

$$T_1 = \mathrm{Re}\left(\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\mathcal{L}_0 U)}{K^2+c}\right)\right), \quad T_2 = \mathrm{Im}\left(\mathcal{F}^{-1}\left(\frac{\mathcal{F}(\mathcal{L}_0 U)}{K^2+c}\right)\right) \tag{7}$$

By decomposing the complex amplitude ψ into its real and imaginary parts, $\psi = U_{re}(x,y) + iU_{im}(x,y)$, and substituting into the operator $\mathcal{L}_0 u$, we get the sub-operators $\mathcal{L}_{\mathcal{R}e}$ and $\mathcal{L}_{\mathcal{I}m}$:

$$\mathcal{L}_{\mathcal{R}e} = -\mu U_{re} + \frac{1}{2}\Delta U_{re} + I(x,y)U_{re} + \beta\left(U_{re}^3 + U_{re}U_{im}^2\right) - \gamma\left(U_{re}^2 + U_{im}^2\right)^2 U_{re} \tag{8}$$

$$\mathcal{L}_{im} = -\mu U_{im} + \frac{1}{2}\Delta U_{im} + I(x,y)U_{im} + \beta\left(U_{im}^3 + U_{im}U_{re}^2\right) - \gamma\left(U_{im}^2 + U_{re}^2\right)^2 U_{im} \tag{9}$$



a) b) c) d)

Figure 1. Two square sublattices stacked on top of each other before twisting (a) and Moiré photonic lattices with twisting angle $\theta = arctan(5/12)$, $\theta = arctan(3/4)$ (commensurate) (b), (c) and $\theta = pi/6$ (incommensurate) (d).

After taking the partial derivatives of $\mathcal{L}_{\mathcal{R}e}$ and $\mathcal{L}_{\mathcal{I}m}$ with respect to both U_{re} and U_{im}, we obtain the elements of the operator $\mathcal{L}_1$:

$$R_{11} = \frac{\partial \mathcal{L}_{\mathcal{R}e}}{\partial U_{re}}(T_1), \quad R_{12} = \frac{\partial \mathcal{L}_{\mathcal{R}e}}{\partial U_{im}}(T_2) \tag{10}$$

$$R_{21} = \frac{\partial \mathcal{L}_{\mathcal{I}m}}{\partial U_{re}}(T_1), \quad R_{22} = \frac{\partial \mathcal{L}_{\mathcal{I}m}}{\partial U_{im}}(T_2) \tag{11}$$

Thus, the operator $\mathcal{L}_1$ is defined as:

$$\mathcal{L}_1 U = R_{11} + R_{12} + i(R_{21} + R_{22}) \quad (12)$$

Then we apply the SOM to find the steady-excited state solutions known as vortex solitons, which were introduced by Yang [24]. This method has been shown to be effective for finding steady-state solutions in a variety of nonlinear wave equations. With the vortex soliton solutions obtained, we can test their stability by performing direct perturbation simulations in real time using the split-step Fourier method.

3. Results and Discussion

In this section, we present the results of the search for topological solitons in high-order nonlinear material with Moiré photonic lattices from commensurate to incommensurate geometries. We look for these topological states in the form $\psi(r,z) = U(r)e^{i\mu z}$ with complex amplitude function *U(r)* and corresponding propagation constant μ. To characterize the properties of the vortex solitons, we based on the dependence of power $P = \iint |U|^2 \, d^2r$ and form factor $\chi = \left(\iint |U|^4 \, d^2r\right)^{\frac{1}{2}} / P$ on the propagation constant μ [25].

3.1. The Different Properties of Fundamental Soliton and Vortex Solitons

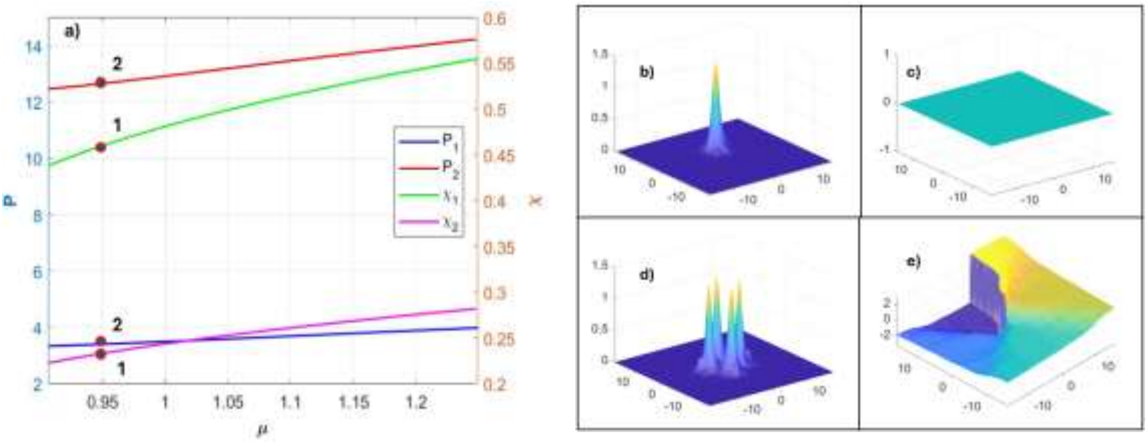


Figure 2. Comparison in power and form factor versus propagation constant μ between fundamental (P_1, χ_1) and vortex solitons (P_2, χ_2) versus propagation constant μ with $\theta = arctan(5/12)$ (a) and profiles of fundamental soliton and phase at point 1 (b), (c) and vortex soliton at point 2 (d), (e).

In Fig. 2, the graph demonstrates the different properties of fundamental and vortex solitons influenced by twist angle $\theta = arctan(5/12)$ with the equal depth of two sublattices $I_1 = I_2 = 2$. The fundamental solitons, which were reported in [25], display lower power when compared to the vortex soliton. As μ increases, the power rises steadily but it remains smaller in magnitude than that of the vortex soliton. The form factor of the fundamental soliton starts relatively high, increasing with μ and the rate of increase is larger than the vortex soliton due to its energy concentration on a single peak, which represents a simpler spatial profile. The vortex soliton requires higher power than the fundamental soliton to maintain the stability of each peak surrounded center. When μ increases, the vortex soliton's power rises more rapidly, which reflects the more complex structure of this type of soliton. To be more specific, Figs. 2b and 2d show the distinct profiles of the fundamental, the vortex soliton and their phases. The fundamental soliton appears as a single peaked structure that is localized both in space and in intensity and this is the simplest soliton solution characterized by a symmetric bell-shaped profile. While the vortex soliton consists of multiple peaks forming a ring-like structure and the peaks are arranged symmetrically around a central core, the overall structure exhibits phase singularities with phase light filed goes from $-\pi$ to π. The phase distribution is key to the vortex soliton, unlike the fundamental soliton, which does not have a constant phase as displayed in Figs. 2c and 2e. Therefore, this is the defining feature of the vortex soliton which results in a twist around the vortex core.

3.2. Investigation of Vortex Soliton in Symmetric Moiré Photonic Lattice ($I_1 = I_2$)

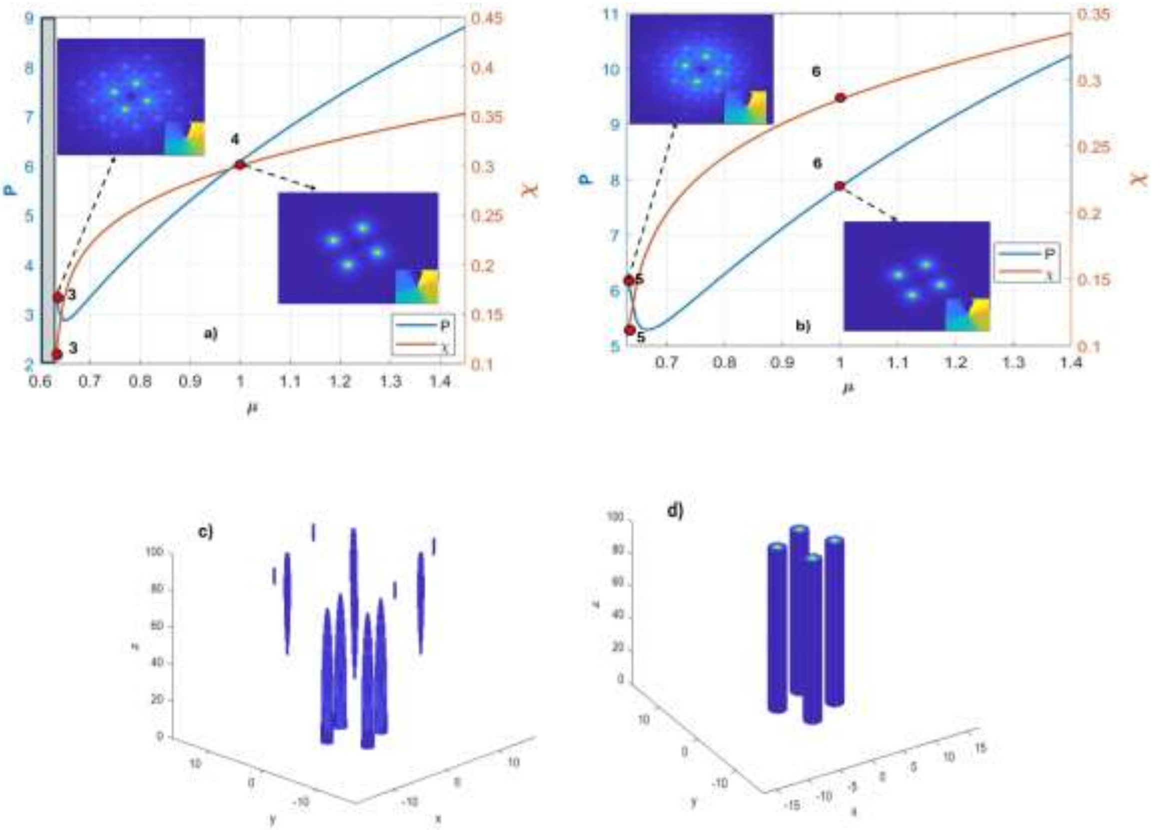


Figure 3. Power and form-factor versus μ in Moiré lattices with commensurate angle $\theta = \arctan(3/4)$ (a), incommensurate angle $\theta = \mathrm{pi}/6$ (b) and beam perturbed propagation (c), (d) at point 3 (unstable vortex soliton) and point 4 (stable vortex soliton).

In the symmetric case, Fig. 3 illustrates the differences in vortex solitons' properties between the periodic and aperiodic angles of the Moiré lattice. In Figure 3a, as increases, the power of the soliton increases significantly; however, initially, the power decreases slightly to stabilize the soliton, and then it increases rapidly, accompanied by an increase in the form factor. Figure 3b displays the required power to stabilize the vortex soliton is higher than that of Fig. 3a, but the form factor is higher than that of the commensurate case. In contrast to the vortex stable soliton at points 4 and 6, points 3 and 5 show the unstable vortex soliton. The delocalization status is caused by the unstable soliton, which has numerous tiny peaks surrounding the main core center peaks. While a lot of power is needed to stabilize the stable vortex solitons and focus them on the center peaks. Therefore, the vortex solitons exhibit more complex behavior with a steeper rise in power and a smoother increase in the form factor. Although the solitons are more strongly localized, the incommensurate angle causes some minor irregularities in the intensity distribution. This leads to a more complex soliton profile, indicating that despite the asymmetry, the incommensurate lattice can cause stronger localization and potentially improved stability.

3.3. Investigation of Vortex Soliton in Asymmetric Moiré Photonic Lattice ($I_1 \neq I_2$)

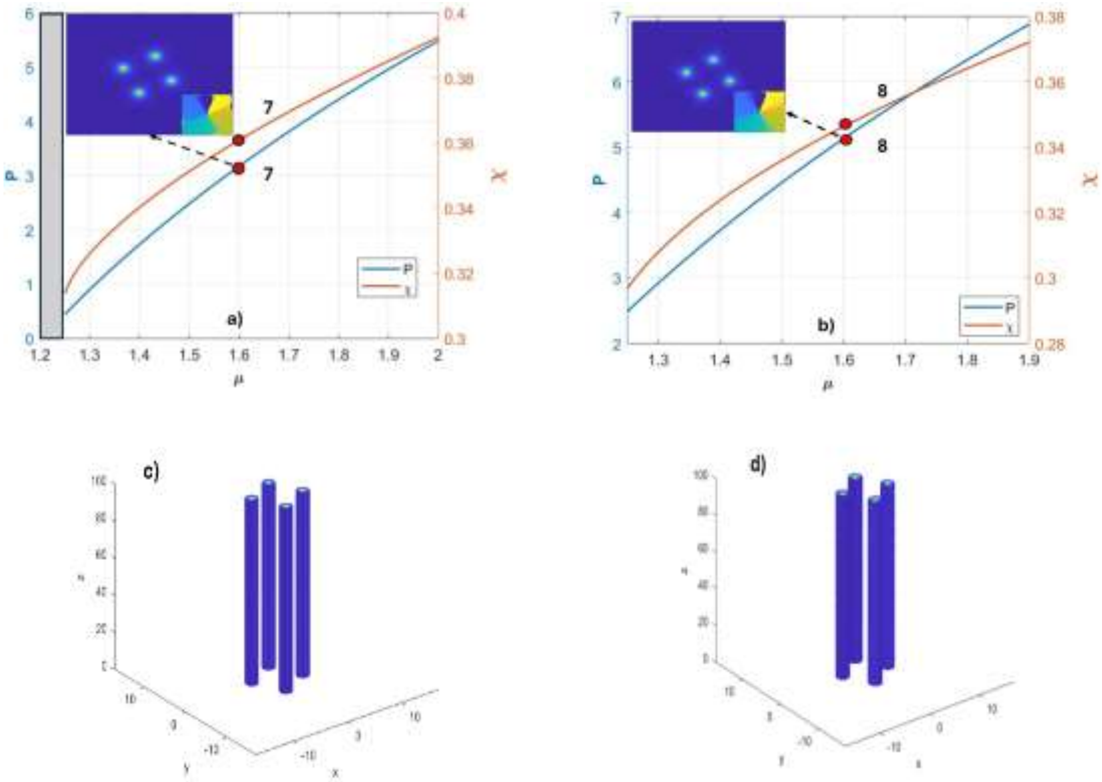


Figure 4. Power and form-factor versus μ in Moiré lattices with $I_1 = 2, I_2 = 4$ with commensurate angle $\theta = \arctan(3/4)$ (a), incommensurate angle $\theta = \mathrm{pi}/6$ (b) and beam perturbed propagation of points 7, 8 respectively (c), (d).

Fig. 4 shows the soliton properties of periodic and aperiodic angles with $I_1 = 2, I_2 = 4$ at the asymmetric case. When increasing the depth of the twisting pattern at $I_2 = 4$, the range of the propagation constant increases from 1.2 to 2 for searching the vortex solitons. In both cases from Fig. 4 signify that the power requirement for stabilizing soliton decreases when compared to symmetric cases from Fig. 3. While the form factor increases strongly higher than that of the symmetric cases, which indicates that the localization is much higher than that of the ones before and lower energy usage.

4. Conclusion

In summary, we demonstrated the possibility of enabling topological solitons in higher-order nonlinear material with third and fifth-order competing nonlinearities, supported by Moiré photonic lattices. The Moiré lattice geometry dominates their soliton properties, demonstrating significantly stronger localization and higher power for incommensurate lattices compared to commensurate ones, illustrating the critical role of lattice design in soliton dynamics. In contrast to previous studies that have concentrated on low-order nonlinearities, this work combined third and fifth-order competing nonlinearities. By using advanced numerical methods, including split-step Fourier method and the squared-operator approach to model and analyze these effects, enabling precise characterization of vortex soliton properties such as power, form factor and phase distribution. The fifth-order nonlinearity enhances the stability and manageability of vortex solitons, making them promising for use in advanced optical systems, including all-optical switches, topological waveguides and robust light-based s ignal processors.

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