VNU Journal of Science: Mathematics – Physics, Vol. 41, No. 2 (2025) 26-35

VNU Journal of Science: Mathematics - Physics

Journal homepage: https://js.vnu.edu.vn/MaP

Original Article

The Complete Convergence for Sequence of Arbitrary Random Variables in Hilbert Space and its Application

Ta Van Chien, Bui Khanh Hang*

VNU University of Science, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam

Received 10th March 2025
Revised 20th March 2025; Accepted 10th June 2025

Abstract: In this work, we establish the complete convergence for sequences of arbitrary random variables taking values in Hilbert space $\mathbb{H}$ with general normalizing sequences. As corollaries, we present some convergence results for $\mathbb{H}$ -valued martingale difference sequences. Finally, the complete convergence of degenerate von Mises statistics is investigated.

Keywords: Complete convergence, Martingale difference, Hilbert space.

1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins [1] as follows: A sequence of random variables $\{X_n, n \geq 1\}$ is said to complete convergence to a constant C if $\sum_{n=1}^{\infty} P\,(|X_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. In the same paper, they proved that a sequence of arithmetic means of i.i.d random variables converges completely to the expected value of the variables provided their variance is finite. The results of Hsu and Robbins are a fundamental concept in probability theory and extended by several authors.

In 2003, Jajte [2] gave a strong law of large numbers for general weighted sums of i.i.d random variables using the class of function ϕ which satisfies the following conditions:

i) For some $d \geq 1, \phi$ is strictly increasing on $[d, \infty)$ with range $[0, \infty)$;

ii) There exist $C > 0$ and a positive integer $k_0 \geq d$ such that $\phi(y+1)/\phi(y) \leq C$ for all $y \geq k_0$;

iii) There exist constants a and b such that for all $s > d, \phi^2(s) \int_s^{\infty} \frac{1}{\phi^2(x)} dx \leq as + b$.

Inspired by Jajte [2], Son et al., [3] developed Jajte's technique to obtain the complete convergence for randomly weighted sums of negatively associated random variables with general normalizing

* Corresponding author.
E-mail address: khanhhang.bui@gmail.com

https//doi.org/10.25073/2588-1124/vnumap.4969

sequences. Recently, Yang (2007) [4] has proved two strong limit theorems for arbitrary stochastic sequences, which generalized the results of Jardas et al., [5] for sequences of independent random variables and the results of Liu and Yang (2003) [6] for arbitrary stochastic sequences. Since then, the results for the convergence of the sequence of arbitrary random variables have been studied by several authors, for instance, W. Yang and X. Yang (2008) [7], Zhang et al., (2014) [8], Wang et al., (2019) [9] and so forth.

The aim of this work is to study the complete convergence for arbitrary stochastic sequences. Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a stochastic sequence on the probability space$(\Omega, \mathcal{F}, P)$, that is, the sequence of σ -fields $\{\mathcal{F}_n, n \geq 0\}$ in $\mathcal{F}$ which is increasing in n (that is $\mathcal{F}_n \uparrow$), and $\mathcal{F}_n$ is adapted to random variables X_n . Let $\mathbb{H}$ be a real separable Hilbert space with norm $\|\cdot\|$. An $\mathcal{F}$ -measurable function from Ω to $\mathbb{H}$ is called $\mathbb{H}$ -valued random variable. The expected value of a $\mathbb{H}$ -valued random variable X, denoted by EX, is defined by be the Bochner integral (when $E\|X\| < \infty$).

In this work, basing on the class of functions ϕ in [3], we study the complete convergence for arbitrary stochastic sequences taking values in Hilbert space $\mathbb{H}$ with general normalizing sequences.

Definition 1. A sequence of $\mathbb{H}$ -valued random variables $\{X_n, n \geq 1\}$is said to complete converegence to a constant Cif

$$\sum_{n=1}^{\infty} P\left(\|X_n - C\| > \varepsilon\right) < \infty, \quad \forall \varepsilon > 0.$$

Let $r \geq 1$. We consider class $\mathcal{K}_r$ and class $\mathcal{H}_r$, which are defined in the following way. The class $\mathcal{K}_r$ consists of all functions $\phi(x)$, which satisfies the following conditions:

i) $\phi(x)$ is strictly increasing on $[0, \infty)$ with range$[0, \infty)$;

ii) There exists positive constant a and a positive number n_0 such that

$$\phi^2(s) \int_s^{\infty} \frac{x^{r-1}}{\phi^2(x)} dx \leq C s^r, \quad s > n_0 \tag{1}$$

It can be seen that the class $\mathcal{K}_r$ includes basic and important normalizing functions such as the function $\phi(x) = x^p$ with $2p > r$ and the regularly varying function with index p with $p > r$. We first obtain the complete convergence for sequences of arbitrary $\mathbb{H}$ -valued random variables with general normalizing sequences $\phi \in \mathcal{K}_r$ (see Theorem 9).

The class $\mathcal{H}_r$ includes all functions $\phi(x)$ which satisfies the following conditions:

iii) There exist positive constants C and a positive integer n_0 such that for all $s > n_0$,

$$\phi(s) \int_1^{s} \frac{x^{r-1}}{\phi(x)} dx \leq C s^r. \tag{2}$$

Class $\mathscr{H}_r$ also contains functions of the form $\phi(x) = x^p$ for any real number p and the regularly varying function with index p with $p \geq r$. We establish the results of the complete convergence for the sequence of martingale difference with $\phi \in \mathcal{K}_r \cap \mathscr{H}_r$ in the Corollary 10.

2. Preliminaries

Let $\mathbb{H}$ be a real separable Hilbert space. Let $(\Omega, \mathscr{F}, P)$ be a probability space and suppose we are given a family of σ -field $\{\mathscr{F}_t, t \geq 0\}$ such that $\mathscr{F}_s \subseteq \mathscr{F}_t \subset \mathscr{F}$ for $0 \leq s \leq t$ and $\bigcap_{\varepsilon>0} \mathscr{F}_{s+\varepsilon} = \mathscr{F}_t$. We assume further that each $\mathscr{F}_t$ is complete relative to the probability measure P.

Definition 2. A stochastic sequence of $\mathbb{H}$ -valued random variable $\{X_n, \mathscr{F}_n, n \geq 1\}$ such that $E\|X_n\| < \infty, \forall n \geq 0$ is called $\mathbb{H}$ -martingale (respectively $\mathbb{H}$ -martingale difference) if $E(X_n|\mathscr{F}_{n-1}) = X_{n-1}$ (respectively $E(X_n|\mathscr{F}_{n-1}) = 0$).

Remark 3. The properties of martingale and difference martingale taking values in the Hilbert space are inherited from the real space. Moreover, the Hilbert space can be seen as a 2-smooth Banach space.

To prove our main results, we need the following lemmas.

Lemma 4. *(See [10], [11])* For any $t > 0$ there exists a positive constant C such that for every $\mathbb{H}$ -valued martingale difference sequence $\{X_n, \mathscr{F}_n, n \geq 1\}$, for all $n \geq 1$

$E\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} X_i\right\|^2\right) \leq C\sum_{i=1}^{n} E\,\|X_i\|^2.$

We recall the definition of regularly and slowly varying functions.

Definition 3. A positive measurable function f defined on $[a,\infty)(a \geq 0)$ is called *regularly varying* at infinity with index ρ, written $f \in \mathcal{RV}_\rho$, if for each $\lambda > 0$,

$$\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)} = \lambda^\rho.$$

In particular, when $\rho = 0$, the function f is called *slowly varying* at infinity, written $f \in \mathcal{RV}$.

Clearly, $x^p, x^p\log_+(x), x^p\log_+(\log_+(x)), x^p\dfrac{\log_+(x)}{\log_+(\log_+(x))}$ are regularly varying functions at infinity with index p where $log_+(x) := max\{log\, x, 1\}$. In particular $log^\beta x, \beta \in \mathbb{R}, \exp\left(log^\beta x\right), 0 < \beta < 1$ are slowly varying at infinity. The following result of Karamata is often applicable.

Lemma 6. (Karamata's theorem, [12]) Let $f \in \mathcal{RV}_\rho$ be locally bounded on $[a,\infty)$. Then

i) For $\sigma \geq -(\rho+1)$

$$\lim_{x\to\infty}\frac{x^{\sigma+1}f(x)}{\int_a^x t^\sigma f(t)dt} = \sigma+\rho+1,$$

ii) For $\sigma < -(\rho+1)$ (and for $\sigma = -(\rho+1)$ if $\int_x^\infty t^{-(\rho+1)} f(t)dt < \infty$)

$$\lim_{x\to\infty}\frac{x^{\sigma+1}f(x)}{\int_x^\infty t^\sigma f(t)dt} = -(\sigma+\rho+1).$$

It essentially says that integrals of regularly varying functions are again regularly varying, or more precisely, one can take the slowly varying function out of the integral. For more details regarding regularly varying functions, the reader may refer to Bingham [13].

Definition 7. *(Bingham [13], Theorem 1.5.13)* Let $\ell(.)$be a slowly varying function. Then, there exists a slowly varying function $\ell^\#(.)$(unique up to asymptotic equivalence) satisfying

$$\lim_{x\to\infty}\ell(x)\ell^\#(x\ell(x)) = 1 \quad \text{and} \quad \lim_{x\to\infty}\ell^\#(x)\ell(x\ell^\#(x)) = 1.$$

The function $\ell^\#$ is called the de Bruijn conjugate of ℓ, and $(\ell,\ell^\#)$ is called a (slowly varying) conjugate pair.

For $a,b > 0$, each of $(\ell(ax),\ell^\#(bx)), (a\ell(x), a^{-1}\ell^\#(x)), ((\ell(x^a))^{1/a}, (\ell^\#(x^a))^{1/a})$ is a conjugate pair by [13] (Proposition 1.5.14). R. Bojanic and Seneta [14] proved that if $\ell(.)$is a slowly varying function satisfying

$$\lim_{x\to\infty}\left(\frac{\ell(\lambda_0 x)}{\ell(x)} - 1\right) log(\ell(x)) = 0, \tag{3}$$

for some $\lambda_0 > 1$, then for every $a \in \mathbb{R}$,

$$\lim_{x\to\infty}\frac{\ell(x\ell^a(x))}{\ell(x)} = 1,$$

Therefore, we can choose (up to asymptotic equivalence) $\ell^{\#}(x) = 1/\ell(x)$. In particular, if $\ell(x) = log(x)$ then $\ell^{\#}(x) = 1/\,log(x)$.

Lemma 8. (Bingham [13], Section 1.7.7) If $f \in \mathcal{RV}_\rho$ with $\rho \neq 0$, then there exists $g \in \mathcal{RV}_{1/\rho}$ such that

$$\lim_{x\to\infty} \frac{f(g(x))}{x} = \lim_{x\to\infty} \frac{g(f(x))}{x} = 1.$$

The function g is determined uniquely up to asymptotic equivalence. In particular, if $f(x) = x^{ab}\ell^a(x^b)$ with $\ell(x)$ is a slowly varying function and $a, b > 0$, then

$$g(x) = x^{\frac{1}{ab}}\ell^{\#\frac{1}{b}}\left(x^{\frac{1}{a}}\right). \tag{4}$$

Throughout this work, $\mathbb{H}$ is a real separable Hilbert space, by saying $\{X_n, \mathcal{F}_n, n \geq 1\}$ is a sequence of $\mathbb{H}$ -valued arbitrary random variables, we mean that is the stochastic sequence taking value on Hilbert space $\mathbb{H}$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive real numbers, we use notion $a_n = O(b_n)$ means that $a_n \leq Cb_n$ for some $0 < C < \infty$. $(\ell, \ell^{\#})$ is a slowly varying conjugate pair. The indicator function of $A$is denoted by $I(A)$. The symbol C denotes a generic positive constant whose value may be different for each appearance.

The organization of the work is as follows. In Section 3, we list our main results on complete convergence for the sequence of arbitrary random variables taking value on a Hilbert space. Moreover, we obtain some conditions for the complete convergence of $\mathbb{H}$ -valued martingale difference. Finally, we present an application of our results to general von Mises statistics in Section 4.

3. The Main Results

We establish the complete convergence for sequence of the stochastic sequence of an arbitrary random variable taking value in $\mathbb{H}$ with general normalizing sequences.

Theorem 9. Let $r \geq 1$ and $\{X, X_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of $\mathbb{H}$ -valued arbitrary random variable and identically distributed. If there exists $\phi \in \mathcal{K}_r$such that

$$E(\phi^{-1}(\|X\|))^r < \infty,$$

then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\max_{1\leq k\leq n} \left\|\sum_{i=1}^{k} (X_i - m_{ni})\right\| \geq \varepsilon\phi(n)\right) < \infty,$$

where $m_{ni} = E(X_i I(\|X_i\| \leq \phi(n))|\mathscr{F}_{i-1})$.

Proof. For all $n \geq 1, 1 \leq i \leq n$, denote

$$Y_{ni} = X_i I(\|X_i\| \leq \phi(n)), \quad U_{nk} = \sum_{i=1}^{k} (Y_{ni} - E(Y_{ni}|\mathcal{F}_{i-1}))$$

For any fixed$\varepsilon > 0$, we see that

$$P\left(\max_{1\leq k\leq n} \left\|\sum_{i=1}^{k} X_i - m_{ni}\right\| \geq \varepsilon\phi(n)\right) \leq \sum_{i=1}^{n} P\left(\|X_i\| > \phi(n)\right)$$
$$+P\left(\max_{1\leq k\leq n} \left\|\sum_{i=1}^{k} X_i\right\| I(\|X_i\| \leq \phi(n)) - E(X_i I(\|X_i\| \leq \phi(n))|\mathcal{F}_{i-1})) \geq \varepsilon\phi(n)\right)$$
$$\leq \sum_{i=1}^{n} P\left(\|X_i\| > \phi(n)\right) + P(\max_{1\leq k\leq n} \|U_{nk}\| > \varepsilon\phi(n))$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}}\sum_{i=1}^{n}P\left(\|X_i\|>\phi(n)\right) &= \sum_{n=1}^{\infty}n^{r-1}\sum_{k=n}^{\infty}P\left(\phi(k)<\|X\|\le\phi(k+1)\right)\\ &= \sum_{k=1}^{\infty}P\left(\phi(k)<\|X\|\le\phi(k+1)\right)\sum_{n=1}^{k}n^{r-1}\\ &\le \sum_{k=1}^{\infty}k^r\,P(k<\phi^{-1}(\|X\|)\le k+1)\\ &\le E(\phi^{-1}(\|X\|))^r<\infty. \end{aligned} \tag{5}$$

Now, we need only to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^{2-r}}P\left(\max_{1\le k\le n}\|U_{nk}\|\ge\varepsilon\phi(n)\right)<\infty.$$

It is easily seen that $\{Y_{ni}-E(Y_{ni}|\mathcal{F}_{i-1}),\ \mathcal{F}_i, i\ge 1\}$, is a sequence of $\mathbb{H}$ -valued martingale difference. By Markov's inequality, Jensen's inequality, Lemma 4, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}}P\left(\max_{1\le k\le n}\|U_{nk}\|\ge\varepsilon\phi(n)\right) &\le \sum_{n=1}^{\infty}\frac{1}{\varepsilon^2 n^{2-r}\phi^2(n)}E\left(\max_{1\le k\le n}\|U_{nk}\|^2\right)\\ &\le \frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}\sum_{i=1}^{n}E\,\|Y_{ni}-E(Y_{ni}|\mathcal{F}_{i-1})\|^2\\ &\le \frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}\sum_{i=1}^{n}E\,\|Y_{ni}\|^2\\ &= \frac{C}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}\sum_{i=1}^{n}E\left(\|X_i\|^2 I(\|X_i\|\le\phi(n))\right).\end{aligned}$$

Under the assumptions $\phi\in\mathcal{K}_r$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^{2-r}\phi^2(n)}\sum_{i=1}^{n}E\left(\|X_i\|^2 I(\|X_i\|\le\phi(n))\right) &\le CE\left(\|X\|^2\sum_{n=1}^{\infty}\frac{n^{r-1}}{\phi^2(n)}I(\|X\|\le\phi(n))\right)\\ &\le CE\left(\phi^2(\phi^{-1}(\|X\|))\sum_{n=[\phi^{-1}(\|X\|)]}^{\infty}\frac{n^{r-1}}{\phi^2(n)}\right)\\ &\le CE(\phi^{-1}(\|X\|))<\infty.\end{aligned}$$

The proof of Theorem 9 is completed.

Example 1. Let l_2 denotes the real separable Hilbert space of all square summable real sequences with the inner product

$$\langle x,y\rangle=\sum_{i=1}^{\infty}x_i\,y_i,$$

for $x=(x_1,x_2,\dots)\in l_2$, $y=(y_1,y_2,\dots)\in l_2$. Let $\{Y_{nk}, n\ge 1, k\ge 1\}$ be an array of real valued identically distributed random variables with the common density function

$$f(x)=\begin{cases}\dfrac{\alpha}{2|x|^{\alpha+1}} & \text{for}\quad |x|>1,\\ 0 & \text{otherwise},\end{cases} \tag{6}$$

where $0<\alpha<2$. Put $X_n^j=a_jY_{nj}$ for $n\ge 1$ and $j\ge 1$, where $a_j\ge 0$ for all $j\ge 1$ and $\sum_{j=1}^{\infty}a_j^{\alpha/2}<\infty$. We shall prove that $\{X_n=(X_n^1,X_n^2,\dots), n\ge 1\}$ is a sequence of l_2-valued random variables, i.e., for each $n\ge 1$,

$$\sum_{j=1}^{\infty}(X_n^j)^2=\sum_{j=1}^{\infty}a_j^2\,Y_{nj}^2<\infty\quad \text{a.s.} \tag{7}$$

Put $\xi_{nj}=a_j^2Y_{nj}^2I_{(|a_jY_{nj}|<1)}$, we have that

$$\sum_{j=1}^{\infty}P\left(a_j^2Y_{nj}^2\ne\xi_{nj}\right)=\sum_{j=1}^{\infty}P\left(|Y_{nj}|>a_j^{-1}\right)\le C\sum_{j=1}^{\infty}a_j^{\alpha/2}<\infty.$$

From the Borel - Cantelli lemma, we have $P(a_j^2Y_{nj}^2\ne\xi_{nj}, i.o.)=0$. Then to prove (7) it is enough to show that

$$\sum_{j=1}^{\infty} \xi_{nj} < \infty \text{ a.s.}$$

We have that

$$\sum_{j=1}^{\infty} E\,|\xi_{nj} - E\xi_{nj}|^2 \leq \sum_{j=1}^{\infty} E\,\xi_{nj}^2 = \sum_{j=1}^{\infty} a_j^4\, E(Y_{nj}^4 I_{(|a_j Y_{nj}|<1)}) \leq C \sum_{j=1}^{\infty} a_j^{\alpha/2} < \infty.$$

It follows from the Khintchine-Kolmogorov convergence theorem that $\sum_{j=1}^{\infty}(\xi_{nj} - E\xi_{nj}) < \infty$ a.s.. Moreover,

$$\sum_{j=1}^{\infty} E\,\xi_{nj} = \sum_{j=1}^{\infty} a_j^2\, E(Y_{nj}^2 I_{(|a_j Y_{nj}|<1)}) \leq C \sum_{j=1}^{\infty} a_j^{\alpha/2} < \infty.$$

Thus, $\sum_{j=1}^{\infty} \xi_{nj} < \infty$ a.s. We consider the standard orthonormal basis of l_2 is $\{e_n, n \geq 1\}$ where e_n denotes the element of l_2 having 1 in its nth position and 0 elsewhere.

When $1 < \alpha < 2$, let $\phi(x) = x^{1/p}$ where $1 \leq rp < \alpha$, we can easily check that $\phi(x) \in \mathcal{K}_r$. Set $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$, then $\{X, X_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of $\mathbb{H}$ -valued arbitrary random variable. We get

$$E(\phi^{-1}(\|X\|))^r = E\left(\sum_{j=1}^{\infty}(a_j Y_{1j})^2\right)^{rp/2} = E|Y_{11}|^{rp}\left(\sum_{j=1}^{\infty} a_j^2\right)^{rp/2} < \infty$$

for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(X_i - m_{ni})\right\| \geq \varepsilon\phi(n)\right) < \infty$$

where $m_{ni} = E(X_i I(\|X_i\| \leq \phi(n))|\mathcal{F}_{i-1})$.

Let $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of $\mathbb{H}$ -valued martingale difference and $\phi \in \mathcal{K}_r \cap \mathcal{H}_r$, we establish the complete convergence for the sequence of martingale difference in the following corollary.

Corollary 10. Let $r \geq 1$ and $\{X, X_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of $\mathbb{H}$ -valued martingale difference and identically distributed. If there exists $\phi \in \mathcal{K}_r \cap \mathcal{H}_r$ such that $E(\phi^{-1}(\|X\|))^r < \infty$ then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} X_i\right\| \geq \varepsilon\phi(n)\right) < \infty.$$

In particular, when $r = 2$ we have

$$\frac{1}{\phi(n)} \max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} X_i\right\| \to 0 \text{ completely as } n \to \infty.$$

Proof. Thanks to Theorem 9, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-r}} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(X_i - m_{ni})\right\| \geq \varepsilon\phi(n)\right) < \infty,$$

where $m_{ni} = E(X_i I(\|X_i\| \leq \phi(n))|\mathscr{F}_{i-1})$. Now, we need to show that

$$\frac{\max_{1\leq k\leq n} \sum_{i=1}^{k} \left\|E(X_i I(\left\|X_j\right\| \leq \phi(n))|\mathscr{F}_{i-1}))\right\|}{\phi(n)} \to 0 \text{ as } n \to \infty. \tag{8}$$

By $\{X, X_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of $\mathbb{H}$ -valued martingale difference and identically distributed, we have

$$\frac{\max_{1\le k\le n}\sum_{i=1}^{k}\|E(X_iI(\|X_i\|\le\phi(n))|\mathcal{F}_{i-1})\|}{\phi(n)}\le\frac{\sum_{i=1}^{n}\|E(X_iI(\|X_i\|>\phi(n))|\mathcal{F}_{i-1})\|}{\phi(n)}$$
$$\le\frac{\sum_{i=1}^{n}\|E(X_iI(\|X_i\|>\phi(n)))\|}{\phi(n)}$$
$$\le\frac{Cn}{\phi(n)}E(\|X\||I(\|X\|>\phi(n))).$$

By $\phi\in\mathcal{H}_r$ and $r\ge 1$ we have

$$\sum_{n=1}^{\infty}\frac{1}{\phi(n)}E(\|X\|I(\|X\|>\phi(n)))=\sum_{n=1}^{\infty}\frac{1}{\phi(n)}\sum_{k=n}^{\infty}E\left(\|X\|I(\phi(k)<\|X\|\le\phi(k+1))\right)$$
$$=\sum_{k=1}^{\infty}E\left(\|X\|I(\phi(k)<\|X\|\le\phi(k+1))\right)\sum_{n=1}^{k}\frac{1}{\phi(n)}$$
$$\le C\sum_{k=1}^{\infty}E\left(\phi(k)I(\phi(k)<\|X\|\le\phi(k+1))\right)\sum_{n=1}^{k}\frac{n^{r-1}}{\phi(n)}$$
$$\le C\sum_{k=1}^{\infty}E\left(k^rI(\phi(k)<\|X\|\le\phi(k+1))\right)$$
$$\le CE(\phi^{-1}(\|X\|))^r<\infty.$$

By Kronecker's lemma, we obtain

$$\frac{Cn}{\phi(n)}E(\|X\|I(\|X\|>\phi(n))\le\frac{C}{\phi(n)}\sum_{k=1}^{n}E\left(\|X\|I(\|X\|>\phi(k))\right)\to 0 \text{ as } n\to\infty.$$

Hence, the proof is completed.

By letting $r=1$ in Corollary 10, we get the Marcinkiewicz-Zygmund type strong law of large numbers for sequences of $\mathbb{H}$ -valued martingale difference in the following corollary. The proof is standard.

Corollary 11. Let $\{X,X_n,\mathcal{F}_n,n\ge 1\}$ be a sequence of $\mathbb{H}$ -valued martingale difference and identically distributed. If there exists $\phi\in\mathcal{K}_1\cap\mathcal{H}_1$ such that $E(\phi^{-1}(\|X\|))<\infty$ then for every$\varepsilon>0$,

$$\frac{1}{\phi(n)}\sum_{i=1}^{n}a_i X_i\to 0 \text{ a.s. as } n\to\infty.$$

Proof. For any $\varepsilon>0$, by applying Corollary 10 with $r=1$, we have

$$\infty>\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}X_i\right\|>\varepsilon\phi(n)\right)=\sum_{l=0}^{\infty}\sum_{n=2^l}^{2^{l+1}-1}\frac{1}{n}P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}X_i\right\|>\varepsilon\phi(n)\right)$$
$$\ge\sum_{l=0}^{\infty}\sum_{n=2^l}^{2^{l+1}-1}\frac{1}{2^{l+1}}P\left(\max_{1\le k\le 2^l}\left\|\sum_{i=1}^{k}X_i\right\|>\varepsilon\phi(2^l)\right)=\frac{1}{2}\sum_{l=0}^{\infty}P\left(\max_{1\le k\le 2^l}\left\|\sum_{i=1}^{k}X_i\right\|>\varepsilon\phi(2^l)\right).$$

Using the Borel-Cantelli lemma, we get

$$\frac{1}{\phi(2^i)}\max_{1\le k\le 2^i}\left\|\sum_{i=1}^{k}X_i\right\|\to 0 \quad \text{a.s. as } \ i\to\infty. \tag{9}$$

For $2^k\le n<2^{k+1}$, we have

$$0 \leq \frac{1}{\phi(n)} \left\| \sum_{i=1}^{n} X_i \right\| \leq \frac{1}{\phi(2^i)} \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} X_i \right\|. \tag{10}$$

The conclusion of the corollary follows from (9) and (10).

4. Application to General von Mises Statistics

Statistics of Cramer-von Mises type are an important tool for testing statistical hypotheses. Next, we will consider general bivariate and degenerate von Mises statistics (V-statistics). Let $h: \mathbb{R}^2 \to \mathbb{R}$ be a symmetric, measurable function. We call

$$V_n = \sum_{i,j=1}^{n} h\,(X_i, X_j) \tag{11}$$

be V-statistic with kernel h. The kernel and related V-statistic are called degenerate if $E(h(x, X_i)) = 0$ for all $x \in \mathbb{R}$. Furthermore, we assume that h is Lipschitz-continuous and positive definite, i.e.

$$\sum_{i,j=1}^{m} c_i\, c_j h(x_i, x_j) \geq 0.$$

for all $c_1, \dots, c_n, x_1, \dots, x_n \in \mathbb{R}$. If additionally $E(h(x, X_i | \mathcal{F}_i)) = 0$ for all $x \in \mathbb{R}$ and $\mathscr{F}_i = \sigma(X_1, X_2, \dots, X_i)$ then by Sun's version of Mercer's theorem (See more detail [15, 16]), we have under these conditions a representation

$$h(x, y) = \sum_{l=1}^{\infty} \lambda_l\, \phi_l(x) \phi_l(y)$$

for orthonormal eigenfunctions $(\phi_l)_{l \in \mathbb{N}}$ with the following properties

- $E(\phi_l(X_n) | \mathcal{F}_n) = 0$ and $E\phi_l^2(X_n) = 1$ for all $l \in \mathbb{N}$,
- $\lambda_l \geq 0$ for all $l \in \mathbb{N}$ and $\sum_{l=1}^{\infty} \lambda_l < \infty$.

Dung and Son [17, 18] gave the almost sure convergence of degenerate von Mises statistics for the sequence of independent and pairwise independent real-valued data with weights being a sequence of real numbers. In the section, we use methods for random variables taking values in Hilbert spaces (as a 2-smooth Hilbert space) to obtain the conditions for the complete convergence of degenerate von Mises statistics with martingale difference real-valued data. By setting $r = 2$ in Corollary 10, we obtain the complete convergence of V-statistic following below.

Theorem 12. Let $\{X_n, \mathcal{F}_n, n \geq 0\}$ be a sequence of real-valued martingale difference and identically distributed. Let h be a Lipschitz-continuous, positive definite kernel function such that

$$E|h(X, X)| < \infty.$$

Then, for any $2p > 1$

$$\frac{1}{n^2} \log_+^{2p}(n) \max_{1 \leq k \leq n} V_k \to 0 \text{ completely as } n \to \infty.$$

Proof. We can treat such V-statistics in the setting of Hilbert spaces. Let $\mathbb{H}$ be a Hibert space of real-valued sequences $y = (y_l)_{l \in \mathbb{N}}$ equipped with the inner product

$$\langle y, z \rangle = \sum_{l=1}^{\infty} \lambda_l\, y_l z_l, \quad \text{and} \quad \|y\|^2 = \sum_{l=1}^{\infty} \lambda_l\, y_l^2.$$

We consider the $\mathbb{H}$ -valued random variables$Y_n = (\phi_l(X_n))_{l\in\mathbb{N}}$. Then $\{Y_n, n \geq 1\}$ is a sequence of $\mathbb{H}$ -valued martingale difference and identically distributed and

$$\begin{aligned}\frac{1}{n^2 \log_+^{2p}(n)} \max_{1\le k\le n} V_k &= \frac{1}{n^2 \log_+^{2p}(n)} \max_{1\le k\le n} \sum_{i,j=1}^{k} h(X_i, X_j) \\ &= \frac{1}{n^2 \log_+^{2p}(n)} \max_{1\le k\le n} \sum_{i,j=1}^{k} \sum_{l=1}^{\infty} \lambda_l \phi_l(X_i)\phi_l(X_j) \\ &= \frac{1}{n^2 \log_+^{2p}(n)} \max_{1\le k\le n} \sum_{l=1}^{\infty} \lambda_l \left(\sum_{i=1}^{k} \phi_l(X_i) \right)^2 \\ &= \left(\frac{1}{n \log_+^{p}(n)} \max_{1\le k\le n} \left\| \sum_{i=1}^{k} Y_i \right\| \right)^2 .\end{aligned}$$

Let $\phi(x) = x\, log_+^p(x)$, by Lemma 6 with the condition $2p > 1$, it is easy check that $\phi \in \mathcal{K}_r \cap \mathcal{H}_r$. From (3) and (4) we have $\phi^{-1}(x) = x/\, log_+^p(x)$. Moreover,

$$E\left(\left(\phi^{-1}(\|X\|)\right)^r\right) = E\left(\frac{\|X\|^2}{\log_+^{rp}(\|X\|)}\right) \le E\left(\|X\|\right)^2 = E\,|h(X,X)| < \infty. \tag{12}$$

Using Corollary 10 with $r = 2$, we obtain

$$\frac{\max_{1\le k\le n} \left\| \sum_{i=1}^{k} Y_i \right\|}{n \log_+^{p}(n)} \to 0 \quad \text{completely as} \quad n \to \infty.$$

This implies that

$$\frac{1}{n^2 \log_+^{2p}(n)} \max_{1\le k\le n} V_k \to 0 \text{ completely as } n \to \infty.$$

Acknowledgments

This research was funded by the VNU University of Science, grant number TN.24.02.

References

[1] P. L. Hsu, H. Robbins, Complete Convergence and the Law of Large Numbers, Proc. Nat. Acad. Sci. USA, Vol. 33, 1947, pp. 25-31.

[2] R. Jajte, On the Strong Law of Large Numbers, The Annals of Probability, Vol. 31, No. 1, 2003, pp. 409-412.

[3] T. C. Son, T. M. Cuong, K. B. Hang, L. V. Dung, On the Baum–Katz Theorem for Randomly Weighted Sums of Negatively Associated Random Variables with General Normalizing Sequences and Applications in Some Random Design Regression Models, Statistical Papers, Springer, Vol. 65, No. 3, 2023, pp. 1869-1900, https://doi.org/10.1007/s00362023-01483-4.

[4] W. Yang, Strong Limit Theorems for Arbitrary Stochastic Sequences, Journal of mathematical analy- sis and applications, Vol. 326, 2007, pp. 1445-1451, https://api.semanticscholar.org/CorpusID: 123447548.

[5] C. Jardas, J. Pecaric, N. Sarapa, A Note on Chung's Strong Law of Large Numbers, Journal of Mathematical Analysis and Applications, Vol. 217, No. 1, 1998, pp. 328-334, https://doi.org/10.1006/jmaa.1998.5740.
[6] W. Liu, W. Yang, A Class of Strong Limit Theorems for the Sequences of Arbitrary Random Variables, Statistics & probability letters, Vol. 64, No. 2, 2003, pp. 121-131, https://doi.org/10.1016/S01677152(03)00104-4.
[7] W. Yang, X. Yang, A Note on Strong Limit Theorems for Arbitrary Stochastic Sequences, Statistics & Probability Letters, Vol. 78, No.14, 2008, pp. 2018-2023, https://doi.org/10.1016/j.spl.2008.01.084.
[8] H. Zhang, R. Hao, W. Yang, Z. Bao, Some Limit Theorems for the Log-optimal Portfolio, Journal of Inequalities and Applications, Vol. 310, 2014, pp. 1-10, https://doi.org/10.1186/1029-242X-2014-310.
[9] Z. Wang, Y. Dong, F. Ding, On Almost Sure Convergence for Sums of Stochastic Sequence, Communications in Statistics-Theory anD Methods, Vol. 48, No. 14, 2019, pp. 3609-3621, https://doi.org/10.1080/03610926.2018.1478102.
[10] D. Qiu, P. Chen, V. Andrei, Complete Moment Convergence for L^P-mixingales, Acta Mathematica Scientia, Vol. 37, No. 5, 2017, pp. 1319-1330, https://doi.org/10.1016/S0252-9602(17)30075-9.
[11] S. Gan, Rosenthal's Inequality and Its Applications for B-valued Random Elements, Acta Math Sci, Vol. 30, No. 2, 2010, pp. 327-334.
[12] F. Cirstea, V. D. Radulescu, Nonlinear Problems with Boundary Blow-up: a Karamata Regular Variation Theory Approach, Asymptotic Analysis, Vol. 46, No. 3, 2006, pp. 275-298, https://doi.org/10.3233/ASY-2006-732.
[13] N. H. Bingham, C. M. Goldie, J. L. Teugels, Regular Variation, Encyclopedia of Mathematics and Its Applications, Series 27, Cambridge University Press, Cambridge, 1987.
[14] R. Bojanic, E. Seneta, Slowly Varying Functions and Asymptotic Relations, J. Math. Anal. Appl, Vol. 34, 1971, pp. 302-315, https://doi.org/10.1016/0022-247X(71)90114-4.
[15] H. Sun, Mercer Theorem for RKHS on Noncompact Sets, Journal of Complexity, Vol. 21, No. 3, 2005, pp. 337-349, https://doi.org/10.1016/j.jco.2004.09.002.
[16] H. Dehling, O. S. Sharipov, M. Wendler, Bootstrap for Dependent Hilbert Space-valued Random Variables with Application to von Mises Statistics, J. Multivariate Anal, Vol. 133C, 2015, pp. 200-215, https://doi.org/10.1016/j.jmva.2014.09.011.
[17] T. C. Son, T. M. Cuong, L. V. Dung, On the Almost Sure Convergence for Sums of Negatively Superadditive Dependent Random Vectors in Hilbert Spaces and Its Application, Commun. Statist. Theor. Methods, Vol. 49, No. 11, 2020, pp. 1-17, https://doi.org/10.1080/03610926.2019.1584304.
[18] T. C. Son, T. M. Cuong, L. V. Dung, T. V. Chien, On the Convergence for Weighted Sums of Hilbert-valued Coordinatewise Pairwise NQD Random Variables and Its Application, Commun. Statist. Theor. Methods, Vol. 52, No. 23, 2022, pp. 8371-8387, https://doi.org/10.1080/03610926.2022.2062604.