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Original Article

Nonlinear Vibration of Three-Phase Composite Cylindrical Panels Utilizing Reddy's Higher-Order Shear Deformation Shell Theory

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Abstract: This paper presents a comprehensive analytical framework to characterize the nonlinear vibration behavior of a three-phase composite. The cylindrical panels are supported by a Pasternak-type elastic foundation and subjected to combined thermal environment and mechanical loads. A sophisticated mathematical model is formulated basing on Reddy's higher-order theory to precisely capture the complex interactions between elastic foundation. The material properties of a three-phase composite are meticulously determined through analytical expressions that nonlinearly account for the interactions between the constituent materials. The volume fractions of the components in the magneto-electro-elastic face sheets are assumed to be equal. Analytical vibration solutions for the laminated plate are obtained by applying Galerkin method in conjunction with fourth-order Runge-Kutta method. Numerical results are provided to clarify the impact of geometric and material parameters, temperature increase, magnetic and electric potentials and elastic foundations on the vibration behavior of a three-phase composite.

Keywords: Vibration; thermal environment; elastic foundations; three-phase composite; Reddy's higher-order shear deformation shell theory.

1. Introduction

In recent years, polymer composite materials have found extensive applications across various sectors, including construction, shipbuilding, and civilian uses such as household goods and industrial plastic production. These materials owe their versatility to reinforcing components like fibers, fabrics, and particles, each playing a distinct role in enhancing the performance of composite. Fibers and fabrics

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are instrumental in boosting the structural load-bearing capacity, while particles contribute to reducing cracking, minimizing plastic deformation, and improving impermeability and fire resistance. By integrating both fibers and particles, three-phase composites are created, offering a superior balance of strength and durability. This synergy enables materials to better meet the evolving demands of modern engineering and design. Therefore, the combination of fibers and particles in composite materials results in a three-phase composite, making the materials more advanced and better suited to meet the demands of modern engineering. This presents a promising research direction with high practical potential. As of late, research on three-phase composite materials and structures has expanded significantly. Sharma et al., [1] demonstrated that incorporating core-shell particles into carbon fiber-reinforced PA6 significantly enhances the material's fatigue life. Liu et al., [2] investigated the nonlinear forced vibration of a novel functionally graded three-phase composite cylindrical shell, anticipated to be a common structure in future carrier rockets, considering the effects of aerodynamic forces, external excitations, and hygrothermal environments. Ghovehoud et al., [3] explored how stiffeners, as well as the geometric and mechanical properties of the core and composite layers, impact the dynamic instability of the plate, in their research on typical three-phase composites, Duc et al., [4] proposed a method to determine the bending deflection of three-phase composite plates with glass fiber reinforcements and TiO_2 particles. The method considers particle-matrix interaction and shear deformation, providing expressions for material properties to support design and optimization.

Vibration analysis of structures is essential for ensuring safety, preventing damage, and enhancing comfort through the control of excessive deflections. It also contributes to improved structural performance and durability by managing dynamic loadings and avoiding resonance. Numerous authors have explored vibration analysis issues for various types of structures, including plates, to achieve these objectives [5-7]. The vibration analysis of structures is influenced by their shape and geometric dimensions, leading to the application of various theoretical frameworks, including classical theory, first-order shear theory, higher-order shear deformation theory, and nonlocal theory [8-10]. Vannin and Duc pioneered a theoretical framework for calculating the elastic modulus of spherical particle-reinforced composites, incorporating critical matrix-filler interactions [11-13]. Their model provides a foundational approach for predicting the mechanical behavior of such heterogeneous materials. Building on this theoretical groundwork, Minh et al., [14-16] conducted experimental studies to fabricate three-phase composites, demonstrating close alignment between empirical results and the proposed theoretical predictions. Furthermore, Minh work expanded to analyze plate bending mechanics, integrating shear deformation and time-dependent creep effects a crucial advancement for applications requiring long-term structural integrity. Thu et al., [17-19] explored the buckling stability of three-phase composite plates under hydrodynamic loading conditions. Employing first-order shear deformation theory, the study accounted for transverse shear strains, offering insights into the dynamic response of composite structures under hydrodynamic loads.

This work employs an analytical approach to investigate the nonlinear vibration behavior of three-phase composite panels subjected to mechanical and thermal loading. The key contributions of this research are as follows:

The material properties of three-phase composites are formulated in a nonlinear manner, explicitly incorporating the volume fractions of fiber and particle reinforcements.

A mathematical model is developed for laminated three-phase composite panels resting on elastic foundations within a thermal environment.

For the first time, Reddy's higher-order shear deformation shell theory is utilized to analyze the vibration characteristics of thick three-phase composite panels, including natural frequencies, phase plane trajectories, and dynamic responses.

2. Modelling and Material Properties of Three-phase Composite Panels

Consider a thick three-phase composite cylindrical panel with dimensions defined by its thickness *h*, length *a*, width *b* and radius of curvature *R*, positioned within the Cartesian coordinate system *Oxyz*, illustrated in Fig. 1. The origin *O* is located at the corner of the panel, with the *Oxy* plane being the mid-plane of the plate (corresponding to $z=0$) and the *z* axis extends through the thickness direction. It is assumed that the layers of the panel are perfectly bonded to each other. The displacement components in the *x*,*y*,*z* directions are denoted by *u*,*v* and *w*,respectively. ϕ_x and ϕ_y represent the rotations of the normal to the middle plane relative to the *y* and *x* axes, respectively. The plate is considered to be placed on a Winkler-Pasternak elastic foundation, where the interaction between the elastic foundations and the panel is modeled by a system of springs with stiffness k_1, combined with a shear layer between the spring system and the plate with modulus k_2.

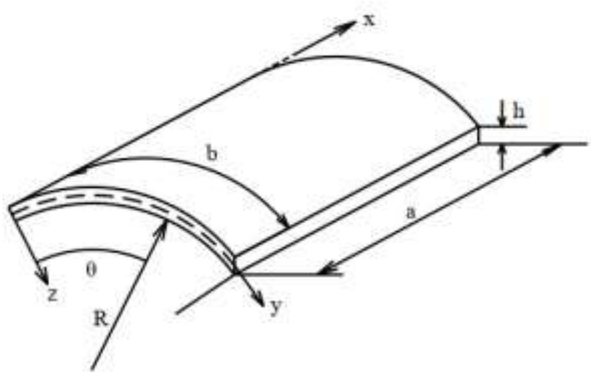

Figure 1. Schematic of three-phase composite cylindrical panels on elastic foundations.

ξ_m, ξ_f and ξ_p are the volume fraction, E_m, E_a, E_c are Young's modulus, ν_m, ν_a, ν_c are Poisson's ratio and $\alpha_m, \alpha_a, \alpha_c$ are thermal expansion coefficients of matrix, fiber and particle components, respectively. The values of material properties including Young's modulus, Poisson's ratio and thermal expansion coefficients for each phase of three-phase composite materials are presented in Table 1.

Table 1. Material properties for each phase of three-phase composite materials [4]

Phase	Young's modulus	Poisson's ratio	Thermal expansion coefficients
Polyester matrix	1.43 GPa	0.345	$14\times10^{-6}K^{-1}$
Glass fiber	22 GPa	0.24	$8\times10^{-6}K^{-1}$
Titanium oxide particle	5.58 GPa	0.20	$10\times10^{-6}K^{-1}$

Vanin and Duc [11, 12] proposed a method to determine the material properties of three-phase composite materials by dividing it into two steps. In Step 1, the matrix and particle components are combined to form a "hypothetical matrix", assumed to be the isotropic material. The Young's modulus and Poisson's ratio of this hypothetical matrix are:

$$\bar{E} = \frac{9\bar{K}\bar{G}}{3\bar{K}+\bar{G}},\ \bar{\nu} = \frac{3\bar{K}-2\bar{G}}{6\bar{K}-2\bar{G}}, \tag{1}$$

with

$$\begin{gathered}\bar{K} = K_m \frac{1+4\xi_c G_m L\left(3K_m\right)^{-1}}{1-4\xi_c G_m L\left(3K_m\right)^{-1}},\ \bar{G} = G_m \frac{1-\xi_c\left(7-5\nu_m\right)H}{1+\xi_c\left(8-10\nu_m\right)H}, \\ H = \frac{G_m/G_c-1}{8-10\nu_m+\left(7-5\nu_m\right)G_m/G_c}, L = \frac{K_c-K_m}{K_c+4G_m/3}.\end{gathered} \tag{2}$$

The hypothetical matrix and fiber component form the three-phase composite materials in the second step. At this point, the three-phase composite materials exhibit transversely isotropic properties with six independent elastic moduli as follows:

$$\begin{gathered}E_{11} = \xi_a E_a + \left(1-\xi_a\right)\bar{E} + \frac{8\bar{G}\xi_a\left(1-\xi_a\right)\left(\nu_a-\bar{\nu}\right)}{2-\xi_a+\bar{x}\xi_a+\left(1-\xi_a\right)\left(x_a-1\right)\dfrac{\bar{G}}{G_a}}, \\ E_{22} = \left\{\frac{\nu_{21}^2}{E_{11}} + \frac{1}{8\bar{G}}\left[\begin{array}{l} 2\dfrac{\left(1-\xi_a\right)(\bar{x}-1)+\left(x_a-1\right)\left(\bar{x}-1+2\xi_a\right)\dfrac{\bar{G}}{G_a}}{2-\xi_a+\bar{x}\xi_a+\left(1-\xi_a\right)\left(x_a-1\right)\dfrac{\bar{G}}{G_a}} \\ +2\dfrac{\bar{x}\left(1-\xi_a\right)+\left(1+\xi_a\bar{x}\right)\dfrac{\bar{G}}{G_a}}{\bar{x}+\xi_a+\left(1-\xi_a\right)\dfrac{\bar{G}}{G_a}} \end{array}\right]\right\}^{-1}, \\ \nu_{21} = \bar{\nu} - \frac{\left(\bar{x}+1\right)\left(\bar{\nu}-\nu_a\right)\xi_a}{2-\xi_a+\bar{x}\xi_a+\left(1-\xi_a\right)\left(x_a-1\right)\dfrac{\bar{G}}{G_a}}, \\ \nu_{23} = -\frac{E_{22}\nu_{21}^2}{E_{11}} + \frac{E_2}{8\bar{G}}\left[\begin{array}{l} 2\dfrac{\left(1-\xi_a\right)\bar{x}+\left(1+\xi_a\bar{x}\right)\dfrac{\bar{G}}{G_a}}{\bar{x}+\xi_a+\left(1-\xi_a\right)\dfrac{\bar{G}}{G_a}} \\ -\dfrac{2\left(1-\xi_a\right)\left(\bar{x}-1\right)+\left(x_a-1\right)\left(\bar{x}-1+2\xi_a\right)\dfrac{\bar{G}}{G_a}}{2-\xi_a+\bar{x}\xi_a+\left(1-\xi_a\right)\left(x_a-1\right)\dfrac{\bar{G}}{G_a}} \end{array}\right], \\ G_{23} = \bar{G}\frac{\bar{x}+\xi_a+\left(1-\xi_a\right)\dfrac{\bar{G}}{G_a}}{\left(1-\xi_a\right)\bar{x}+\left(1+\bar{x}\xi_a\right)\dfrac{\bar{G}}{G_a}}, G_{12} = \bar{G}\frac{1+\xi_a+\left(1-\xi_a\right)\dfrac{\bar{G}}{G_a}}{1-\xi_a+\left(1+\xi_a\right)\dfrac{\bar{G}}{G_a}},\end{gathered} \tag{3}$$

where

$$\bar{x} = 3 - 4\bar{\nu}, x_a = 3 - 4\nu_a. \tag{4}$$

The thermal expansion coefficient of the "hypothetical matrix" are determined as follows [14]

$$\alpha^* = \alpha_m + (\alpha_c - \alpha_m)\frac{K_c(3K_m + 4G_m)\xi_c}{K_m(3K_c + 4G_m) + 4(K_c - K_m)G_m\xi_c}. \tag{5}$$

Afterward, two thermal expansion coefficients of the three-phase composite material are be determined from the thermal expansion coefficients of the "hypothetical matrix" and the fiber component from following expressions:

$$\alpha_{11} = \alpha^* - (\alpha^* - \alpha_a)\xi_a E_{11}^{-1}\left[E_a + \frac{8G_a(\nu_a - \nu)(1-\xi_a)(1+\nu_a)}{2 - \xi_a + \bar{x}\xi_a + (1-\xi_a)(x_a - 1)\dfrac{\bar{G}}{G_a}}\right],$$
$$\alpha_{22} = \alpha^* + (\alpha^* - \alpha_{11})\nu_{21} - (\alpha - \alpha_a)(1+\nu_a)\frac{\nu - \nu_{12}}{\nu - \nu_a}. \tag{6}$$

It is assumed that Pasternak-type elastic foundations support the panels. The interaction between the sandwich plate and the elastic foundations is defined as follows:

$$q_e = k_1\omega - k_2\nabla^2\omega \tag{7}$$

where $\nabla^2 = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}$ is the deflection of the sandwich panel, k_1 and k_2 are Winkler foundation stiffness and shear layer stiffness of Pasternak foundation, respectively.

3. Basic Equations

In this study, Reddy's higher order shear deformation plate theory [20] is used to establish the fundamental equations for investigating the vibration of three-phase composite cylindrical panels. The strain components at points a distance *z* from the mid-plane is determined as:

$$\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix} = \begin{pmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{pmatrix} + z\begin{pmatrix}k_x^1\\ k_y^1\\ k_{xy}^1\end{pmatrix} + z^3\begin{pmatrix}k_x^3\\ k_y^3\\ k_{xy}^3\end{pmatrix}, \begin{pmatrix}\gamma_{xz}\\ \gamma_{yz}\end{pmatrix} = \begin{pmatrix}\gamma_{xz}^0\\ \gamma_{yz}^0\end{pmatrix} + z^2\begin{pmatrix}k_{xz}^2\\ k_{yz}^2\end{pmatrix}, \tag{8}$$

in which

$$\begin{pmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{pmatrix} = \begin{pmatrix}\dfrac{\partial u}{\partial x} - \dfrac{w}{R} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2\\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\end{pmatrix}, \begin{pmatrix}k_x^1\\ k_y^1\\ k_{xy}^1\end{pmatrix} = \begin{pmatrix}\dfrac{\partial\phi_x}{\partial x}\\ \dfrac{\partial\phi_y}{\partial y}\\ \dfrac{\partial\phi_x}{\partial y} + \dfrac{\partial\phi_y}{\partial x}\end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} k_x^3 \\ k_y^3 \\ k_{xy}^3 \end{pmatrix} = -c_1 \begin{pmatrix} \dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 \mathrm{w}}{\partial x^2} \\ \dfrac{\partial \phi_y}{\partial y} + \dfrac{\partial^2 \mathrm{w}}{\partial y^2} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + 2\dfrac{\partial^2 \mathrm{w}}{\partial x \partial y} \end{pmatrix}, \begin{pmatrix} \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{pmatrix} = \begin{pmatrix} \phi_x + \dfrac{\partial \mathrm{w}}{\partial x} \\ \phi_y + \dfrac{\partial \mathrm{w}}{\partial y} \end{pmatrix}, \begin{pmatrix} k_{xz}^2 \\ k_{yz}^2 \end{pmatrix} = -3c_1 \begin{pmatrix} \phi_x + \dfrac{\partial \mathrm{w}}{\partial x} \\ \phi_y + \dfrac{\partial \mathrm{w}}{\partial y} \end{pmatrix},$$

with $c_1 = 4/3h^2$; ε_x^0 and ε_y^0 are the normal strain components; $\gamma_{xy}^0, \gamma_{xz}^0$ and γ_{yz}^0 are the shear strain components; u, v and w are the mid-surface displacement components in the coordinate directions x,y and z; ϕ_x and ϕ_y are the rotations of the mid-surface normal about the axes x and y, respectively.

The constitutive equations for the $kth(k=\overline{1,N})$ layer with fiber angle θ_k of the three-phase composite cylindrical panels can be expressed as follows

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{pmatrix}_k = \begin{pmatrix} \overline{Q_{11}^k} & \overline{Q_{12}^k} & \overline{Q_{16}^k} & 0 & 0 \\ \overline{Q_{12}^k} & \overline{Q_{22}^k} & \overline{Q_{26}^k} & 0 & 0 \\ \overline{Q_{16}^k} & \overline{Q_{26}^k} & \overline{Q_{66}^k} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q_{44}^k} & \overline{Q_{45}^k} \\ 0 & 0 & 0 & \overline{Q_{54}^k} & \overline{Q_{55}^k} \end{pmatrix}_k \begin{pmatrix} \varepsilon_x - \alpha_{11}\Delta T \\ \varepsilon_y - \alpha_{22}\Delta T \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix}_k, \tag{10}$$

in which ΔT is the temperature difference of the environment containing the plate from the initial value, where the cylindrical panels have no thermal deformation, to the final value, also known as the temperature increment. $\overline{Q_{ij}^k}$ are the components of the stiffness matrix which are determined as follows:

$$\begin{aligned}
&\overline{Q_{11}^k} = Q_{11}^k cos\theta_k{}^4 + Q_{22}^k sin^4\theta_k + 2\left(Q_{12}^k + 2Q_{66}^k\right) sin^2\theta_k cos^2\theta_k, \\
&\overline{Q_{12}^k} = Q_{12}^k \left(cos^4\theta_k + sin^4\theta_k\right) + \left(Q_{11}^k + Q_{22}^k - 4Q_{66}^k\right) sin^2\theta_k cos^2\theta_k, \\
&\overline{Q_{16}^k} = \left(Q_{12}^k - Q_{22}^k + 2Q_{66}^k\right) sin^3\theta_k cos\theta_k + \left(Q_{11}^k - Q_{12}^k - 2Q_{66}^k\right) sin\theta_k cos^3\theta_k, \\
&\overline{Q_{22}^k} = Q_{11}^k sin^4\theta_k + Q_{22}^k cos^4\theta_k + 2\left(Q_{12}^k + 2Q_{66}^k\right) sin^2\theta_k cos^2\theta_k, \\
&\overline{Q_{26}^k} = \left(Q_{11}^k - Q_{12}^k - 2Q_{66}^k\right) sin^3\theta_k cos\theta_k + \left(Q_{12}^k - Q_{22}^k + 2Q_{66}^k\right) sin\theta_k cos^3\theta_k, \\
&\overline{Q_{44}^k} = cos^2\theta_k Q_{44}^k + sin^2\theta_k Q_{55}^k, \\
&\overline{Q_{45}^k} = -cos\theta_k sin\theta_k Q_{44}^k + cos\theta_k sin\theta_k Q_{55}^k, \\
&\overline{Q_{55}^k} = sin^2\theta_k Q_{44}^k + cos^2\theta_k Q_{55}^k, \\
&\overline{Q_{66}^k} = Q_{66}^k \left(sin^4\theta_k + cos^4\theta_k\right) + \left[Q_{11}^k + Q_{22}^k - 2\left(Q_{12}^k + Q_{66}^k\right)\right] sin^2\theta_k cos^2\theta_k,
\end{aligned} \tag{11}$$

with

$$\begin{aligned}
&Q_{11}^k = \frac{E_{11}^k}{1 - v_{12}^k v_{21}^k}, Q_{22}^k = \frac{E_{22}^k}{1 - v_{12}^k v_{21}^k}, \\
&Q_{21}^k = Q_{12}^k = \frac{v_{12}^k E_{22}^k}{1 - v_{12}^k v_{21}^k}, Q_{66}^k = G_{12}^k, Q_{44}^k = G_{23}^k, Q_{55}^k = G_{13}^k.
\end{aligned} \tag{12}$$

The results of internal force and moment for N layers are calculated as follows:

$$(N_i, M_i, P_i) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_i \left(1, z, z^3\right) dz, \ i = x, y,$$
$$(Q_i, K_i) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_{iz} \left(1, z^2\right) dz, \ i = x, y, \tag{13}$$

in which

$$z_k = -\frac{h}{2} + \frac{(k-1)}{N} h, \quad z_{k+1} = -\frac{h}{2} + \frac{k}{N} h, \quad k = \overline{1, N}. \tag{14}$$

Substituting Eqs. (8) and (9) into Eq. (10), then substituting the results into Eq. (13), the expressions for internal forces and moments are obtained as

$$\begin{aligned}(N_x, N_y, N_{xy}) &= (A_{11}, A_{12}, A_{16})\varepsilon_x^0 + (A_{12}, A_{22}, A_{26})\varepsilon_y^0 + (A_{16}, A_{26}, A_{66})\gamma_{xy}^0 + (B_{11}, B_{12}, B_{16})k_x^1 \\ &+ (B_{12}, B_{22}, B_{26})k_y^1 + (B_{16}, B_{26}, B_{66})k_{xy}^1 + (E_{11}, E_{12}, E_{16})k_x^3 + (E_{12}, E_{22}, B_{26})k_y^3 + (E_{16}, E_{26}, E_{66})k_{xy}^3 \\ &- \Delta T\left[\alpha_1 (A_{11}, A_{12}, A_{16}) + \alpha_2 (A_{12}, A_{22}, A_{26})\right],\end{aligned} \tag{15a}$$

$$\begin{aligned}(M_x, M_y, M_{xy}) &= (B_{11}, B_{12}, B_{16})\varepsilon_x^0 + (B_{12}, B_{22}, B_{26})\varepsilon_y^0 + (B_{16}, B_{26}, B_{66})\gamma_{xy}^0 + (D_{11}, D_{12}, D_{16})k_x^1 \\ &+ (D_{12}, D_{22}, D_{26})k_y^1 + (D_{16}, D_{26}, D_{66})k_{xy}^1 + (F_{11}, F_{12}, F_{16})k_x^3 + (F_{12}, F_{22}, F_{26})k_y^3 + (F_{16}, F_{26}, F_{66})k_{xy}^3 \\ &- \Delta T\left[\alpha_1 (B_{11}, B_{12}, B_{16}) + \alpha_2 (B_{12}, B_{22}, B_{26})\right],\end{aligned} \tag{15b}$$

$$\begin{aligned}(P_x, P_y, P_{xy}) &= (E_{11}, E_{12}, E_{16})\varepsilon_x^0 + (E_{12}, E_{22}, E_{26})\varepsilon_y^0 + (E_{16}, E_{26}, E_{66})\gamma_{xy}^0 + (F_{11}, F_{12}, F_{16})k_x^1 \\ &+ (F_{12}, F_{22}, F_{26})k_y^1 + (F_{16}, F_{26}, F_{66})k_{xy}^1 + (H_{11}, H_{12}, H_{16})k_x^3 + (H_{12}, H_{22}, H_{26})k_y^3 + (H_{16}, H_{26}, H_{66})k_{xy}^3 \\ &- \Delta T\left[\alpha_1 (E_{11}, E_{12}, E_{16}) + \alpha_2 (E_{12}, E_{22}, E_{26})\right],\end{aligned} \tag{15c}$$

$$(Q_y, Q_x) = (A_{44}, A_{54})\gamma_{yz}^0 + (A_{45}, A_{55})\gamma_{zx}^0 + (D_{44}, D_{54})k_{yz}^2 + (D_{45}, D_{55})k_{zx}^2, \tag{15d}$$

$$(K_y, K_x) = (D_{44}, D_{54})\gamma_{yz}^0 + (D_{45}, D_{55})\gamma_{zx}^0 + (F_{44}, F_{54})k_{yz}^2 + (F_{45}, F_{55})k_{zx}^2, \tag{15e}$$

where

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^f \left(1, z, z^2, z^3, z^4, z^6\right) dz, \ i, j = (1, 2, 6),$$
$$\left(A_{kl}, D_{kl}, F_{kl}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{kl}^f \left(1, z^2, z^4\right) dz, \ k, l = (4, 5). \tag{16}$$

The equations of motion of the cylindrical panels subjected to uniformly distributed external pressure q and viscous damping coefficient ε are written according to Hamilton’s principle and Reddy's higher order shear deformation shell theory as [20]:

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=\overline{j}_1\frac{\partial^2 u}{\partial t^2}+\overline{j}_2\frac{\partial^2 \phi_x}{\partial t^2}-\overline{j}_3\frac{\partial^3 w}{\partial t^2\partial x}, \tag{17a}$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{y}}{\partial y}=\overline{j_1^*}\frac{\partial^2 v}{\partial t^2}+\overline{j_2^*}\frac{\partial^2 \phi_y}{\partial t^2}-\overline{j_3^*}\frac{\partial^3 w}{\partial t^2\partial y}, \tag{17b}$$

$$\begin{aligned}&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}-3c_1\left(\frac{\partial K_x}{\partial x}+\frac{\partial K_y}{\partial y}\right)+c_1\left(\frac{\partial^2 P_x}{\partial x^2}+2\frac{\partial^2 P_{xy}}{\partial x\partial y}+\frac{\partial^2 P_y}{\partial y^2}\right)+\frac{N_x}{R}\\&+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2}+q-k_1w+k_2\nabla^2 w\\&=j_1\frac{\partial^2 w}{\partial t^2}+2\varepsilon j_1\frac{\partial w}{\partial t}+\overline{j}_3\frac{\partial^3 u}{\partial t^2\partial x}+\overline{j}_5\frac{\partial^3 \phi_x}{\partial t^2\partial x}+\overline{j_3}^*\frac{\partial^3 v}{\partial t^2\partial y}+\overline{j_5^*}\frac{\partial^3 \phi_y}{\partial t^2\partial y}-c_1^2 j_7\left(\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\partial^4 w}{\partial t^2\partial y^2}\right),\end{aligned} \tag{17c}$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x+3c_1K_x-c_1\left(\frac{\partial P_x}{\partial x}+\frac{\partial P_{xy}}{\partial y}\right)=\overline{j}_2\frac{\partial^2 u}{\partial t^2}+\overline{j}_4\frac{\partial^2 \phi_x}{\partial t^2}-\overline{j}_5\frac{\partial^3 w}{\partial t^2\partial x}, \tag{17d}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{y}}{\partial y}-Q_y+3c_1K_y-c_1\left(\frac{\partial P_{xy}}{\partial x}+\frac{\partial P_{y}}{\partial y}\right)=\overline{j_2^*}\frac{\partial^2 v}{\partial t^2}+\overline{j_4^*}\frac{\partial^2 \phi_y}{\partial t^2}-\overline{j_5^*}\frac{\partial^3 w}{\partial t^2\partial y}, \tag{17e}$$

in which

$$\begin{aligned}&\left(j_1,j_2,j_3,j_4,j_5,j_7\right)=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\rho_f\left(z\right)\left(1,z,z^2,z^3,z^4,z^6\right)dz\\&\overline{j}_1=j_1+\frac{2j_2}{R},\overline{j_1^*}=j_1,\overline{j}_2=j_2+\frac{j_3}{R}-c_1j_4-\frac{c_1j_5}{R},\overline{j_2^*}=j_2-c_1j_4,\\&\overline{j}_3=c_1j_4+\frac{c_1j_5}{R},\overline{j_3^*}=c_1j_4,\overline{j}_4=\overline{j_4^*}=j_3-2c_1j_5+c_1^2j_7,\overline{j}_5=\overline{j_5^*}=c_1j_5-c_1^2j_7.\end{aligned} \tag{18}$$

The middle surface strains can be expressed from Eq. (15a) as:

$$\begin{aligned}\varepsilon_x^0&=A_{11}^*\frac{\partial^2 f}{\partial y^2}+A_{12}^*\frac{\partial^2 f}{\partial x^2}-A_{13}^*\frac{\partial^2 f}{\partial x\partial y}+A_{14}^*\frac{\partial \phi_x}{\partial x}-c_1A_{15}^*\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+A_{16}^*\frac{\partial \phi_y}{\partial y}\\&-c_1A_{17}^*\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+A_{18}^*\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-c_1A_{19}^*\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)+\Delta T\alpha_1,\\ \varepsilon_y^0&=A_{21}^*\frac{\partial^2 f}{\partial y^2}+A_{22}^*\frac{\partial^2 f}{\partial x^2}-A_{23}^*\frac{\partial^2 f}{\partial x\partial y}+A_{24}^*\frac{\partial \phi_x}{\partial x}-c_1A_{25}^*\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+A_{26}^*\frac{\partial \phi_y}{\partial y}\\&-c_1A_{27}^*\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+A_{28}^*\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)-c_1A_{29}^*\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)+\Delta T\alpha_2,\end{aligned} \tag{19}$$

$$\gamma_{xy}^0 = A_{31}^*\frac{\partial^2 f}{\partial y^2} + A_{32}^*\frac{\partial^2 f}{\partial x^2} - A_{33}^*\frac{\partial^2 f}{\partial x\partial y} + A_{34}^*\frac{\partial \phi_x}{\partial x} - c_1 A_{35}^*\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + A_{36}^*\frac{\partial \phi_y}{\partial y}$$
$$-c_1 A_{37}^*\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + A_{38}^*\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - c_1 A_{39}^*\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x\partial y}\right),$$

where the coefficients $A_{ij}^*\left(i=\overline{1,3};\, j=\overline{1,9}\right)$ are expressed in Appendix A.

To reduce the number of unknowns and equations, the stress function $f(t,x,y)$ is introduced to satisfy the condition as:

$$N_x = \frac{\partial^2 f}{\partial y^2}, N_y = \frac{\partial^2 f}{\partial x^2}, N_{xy} = -\frac{\partial^2 f}{\partial x\partial y}. \tag{20}$$

Substituting Eq. (20) into Eqs. (17a) and (17b), collecting the second order derivatives of u and v , then substituting them into Eqs. (17c) – (17e) yeilds

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - 3c_1\left(\frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y}\right) + c_1\left(\frac{\partial^2 P_x}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x\partial y} + \frac{\partial^2 P_y}{\partial y^2}\right) + \frac{N_x}{R} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}$$
$$+\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + q - k_1 w + k_2\nabla^2 w = j_1\frac{\partial^2 w}{\partial t^2} + 2\varepsilon j_1\frac{\partial w}{\partial t} + \bar{\bar{j}}_5\frac{\partial^3\phi_x}{\partial t^2\partial x} + \bar{\bar{j}}_5^*\frac{\partial^3\phi_y}{\partial t^2\partial y} + \bar{\bar{j}}_7\frac{\partial^4 w}{\partial t^2\partial x^2} + \bar{\bar{j}}_7^*\frac{\partial^4 w}{\partial t^2\partial y^2},$$
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + 3c_1 K_x - c_1\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_{xy}}{\partial y}\right) = \bar{\bar{j}}_3\frac{\partial^2\phi_x}{\partial t^2} - \bar{\bar{j}}_5\frac{\partial^3 w}{\partial t^2\partial x}, \tag{21}$$
$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + 3c_1 K_y - c_1\left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_y}{\partial y}\right) = \bar{\bar{j}}_3^*\frac{\partial^2\phi_y}{\partial t^2} - \bar{\bar{j}}_5^*\frac{\partial^3 w}{\partial t^2\partial y},$$

where

$$\bar{\bar{j}}_3 = \bar{j}_4 - \frac{\left(\bar{j}_2\right)^2}{\bar{j}_1}, \bar{\bar{j}}_3^* = \bar{j}_4^* - \frac{\left(\bar{j}_2^*\right)^2}{\bar{j}_1^*},\ \bar{\bar{j}}_5 = \bar{j}_5 - \frac{\bar{j}_2\bar{j}_3}{\bar{j}_1}, \bar{\bar{j}}_5^* = \bar{j}_5^* - \frac{\bar{j}_2^*\bar{j}_3^*}{\bar{j}_1^*},$$
$$\bar{\bar{j}}_7 = \frac{\left(\bar{j}_3\right)^2}{\bar{j}_1} - c_1^2 j_7, \bar{\bar{j}}_7^* = \frac{\left(\bar{j}_3^*\right)^2}{\bar{j}_1^*} - c_1^2 j_7. \tag{22}$$

Substituting Eq. (19) into equations (15b) – (15e), and then substituting the results into Eq. (21) to obtain the following motion equations

$$U_{11}(w) + U_{12}(\phi_x) + U_{13}(\phi_y) + U_{14}(f) + S(w,f) + q \tag{23}$$
$$= j_1\frac{\partial^2 w}{\partial t^2} + 2\varepsilon j_1\frac{\partial w}{\partial t} + \bar{\bar{j}}_5\frac{\partial^3\phi_x}{\partial t^2\partial x} + \bar{\bar{j}}_5^*\frac{\partial^3\phi_y}{\partial t^2\partial y} + \bar{\bar{j}}_7\frac{\partial^4 w}{\partial t^2\partial x^2} + \bar{\bar{j}}_7^*\frac{\partial^4 w}{\partial t^2\partial y^2},$$
$$U_{21}(w) + U_{22}(\phi_x) + U_{23}(\phi_y) + U_{24}(f) = \bar{\bar{j}}_3\frac{\partial^2\phi_x}{\partial t^2} - \bar{\bar{j}}_5\frac{\partial^3 w}{\partial t^2\partial x},$$
$$U_{31}(w) + U_{32}(\phi_x) + U_{33}(\phi_y) + U_{34}(f) = \bar{\bar{j}}_3^*\frac{\partial^2\phi_y}{\partial t^2} - \bar{\bar{j}}_5^*\frac{\partial^3 w}{\partial t^2\partial y},$$

where

$$
\begin{aligned}
&U_{11}(w)=I_{11}^{*}\frac{\partial^2 w}{\partial x^2}+I_{12}^{*}\frac{\partial^2 w}{\partial x\partial y}+I_{13}^{*}\frac{\partial^2 w}{\partial y^2}+I_{14}^{*}\frac{\partial^4 w}{\partial x^4}+I_{15}^{*}\frac{\partial^4 w}{\partial x^3\partial y}+I_{16}^{*}\frac{\partial^4 w}{\partial x^2\partial y^2}\\
&+I_{17}^{*}\frac{\partial^4 w}{\partial x\partial y^3}+I_{18}^{*}\frac{\partial^4 w}{\partial y^4}-k_1 w+k_2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right),\\
&U_{12}(\phi_x)=I_{21}^{*}\frac{\partial \phi_x}{\partial x}+I_{22}^{*}\frac{\partial \phi_x}{\partial y}+I_{23}^{*}\frac{\partial^3 \phi_x}{\partial x^3}+I_{24}^{*}\frac{\partial^3 \phi_x}{\partial x^2\partial y}+I_{25}^{*}\frac{\partial^3 \phi_x}{\partial x\partial y^2}+I_{26}^{*}\frac{\partial^3 \phi_x}{\partial y^3},\\
&U_{13}(\phi_y)=I_{31}^{*}\frac{\partial \phi_y}{\partial x}+I_{32}^{*}\frac{\partial \phi_y}{\partial y}+I_{33}^{*}\frac{\partial^3 \phi_y}{\partial x^3}+I_{34}^{*}\frac{\partial^3 \phi_y}{\partial x^2\partial y}+I_{35}^{*}\frac{\partial^3 \phi_y}{\partial x\partial y^2}+I_{36}^{*}\frac{\partial^3 \phi_y}{\partial y^3},\\
&U_{14}(f)=I_{41}^{*}\frac{\partial^4 f}{\partial x^4}+I_{42}^{*}\frac{\partial^4 f}{\partial x^3\partial y}+I_{43}^{*}\frac{\partial^4 f}{\partial x^2\partial y^2}+I_{44}^{*}\frac{\partial^4 f}{\partial x\partial y^3}+I_{45}^{*}\frac{\partial^4 f}{\partial y^4}+\frac{\partial^2 f}{\partial y^2}\frac{1}{R},\\
&S(w,f)=\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 w}{\partial y^2},\\
&U_{21}(w)=I_{51}^{*}\frac{\partial w}{\partial x}+I_{52}^{*}\frac{\partial w}{\partial y}+I_{53}^{*}\frac{\partial^3 w}{\partial x^3}+I_{54}^{*}\frac{\partial^3 w}{\partial x^2\partial y}+I_{55}^{*}\frac{\partial^3 w}{\partial x\partial y^2}+I_{56}^{*}\frac{\partial^3 w}{\partial y^3},\\
&U_{22}(\phi_x)=I_{61}^{*}\phi_x+I_{62}^{*}\frac{\partial^2 \phi_x}{\partial x^2}+I_{63}^{*}\frac{\partial^2 \phi_x}{\partial x\partial y}+I_{64}^{*}\frac{\partial^2 \phi_x}{\partial y^2},\\
&U_{23}(\phi_y)=I_{71}^{*}\phi_y+I_{72}^{*}\frac{\partial^2 \phi_y}{\partial x^2}+I_{73}^{*}\frac{\partial^2 \phi_y}{\partial x\partial y}+I_{74}^{*}\frac{\partial^2 \phi_y}{\partial y^2},\\
&U_{24}(f)=I_{81}^{*}\frac{\partial^3 f}{\partial x^3}+I_{82}^{*}\frac{\partial^3 f}{\partial x^2\partial y}+I_{83}^{*}\frac{\partial^3 f}{\partial x\partial y^2}+I_{84}^{*}\frac{\partial^3 f}{\partial y^3},\\
&U_{31}(w)=I_{91}^{*}\frac{\partial w}{\partial x}+I_{92}^{*}\frac{\partial w}{\partial y}+I_{93}^{*}\frac{\partial^3 w}{\partial x^3}+I_{94}^{*}\frac{\partial^3 w}{\partial x^2\partial y}+I_{95}^{*}\frac{\partial^3 w}{\partial x\partial y^2}+I_{96}^{*}\frac{\partial^3 w}{\partial y^3},\\
&U_{32}(\phi_x)=I_{101}^{*}\phi_x+I_{102}^{*}\frac{\partial^2 \phi_x}{\partial x^2}+I_{103}^{*}\frac{\partial^2 \phi_x}{\partial x\partial y}+I_{104}^{*}\frac{\partial^2 \phi_x}{\partial y^2},\\
&U_{33}(\phi_y)=I_{111}^{*}\phi_y+I_{112}^{*}\frac{\partial^2 \phi_y}{\partial x^2}+I_{113}^{*}\frac{\partial^2 \phi_y}{\partial x\partial y}+I_{114}^{*}\frac{\partial^2 \phi_y}{\partial y^2},\\
&U_{34}(f)=I_{121}^{*}\frac{\partial^3 f}{\partial x^3}+I_{122}^{*}\frac{\partial^3 f}{\partial x^2\partial y}+I_{123}^{*}\frac{\partial^3 f}{\partial x\partial y^2}+I_{124}^{*}\frac{\partial^3 f}{\partial y^3},
\end{aligned}
\tag{24}
$$

with coefficients $I_{ij}^{*}\,(i=1,\ j=\overline{1,8};\ i=2,3,5,9,\ j=\overline{1,6};\ i=4,\ j=\overline{1,5};\ i=\overline{6,8},\overline{10,12},\ j=\overline{1,4})$ are expressed in Appendix B.

In this study, the imperfection in the initial shape of the three-phase composite plate is considered and characterized by the function $w^{*}(x,y)$, which is small compared to the thickness of the plate. In this case, the motion equations (23) becomes

$$\begin{aligned}
&U_{11}(w)+U_{12}(\phi_x)+U_{13}(\phi_y)+U_{14}(f)+S(w,f)+U_{11}^*(w^*)+S^*(w^*,f)+q \\
&= j_1\frac{\partial^2 w}{\partial t^2}+2\varepsilon j_1\frac{\partial w}{\partial t}+\bar{\bar{j}}_5\frac{\partial^3\phi_x}{\partial t^2\partial x}+\bar{\bar{j}}_5^*\frac{\partial^3\phi_y}{\partial t^2\partial y}+\bar{\bar{j}}_7\frac{\partial^4 w}{\partial t^2\partial x^2}+\bar{\bar{j}}_7^*\frac{\partial^4 w}{\partial t^2\partial y^2}, \\
&U_{21}(w)+U_{22}(\phi_x)+U_{23}(\phi_y)+U_{24}(f)+U_{21}^*(w^*)=\bar{\bar{j}}_3\frac{\partial^2\phi_x}{\partial t^2}-\bar{\bar{j}}_5\frac{\partial^3 w}{\partial t^2\partial x}, \\
&U_{31}(w)+U(\phi_x)+U_{33}(\phi_y)+U_{34}(f)+U_{31}^*(w^*)=\bar{\bar{j}}_3^*\frac{\partial^2\phi_y}{\partial t^2}-\bar{\bar{j}}_5^*\frac{\partial^3 w}{\partial t^2\partial y},
\end{aligned} \tag{25}$$

with

$$\begin{aligned}
&U_{11}^*(w^*)=I_{11}^*\frac{\partial^2 w^*}{\partial x^2}+I_{12}^*\frac{\partial^2 w^*}{\partial x\partial y}+I_{13}^*\frac{\partial^2 w^*}{\partial y^2}+I_{14}^*\frac{\partial^4 w^*}{\partial x^4}+I_{15}^*\frac{\partial^4 w^*}{\partial x^3\partial y}+I_{16}^*\frac{\partial^4 w^*}{\partial x^2\partial y^2} \\
&+I_{17}^*\frac{\partial^4 w^*}{\partial x\partial y^3}+I_{18}^*\frac{\partial^4 w^*}{\partial y^4}, \\
&U_{21}^*(w^*)=I_{51}^*\frac{\partial w^*}{\partial x}+I_{52}^*\frac{\partial w^*}{\partial y}+I_{53}^*\frac{\partial^3 w^*}{\partial x^3}+I_{54}^*\frac{\partial^3 w^*}{\partial x^2\partial y}+I_{55}^*\frac{\partial^3 w^*}{\partial x\partial y^2}+I_{56}^*\frac{\partial^3 w^*}{\partial y^3}, \\
&U_{31}^*(w^*)=I_{91}^*\frac{\partial w^*}{\partial x}+I_{92}^*\frac{\partial w^*}{\partial y}+I_{93}^*\frac{\partial^3 w^*}{\partial x^3}+I_{94}^*\frac{\partial^3 w^*}{\partial x^2\partial y}+I_{95}^*\frac{\partial^3 w^*}{\partial x\partial y^2}+I_{96}^*\frac{\partial^3 w^*}{\partial y^3}, \\
&S^*(w^*,f)=\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 w^*}{\partial x^2}-2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial^2 w^*}{\partial x\partial y}+\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 w^*}{\partial y^2}.
\end{aligned} \tag{26}$$

The geometric compatibility equation represents the relationship between the strain components at the mid-surface of the imperfect plate, determined from Eq. (9) as

$$\frac{\partial^2\varepsilon_x^0}{\partial y^2}+\frac{\partial^2\varepsilon_y^0}{\partial x^2}-\frac{\partial^2\gamma_{xy}^0}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 w^*}{\partial x\partial y}-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial y^2}-\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w^*}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\frac{1}{R}. \tag{27}$$

Substituting Eq. (19) into Eq. (27) gives

$$\begin{aligned}
&k_{01}\frac{\partial^4 f}{\partial x^4}+k_{02}\frac{\partial^4 f}{\partial x^3\partial y}+k_{03}\frac{\partial^4 f}{\partial x^2\partial y^2}+k_{04}\frac{\partial^4 f}{\partial x\partial y^3}+k_{05}\frac{\partial^4 f}{\partial y^4}+k_{06}\frac{\partial^3\phi_x}{\partial x^3}+k_{07}\frac{\partial^3\phi_x}{\partial x^2\partial y}+k_{08}\frac{\partial^3\phi_x}{\partial x\partial y^2} \\
&+k_{09}\frac{\partial^3\phi_x}{\partial y^3}+k_{10}\frac{\partial^3\phi_y}{\partial x^3}+k_{11}\frac{\partial^3\phi_y}{\partial x^2\partial y}+k_{12}\frac{\partial^3\phi_y}{\partial x\partial y^2}+k_{13}\frac{\partial^3\phi_y}{\partial y^3}+k_{14}\frac{\partial^4 w}{\partial x^4}+k_{15}\frac{\partial^4 w}{\partial x^3\partial y}+k_{16}\frac{\partial^4 w}{\partial x^2\partial y^2} \\
&+k_{17}\frac{\partial^4 w}{\partial x\partial y^3}+k_{18}\frac{\partial^4 w}{\partial y^4}-\left(\frac{\partial^2 w}{\partial x\partial y}^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 w^*}{\partial x\partial y}-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w^*}{\partial y^2}-\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w^*}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\frac{1}{R}\right)=0,
\end{aligned} \tag{28}$$

in which

$$\begin{aligned}
&k_{01} = A_{22}^*, k_{02} = -\left(A_{32}^* + A_{23}^*\right), k_{03} = A_{12}^* + A_{21}^* + A_{33}^*, k_{04} = -\left(A_{31}^* + A_{13}^*\right), k_{05} = A_{11}^*,\\
&k_{06} = A_{24}^* - c_1 A_{25}^*, k_{07} = A_{28}^* - A_{34}^* - c_1\left(A_{29}^* - A_{35}^*\right), k_{08} = A_{14}^* - A_{38}^* - c_1\left(A_{15}^* - A_{39}^*\right),\\
&k_{09} = A_{18}^* - c_1 A_{19}^*, k_{10} = A_{28}^* - c_1 A_{29}^*, k_{11} = A_{26}^* - A_{38}^* - c_1\left(A_{27}^* - A_{39}^*\right),\\
&k_{12} = A_{18}^* - A_{36}^* - c_1\left(A_{19}^* - A_{37}^*\right), k_{13} = A_{16}^* - c_1 A_{17}^*, k_{14} = -c_1 A_{25}^*, k_{15} = -c_1\left(2A_{29}^* - A_{35}^*\right),\\
&k_{16} = -c_1\left(A_{15}^* + A_{27}^* - 2A_{39}^*\right), k_{17} = -c_1\left(2A_{19}^* - A_{37}^*\right), k_{18} = -c_1 A_{17}^*.
\end{aligned} \tag{29}$$

Four edges of the three-phase composite cylindrical panel are assumed to be simply supported and immovable within the plane *Oxy* of the cylindrical panel. These boundary conditions are specifically expressed as follows:

$$\begin{aligned}
&w = u = \phi_y = M_x = N_{xy} = 0,\ N_x = N_{x0}\ \text{at}\ x = 0, a,\\
&w = v = \phi_x = M_y = N_{xy} = 0,\ N_y = N_{y0}\ \text{at}\ y = 0, b,
\end{aligned} \tag{30}$$

The boundary condition (30) serves as the basis for selecting the form of the deflection and rotation angle functions. In this study, these unknown functions are chosen in the form of single-term double trigonometric functions that precisely satisfy the boundary conditions for displacement and approximately satisfy the boundary conditions for internal forces and moments. Specifically, the chosen forms of the solutions are

$$\begin{aligned}
&w(x,y,t) = W(t) sin\lambda_m x cos\delta_n y,\\
&\phi_x(x,y,t) = \Phi_x(t) cos\lambda_m x sin\delta_n y,\\
&\phi_y(x,y,t) = \Phi_y(t) sin\lambda_m x cos\delta_n y,
\end{aligned} \tag{31}$$

where $\lambda_m = \dfrac{m\pi}{a}, \delta_n = \dfrac{n\pi}{b}$, with m, n are the natural numbers of half waves in the corresponding direction $x, y,$ and W, Φ_x, Φ_x are amplitude functions dependent on time.

To be able to replace w with $w + w^*$ and to consider the greatest impact of initial imperfection, w^* is assumed to have the same form as the deflection function as

$$w^*(x,y,t) = \mu h sin\lambda_m x sin\delta_n y, \tag{32}$$

with $\mu(0 \le \mu \le 1)$ is initial imperfection parameter.

Substituting Eqs. (31) and (32) into Eq. (28), we obtain the general form of the stress function $f(x,y,t)$ as follows

$$f = A_1 cos2\lambda_m x + A_2 cos2\delta_n y + A_3 sin\lambda_m x sin\lambda_m x + A_4 cos\lambda_m x cos\lambda_m x + \frac{1}{2}N_{x0}y^2 + \frac{1}{2}N_{y0}x^2, \tag{33}$$

in which

$$\begin{aligned}
&A_1 = \frac{W(t)\delta_n^2\left(2\mu h + W(t)\right)}{32\lambda_n^2 A_{22}^*}, A_2 = \frac{W(t)\lambda_n^2\left(2\mu h + W(t)\right)}{32\delta_n^2 A_{11}^*},\\
&A_3 = \frac{K_{01}}{\Delta_k}\Phi_x(t) + \frac{K_{02}}{\Delta_k}\Phi_y(t) + \frac{K_{03}}{\Delta_k}W(t), A_4 = \frac{K_{04}}{\Delta_k}\Phi_x(t) + \frac{K_{05}}{\Delta_k}\Phi_y(t) + \frac{K_{06}}{\Delta_k}W(t),
\end{aligned} \tag{34}$$

with the coefficients $K_{0i}\left(i=\overline{1,6}\right)$ are expressed in Appendix C.

Then, substituting equations (31) – (33) into the system of motion equations (25) and applying the Galerkin method, we obtain

$$\begin{aligned}
&l_{11}W+l_{12}\Phi_x+l_{13}\Phi_y+l_{14}\left(W+\mu h\right)\Phi_x+l_{15}\left(W+\mu h\right)\Phi_y+\left[r_1-N_{x0}\lambda_m^2-N_{y0}\delta_n^2\right]\left(W+\mu h\right)\\
&+r_2W\left(W+\mu h\right)+r_3W\left(W+2\mu h\right)+r_4W\left(W+\mu h\right)\left(W+2\mu h\right)+r_5\frac{1}{R}N_{x0}+r_5q\\
&=\left(j_1-\lambda^2\overline{\overline{j_7}}-\delta^2\overline{\overline{j_7^*}}\right)\frac{\partial^2W}{\partial t^2}+2\varepsilon j_1\frac{\partial W}{\partial t}-\lambda\overline{\overline{j_5}}\frac{\partial^2\Phi_x}{\partial t^2}-\delta\overline{\overline{j_5^*}}\frac{\partial^2\Phi_y}{\partial t^2},\\
&l_{21}W+l_{22}\Phi_x+l_{23}\Phi_y+r_6(W+\mu h)+r_7W(W+2\mu h)=\overline{\overline{j_3}}\frac{\partial^2\Phi_x}{\partial t^2}-\lambda_m\overline{\overline{j_5}}\frac{\partial^2W}{\partial t^2},\\
&l_{31}W+l_{32}\Phi_x+l_{33}\Phi_y+r_8(W+\mu h)+r_9W(W+2\mu h)=\overline{\overline{j_3^*}}\frac{\partial^2\Phi_y}{\partial t^2}-\delta_n\overline{\overline{j_5^*}}\frac{\partial^2W}{\partial t^2},
\end{aligned}\tag{35}$$

where coefficients $l_{ij}\ (i=1,\ j=\overline{1,5};\ i=2,3,\ j=\overline{1,3}), r_k\ (k=\overline{1,9})$ are expressed in Appendix D.

The condition that the four edges $x=0,a$ and $y=0,b$ of the cylindrical panel immovable in the *xOy* plane cannot be directly satisfied but can only be satisfied in the average sense as follows

$$\int_0^b\int_0^a\frac{\partial u}{\partial x}dxdy=0,\int_0^a\int_0^b\frac{\partial v}{\partial x}dydx=0.\tag{36}$$

The derivatives of the displacement components with respect to the *x* and *y* directions are determined from Eqs. (9) and (19) as

$$\begin{aligned}
&\frac{\partial u}{\partial x}=A_{11}^*\frac{\partial^2 f}{\partial y^2}+A_{12}^*\frac{\partial^2 f}{\partial x^2}-A_{13}^*\frac{\partial^2 f}{\partial x\partial y}+A_{14}^*\frac{\partial\phi_x}{\partial x}-c_1A_{15}^*\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2w}{\partial x^2}\right)+A_{16}^*\frac{\partial\phi_y}{\partial y}-c_1A_{17}^*\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2w}{\partial y^2}\right)\\
&+A_{18}^*\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)-c_1A_{19}^*\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2w}{\partial x\partial y}\right)+\Delta T\alpha_1+\frac{\left(w+w^*\right)}{R}-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x},\\
&\frac{\partial v}{\partial y}=A_{21}^*\frac{\partial^2 f}{\partial y^2}+A_{22}^*\frac{\partial^2 f}{\partial x^2}-A_{23}^*\frac{\partial^2 f}{\partial x\partial y}+A_{24}^*\frac{\partial\phi_x}{\partial x}-c_1A_{25}^*\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2w}{\partial x^2}\right)+A_{26}^*\frac{\partial\phi_y}{\partial y}-c_1A_{27}^*\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2w}{\partial y^2}\right)\\
&+A_{28}^*\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)-c_1A_{29}^*\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2w}{\partial x\partial y}\right)+\Delta T\alpha_2-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2-\frac{\partial w}{\partial y}\frac{\partial w^*}{\partial y}.
\end{aligned}\tag{37}$$

Substituting Eqs. (31) – (33) into Eq. (37) and then substituting the results into Eq. (36), we can collect the fictitious compressive edge loadings as

$$\begin{aligned}
N_{x0}&=g_1W+g_2\Phi_x+g_3\Phi_y+g_4\left(W+2\mu h\right)W+g_5\left(W+\mu h\right)+g_6\Delta T,\\
N_{y0}&=f_1W+f_2\Phi_x+f_3\Phi_y+f_4\left(W+2\mu h\right)W+f_5\left(W+\mu h\right)+f_6\Delta T.
\end{aligned}\tag{38}$$

Suppose the uniformly distributed load has a harmonic form $q=Q\sin\Omega t$, where Q is the amplitude and Ω is the frequency, and substituting Eq. (38) into Eq. (35) we get

$$
\begin{aligned}
&l_{11}^1 W + l_{12}^1 \Phi_x + l_{13}^1 \Phi_y + l_{14}^1 \left(W + \mu h\right)\Phi_x + l_{15}^1 \left(W + \mu h\right)\Phi_y + r_1^1 \left(W + \mu h\right) + r_2^1 W\left(W + \mu h\right) \\
&+ r_3^1 W\left(W + 2\mu h\right) + r_4^1 W\left(W + 2\mu h\right)\left(W + \mu h\right) + r_5^1\left(W + \mu h\right)\left(W + \mu h\right) + r_5 \frac{1}{R} g_6 \Delta T + r_5 Q \sin\Omega t \\
&= \left(j_1 - \lambda_m^2 \overline{\overline{j_7}} - \delta_n^2 \overline{\overline{j_7^*}} \right)\frac{\partial^2 W}{\partial t^2} + 2\varepsilon j_1 \frac{\partial W}{\partial t} - \lambda_m \overline{\overline{j_5}} \frac{\partial^2 \Phi_x}{\partial t^2} - \delta_n \overline{\overline{j_5^*}} \frac{\partial^2 \Phi_y}{\partial t^2}, \\
&l_{21} W + l_{22}\Phi_x + l_{23}\Phi_y + r_6 (W + \mu h) + r_7 W(W + 2\mu h) = \overline{\overline{j_3}} \frac{\partial^2 \Phi_x}{\partial t^2} - \lambda_m \overline{\overline{j_5}} \frac{\partial^2 W}{\partial t^2}, \\
&l_{31} W + l_{32}\Phi_x + l_{33}\Phi_y + r_8 (W + \mu h) + r_9 W(W + 2\mu h) = \overline{\overline{j_3^*}} \frac{\partial^2 \Phi_y}{\partial t^2} - \delta_n \overline{\overline{j_5^*}} \frac{\partial^2 W}{\partial t^2},
\end{aligned} \tag{39}
$$

in which

$$
\begin{aligned}
&l_{11}^1 = \left(l_{11} + n_5 \frac{1}{R} g_1 \right), l_{12}^1 = \left(l_{12} + n_5 \frac{1}{R} g_2 \right), l_{13}^1 = \left(l_{13} + n_5 \frac{1}{R} g_3 \right), l_{14}^1 = l_{14} - \left(g_2 \lambda_m^2 + f_2 \delta_n^2 \right), \\
&l_{15}^1 = l_{15} - \left(g_3 \lambda_m^2 + f_3 \delta_n^2 \right), r_1^1 = r_1 - \left(g_6 \lambda_m^2 + f_6 \delta_n^2 \right)\Delta T + r_5 \frac{1}{R} g_5, r_2^1 = r_2 - \left(g_1 \lambda_m^2 + f_1 \delta_n^2 \right), \\
&r_3^1 = r_3 + r_5 \frac{1}{R} g_4, r_4^1 = r_4 - \left(g_4 \lambda_m^2 + f_4 \delta_n^2 \right), r_5^1 = -\left(g_5 \lambda_m^2 + f_5 \delta_n^2 \right).
\end{aligned} \tag{40}
$$

Eq. (39) is a nonlinear differential equation used to determine the vibration characteristics of the three-phase composite cylindrical panel. The dynamic response of the plate is determined using the Galerkin method with the initial condition $W\big|_{t=0} = \Phi_x\big|_{t=0} = \Phi_y\big|_{t=0} = \left.\frac{dW}{dt}\right|_{t=0} = \left.\frac{d\Phi_x}{dt}\right|_{t=0} = \left.\frac{d\Phi_y}{dt}\right|_{t=0} = 0$. From Eq. (39), the natural frequency of the cylindrical panel is determined as the smallest value among the solutions of the following equation

$$
\begin{vmatrix}
l_{11}^1 + r_1^1 + \left(j_1 - \lambda_m^2 \overline{\overline{j_7}} - \delta_n^2 \overline{\overline{j_7^*}} \right)\omega^2 & l_{12}^1 - \lambda_m \overline{\overline{j_5}}\omega^2 & l_{13}^1 - \delta_n \overline{\overline{j_5^*}}\omega^2 \\
l_{21} + r_6 - \lambda_m \overline{\overline{j_5}}\omega^2 & l_{22} + \overline{\overline{j_3}}\omega^2 & l_{23} \\
l_{31} + r_8 - \delta_n \overline{\overline{j_5^*}}\omega^2 & l_{32} & l_{33} + \overline{\overline{j_3^*}}\omega^2
\end{vmatrix} = 0. \tag{41}
$$

4. Results and Discussion

To verify the reliability of the present results, the dimensionless natural frequencies $\bar{\omega} = 2h\omega\sqrt{2\rho\left(1+v\right)/E}$ of a homogeneous isotropic plate are determined and compared with numerical results of Farsangi et al. [21] based on the Mindlin plate theory and the analytical results of Srinivas et al. [22] using the three-dimensional linear theory for small deformation cases. Four cases of vibration modes are considered. Two cases of the geometric parameter are $a/h = 24$ and $a/h = 40$. From the results in Table 2, it can be seen that the present results show good agreement with the published results of other authors. The maximum error is of 6.32%, which can be explained by the differences in the theories used.

Table 2. Comparison of the dimensionless natural frequencies $\bar{\omega} = 2h\omega\sqrt{2\rho(1+v)/E}$
of homogeneous isotropic plate

Mode	Source / a/h	Farsangi et al. [21]	Srinivas et al. [22]	Present	Maximum error
1	40	0.0589	0.0589	0.0554	6.32%
	24	0.1576	0.1581	0.1501	5.33%
2	40	0.0930	0.0931	0.0879	5.92%
	24	0.2444	0.2455	0.2345	4.69%
3	40	0.1481	0.1485	0.1408	5.47%
	24	0.3788	0.3811	0.3673	3.76%
4	40	0.2218	0.2226	0.2123	4.85%
	24	0.5497	0.5544	0.5391	2.84%

The effects of the elastic foundations coefficients k_1, k_2, temperature increment ΔT and the ratio b/h on the natural frequency *(rad/s)* of three-phase composite plate is indicated in Table 3. The cylindrical panel is considered square $(a = b)$. The fiber and particle volume fractions are $\xi_a = 0.2$ and $\xi_c = 0.3$, respectively. The results show that the natural frequency of the three-phase composite cylindrical panel increases significantly when the values of the elastic foundations coefficients increase. This is explained due to the positive effect of the elastic foundations in increasing the stiffness of the three-phase composite cylindrical panel, thereby increasing the natural frequency. Furthermore, we observe that the influence of the Pasternak foundation coefficient on the natural frequency of the cylindrical panel is significantly larger than that of the Winkler foundation coefficient. Additionally, increasing the value of the temperature increment ΔT, the stiffness of the three-phase composite cylindrical panel decreases. As a result, the natural frequency of the cylindrical panel decreases with increasing temperature increment. The results in Table 3 also show that the ratio b/h positively affects the natural frequency, an increase in b/h ratio leads to an increase in natural frequency.

Table 3. Effect of the elastic foundation's coefficients k_1, k_2, temperature increment ΔT and the ratio b/h on the natural frequency of three-phase composite cylindrical panel

ΔT	b/h	$k_1(GPa/m), k_2(GPa.m)$				
		(0.1, 0.01)	(0.1, 0.02)	(0.1, 0.05)	(0.2, 0.02)	(0.2, 0.05)
100K	10	2525.5	2683.7	3110.4	2760.3	3176.8
	20	2758.7	3043.4	3770.7	3177.9	3880.1
	30	2975.6	3366.4	4332.4	3548.1	4475.0
300K	10	2496.2	2656.2	3086.7	2733.6	3153.6
	20	2731.9	3019.2	3751.1	3154.7	3861.1
	30	2950.7	3344.5	4315.4	3527.3	4458.5
500K	10	2466.6	2628.4	3062.8	2706.6	3130.2
	20	2704.8	2994.7	3731.4	3131.3	3841.9
	30	2925.7	3322.4	4298.3	3506.3	4442.0

Table 4 presents data on the effect of fiber and particle volume fraction, along with the a/b ratio, on the natural frequency of a cylindrical panel in the absence of elastic foundations. Below there are several key academic observations based on the data. As the value of ξ_c increases, the natural frequency of the cylindrical panel generally decreases. This trend can be attributed to the increased particle content, which reduces the overall stiffness of the cylindrical panel, leading to a decline in natural frequency. This aligns with the fact that the fiber component significantly enhances the stiffness of materials and structures, while the particle component has a relatively minor impact on stiffness but improves other mechanical properties, such as impermeability and fire resistance. When the ξ_a ratio increases, the natural frequency decreases, indicating that the fiber's contribution positively impacts the stiffness of the plate. For each value of ξ_a, the natural frequency decreases as the a/b ratio increases. This can be explained by the fact that as the plate becomes longer, the structural stability of the plate diminishes, leading to a reduction in natural frequency. The combined effect of ξ_a and aspect a/b on the natural frequency is significant.

Table 4. Effect of fiber and particle volume fraction and a/b ratio on the natural frequency of cylindrical panel

ξ_a	a/b	ξ_c					
		0.05	0.1	0.15	0.2	0.25	0.3
0.05	1	5441.5	5197.5	4989.4	4810.1	4654.2	4517.8
	2	11166	10644	10196	9808.2	9468.4	9168.6
	3	18705	17815	17049	16384	15800	1528.3
0.1	1	5352.3	5125.3	4930.8	4762.5	4615.8	4487.1
	2	10959	10472	10052	9686.6	9365.3	9081.1
	3	18338	17506	16788	16160	15607	15115
0.15	1	5273.5	5061.9	4879.7	4721.5	4583.2	4461.8
	2	10772	10316	9921.6	9576.5	9272.3	9002.5
	3	18002	17223	16547	15954	15429	14962
0.2	1	5204.8	5007.0	4836.1	4687.2	4556.7	4442.0
	2	10603	10175	9803.8	9477.7	9189.4	8933.1
	3	17695	16964	16326	15765	15266	14822

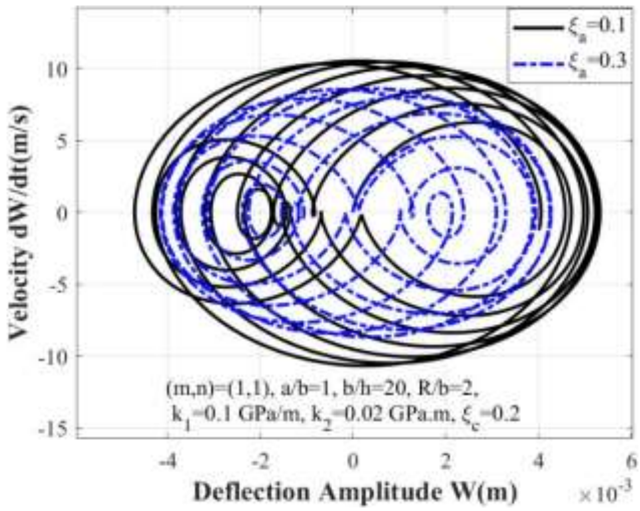

Figure 2. Effect of fiber volume fraction on the phase plane trajectory of three-phase composite cylindrical panel.

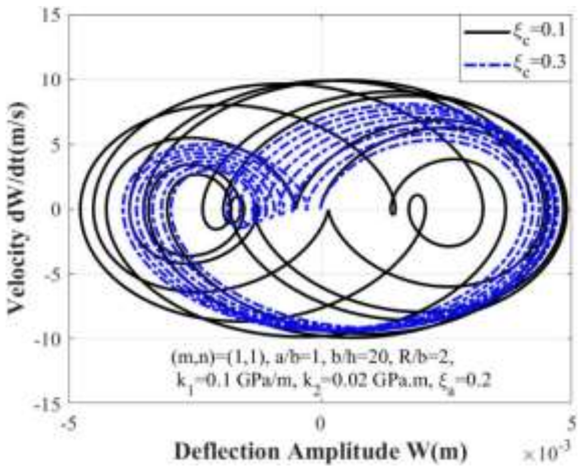

Figure 3. Effect of particle volume fraction on the phase plane trajectory of three-phase composite cylindrical panel.

Figures 2 and 3 illustrate the influence of fiber volume fraction ξ_a and particle volume fraction ξ_c on the phase plane trajectory of the three-phase composite cylindrical panel. When the fiber volume fraction is varied from 0.1 to 0.3, while maintaining the other variable constant, the geometric parameters considered are $a/b=1$ and $b/h=20$. As observed, both the deflection amplitude W and the velocity dW/dt initiate from zero. The phase plane trajectory narrows with an increase in the fiber volume fraction and expands as the particle volume fraction increases. Furthermore, the plate's velocity decreases with an increase in the fiber volume fraction and conversely increases with an increase in the particle volume fraction. Thus, the findings presented in Figs. 2 and 3 reaffirm the positive contribution of the fiber component and the negative contribution of the particle component to the stiffness of the three-phase composite cylindrical panel.

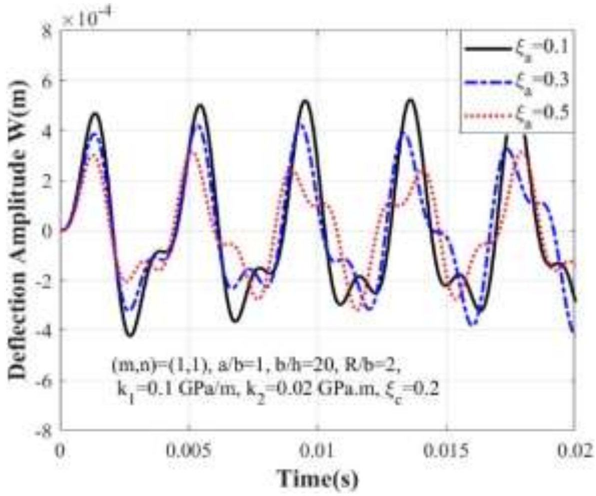

Figure 4. Effect of fiber volume fraction on the dynamic response of three-phase composite cylindrical panel.

Figures 4 and 5 respectively illustrate the influence of fiber and particle volume fractions ξ_a and ξ_c on the dynamic response of a three-phase composite cylindrical panel without elastic foundations. The plate is subjected to an external uniformly distributed load with amplitude of $Q = 1{,}600 N/m^2$ and frequency of Ω = 1,500 rad/s. Interestingly, the obtained results show that the deflection amplitude decreases when the fiber volume fraction increases while keeping the particle volume fraction constant or when the particle volume fraction increases while keeping the fiber volume fraction constant. It seems that both fiber and particle components have a positive impact on the dynamic response of the three-phase composite cylindrical panel, even though the particle component previously had a negative effect on the natural frequency.

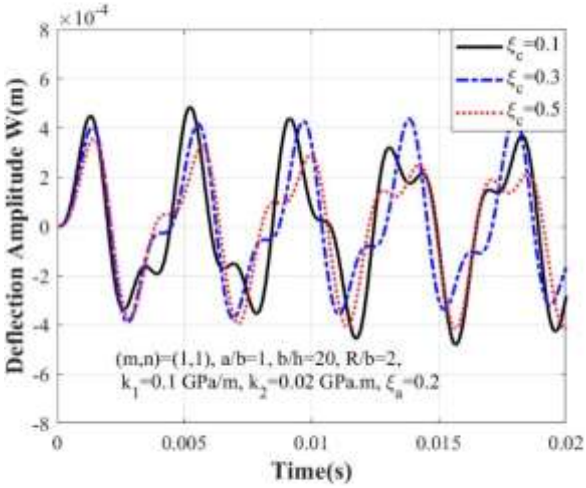

Figure 5. Effect of particle volume fraction on the dynamic response of three-phase composite cylindrical panel.

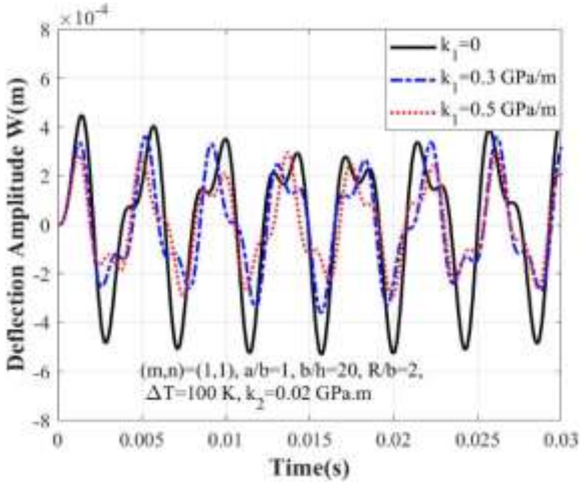

Figure 6. Effect of Winkler foundation stiffness on the dynamic response of three-phase composite cylindrical panel.

Figures 6 and 7 depict the effects of a Winkler foundation with stiffness k_1 and Pasternak foundation with modulus k_2 on the dynamic response of a three-phase composite. With a temperature increment is $\Delta T = 100 K$ and specified geometric parameters are $a/b = 1, b/h = 20$. It is evident that as the stiffness

k_1 and modulus k_2 of the elastic foundation increase, the deflection amplitude of the plate significantly decreases, particularly with k_2 showing a stronger effect. This phenomenon can be attributed to the fact that, when the elastic foundation is modeled with independent spring systems of stiffness k_1 and shear layers with modulus k_2, the reaction force from the foundation opposing the external pressure on the plate's surface diminishes the vibration of the cylindrical panel.

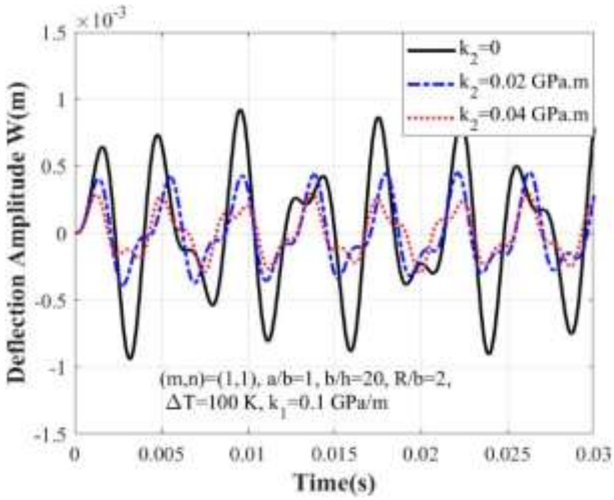

Figure 7. Effect of Pasternak foundation modulus on the dynamic response of three-phase composite cylindrical panel.

5. Conclusions

This study employs Reddy's higher-order shear deformation plate theory to derive the fundamental equations and determine the vibrational characteristics of a three-phase composite cylindrical panel including the natural frequency, amplitude-time relationship, and phase plane trajectory. The key findings are as follows:

- The natural frequency of the cylindrical panel is significantly influenced by the elastic foundations. As the elastic foundation coefficients increase, so does the natural frequency, which highlights the crucial role of foundation stiffness in vibration control.

- The increase in temperature leads to a reduction in the stiffness of the cylindrical panel, causing a corresponding decrease in natural frequency and a significant increase in deflection amplitude. This behavior is consistent across both the natural frequency and dynamic response analysis, where higher temperatures reduce the structural integrity of the cylindrical panel.

- Higher mode numbers correspond to more complex vibration patterns, which increase the stiffness of the cylindrical panel, leading to an increase in the natural frequency. This relationship emphasizes the importance of considering mode shapes in vibration analysis.

- An increase in both fiber and particle volume fractions results in a reduction in the deflection amplitude of the cylindrical panel, with fibers exerting a more pronounced effect than particles. This suggests that fibers contribute more effectively to enhancing the stiffness and mitigating dynamic deflections.

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Appendixes

AppendixA

$$\Delta = A_{11}A_{22}A_{66} - A_{11}A_{26}^2 - A_{66}A_{12}^2 + 2A_{12}A_{16}A_{26} - A_{22}A_{16}^2, \bar{A}_x = \left(-A_{12}A_{26}, A_{66}A_{12}, A_{16}A_{22}, -A_{16}A_{26}, -A_{22}A_{66}, A_{26}^2\right),$$

$$A_{11}^* = \frac{A_{22}A_{66} - A_{26}^2}{\Delta}, A_{12}^* = \frac{-A_{12}A_{66} + A_{16}A_{26}}{\Delta}, A_{13}^* = \frac{A_{12}A_{26} - A_{16}A_{22}}{\Delta}, A_{14}^* = \frac{\bar{A}_x\left(B_{16}, B_{12}, B_{16}, B_{12}, B_{11}, B_{11}\right)}{\Delta},$$

$$A_{15}^* = \frac{\bar{A}_x\left(E_{16}, E_{12}, E_{16}, E_{12}, E_{11}, E_{11}\right)}{\Delta}, A_{16}^* = \frac{\bar{A}_x\left(B_{26}, B_{22}, B_{26}, B_{22}, B_{12}, B_{12}\right)}{\Delta}, A_{17}^* = \frac{\bar{A}_x\left(E_{26}, E_{22}, E_{26}, E_{22}, E_{12}, E_{12}\right)}{\Delta},$$

$$A_{18}^* = \frac{\bar{A}_x\left(B_{66}, B_{26}, B_{66}, B_{26}, B_{16}, B_{16}\right)}{\Delta}, A_{19}^* = \frac{\bar{A}_x\left(E_{66}, E_{26}, E_{66}, E_{26}, E_{16}, E_{16}\right)}{\Delta},$$

$$\bar{A}_y = \left(A_{11}A_{26}, -A_{11}A_{66}, -A_{12}A_{16}, A_{12}A_{66}, A_{16}^2, -A_{16}A_{26}\right), A_{21}^* = \frac{-A_{12}A_{66} + A_{16}A_{26}}{\Delta}, A_{22}^* = \frac{A_{11}A_{66} - A_{16}^2}{\Delta},$$

$$A_{23}^* = \frac{-A_{11}A_{26} + A_{12}A_{16}}{\Delta}, A_{24}^* = \frac{\bar{A}_y\left(B_{16}, B_{12}, B_{16}, B_{11}, B_{12}, B_{11}\right)}{\Delta}, A_{25}^* = \frac{\bar{A}_y\left(E_{16}, E_{12}, E_{16}, E_{11}, E_{12}, E_{11}\right)}{\Delta},$$

$$A_{26}^* = \frac{\bar{A}_y\left(B_{26}, B_{22}, B_{26}, B_{12}, B_{22}, B_{12}\right)}{\Delta}, A_{27}^* = \frac{\bar{A}_y\left(E_{26}, E_{22}, E_{26}, E_{12}, E_{22}, E_{12}\right)}{\Delta}, A_{28}^* = \frac{\bar{A}_y\left(B_{66}, B_{26}, B_{66}, B_{16}, B_{26}, B_{16}\right)}{\Delta},$$

$$A_{29}^* = \frac{\bar{A}_y\left(E_{66}, E_{26}, E_{66}, E_{16}, E_{26}, E_{16}\right)}{\Delta},$$

$$\bar{A}_{xy} = \left(A_{11}A_{22}, -A_{11}A_{26}, -A_{12}^2, A_{12}A_{16}, A_{12}A_{26}, -A_{16}A_{22}\right), A_{31}^* = -\frac{-A_{12}A_{26} + A_{16}A_{22}}{\Delta}, A_{32}^* = -\frac{A_{11}A_{26} - A_{12}A_{16}}{\Delta},$$

$$A_{33}^* = -\frac{-A_{11}A_{22} + A_{12}^2}{\Delta}, A_{34}^* = \frac{-\bar{A}_{xy}\left(B_{16}, B_{12}, B_{16}, B_{12}, B_{11}, B_{11}\right)}{\Delta}, A_{35}^* = \frac{-\bar{A}_{xy}\left(E_{16}, E_{12}, E_{16}, E_{12}, E_{11}, E_{11}\right)}{\Delta},$$

$$A_{36}^* = \frac{-\bar{A}_{xy}\left(B_{26}, B_{22}, B_{26}, B_{22}, B_{12}, B_{12}\right)}{\Delta}, A_{37}^* = \frac{-\bar{A}_{xy}\left(E_{26}, E_{22}, E_{26}, E_{22}, E_{12}, E_{12}\right)}{\Delta},$$

$$A_{38}^* = \frac{-\bar{A}_{xy}\left(B_{66}, B_{26}, B_{66}, B_{26}, B_{16}, B_{16}\right)}{\Delta}, A_{39}^* = \frac{-\bar{A}_{xy}\left(E_{66}, E_{26}, E_{66}, E_{26}, E_{16}, E_{16}\right)}{\Delta}.$$

Appendix B

$I_{11} = B_{11}A^*_{11} + B_{12}A^*_{21} + B_{16}A^*_{31}, I_{12} = B_{11}A^*_{12} + B_{12}A^*_{22} + B_{16}A^*_{32}, I_{13} = B_{11}A^*_{13} + B_{12}A^*_{23} + B_{16}A^*_{33},$

$I_{14} = B_{11}A^*_{14} + B_{12}A^*_{24} + B_{16}A^*_{34} + D_{11}, I_{15} = B_{11}A^*_{15} + B_{12}A^*_{25} + B_{16}A^*_{35} + F_{11},$

$I_{16} = B_{11}A^*_{16} + B_{12}A^*_{26} + B_{16}A^*_{36} + D_{12}, I_{17} = B_{11}A^*_{17} + B_{12}A^*_{27} + B_{16}A^*_{37} + F_{12},$

$I_{18} = B_{11}A^*_{18} + B_{12}A^*_{28} + B_{16}A^*_{38} + D_{16}, I_{19} = B_{11}A^*_{19} + B_{12}A^*_{29} + B_{16}A^*_{39} + F_{16},$

$I_{21} = B_{12}A^*_{11} + B_{22}A^*_{21} + B_{26}A^*_{31}, I_{22} = B_{12}A^*_{12} + B_{22}A^*_{22} + B_{26}A^*_{32}, I_{23} = B_{12}A^*_{13} + B_{22}A^*_{23} + B_{26}A^*_{33},$

$I_{24} = B_{12}A^*_{14} + B_{22}A^*_{24} + B_{26}A^*_{34} + D_{12}, I_{25} = B_{12}A^*_{15} + B_{22}A^*_{25} + B_{26}A^*_{35} + F_{12},$

$I_{26} = B_{12}A^*_{16} + B_{22}A^*_{26} + B_{26}A^*_{36} + D_{22}, I_{27} = B_{12}A^*_{17} + B_{22}A^*_{27} + B_{26}A^*_{37} + F_{22},$

$I_{28} = B_{12}A^*_{18} + B_{22}A^*_{28} + B_{26}A^*_{38} + D_{26}, I_{29} = B_{12}A^*_{19} + B_{22}A^*_{29} + B_{26}A^*_{39} + F_{26},$

$I_{31} = B_{16}A^*_{11} + B_{26}A^*_{21} + B_{66}A^*_{31}, I_{32} = B_{16}A^*_{12} + B_{26}A^*_{22} + B_{66}A^*_{32}, I_{33} = B_{16}A^*_{13} + B_{26}A^*_{23} + B_{66}A^*_{33},$

$I_{34} = B_{16}A^*_{14} + B_{26}A^*_{24} + B_{66}A^*_{34} + D_{16}, I_{35} = B_{16}A^*_{15} + B_{26}A^*_{25} + B_{66}A^*_{35} + F_{16},$

$I_{36} = B_{16}A^*_{16} + B_{26}A^*_{26} + B_{66}A^*_{36} + D_{26}, I_{37} = B_{16}A^*_{17} + B_{26}A^*_{27} + B_{66}A^*_{37} + F_{26},$

$I_{38} = B_{16}A^*_{18} + B_{26}A^*_{28} + B_{66}A^*_{38} + D_{66}, I_{39} = B_{16}A^*_{19} + B_{26}A^*_{29} + B_{66}A^*_{39} + F_{66},$

$I_{41} = E_{11}A^*_{11} + E_{12}A^*_{21} + E_{16}A^*_{31}, I_{42} = E_{11}A^*_{12} + E_{12}A^*_{22} + E_{16}A^*_{32}, I_{43} = E_{11}A^*_{13} + E_{12}A^*_{23} + E_{16}A^*_{33},$

$I_{44} = E_{11}A^*_{14} + E_{12}A^*_{24} + E_{16}A^*_{34} + F_{11}, I_{45} = E_{11}A^*_{15} + E_{12}A^*_{25} + E_{16}A^*_{35} + H_{11},$

$I_{46} = E_{11}A^*_{16} + E_{12}A^*_{26} + E_{16}A^*_{36} + F_{12}, I_{47} = E_{11}A^*_{17} + E_{12}A^*_{27} + E_{16}A^*_{37} + H_{12},$

$I_{48} = E_{11}A^*_{18} + E_{12}A^*_{28} + E_{16}A^*_{38} + F_{16}, I_{49} = E_{11}A^*_{19} + E_{12}A^*_{29} + E_{16}A^*_{39} + H_{16},$

$I_{51} = E_{12}A^*_{11} + E_{22}A^*_{21} + E_{26}A^*_{31}, I_{52} = E_{12}A^*_{12} + E_{22}A^*_{22} + E_{26}A^*_{32}, I_{53} = E_{12}A^*_{13} + E_{22}A^*_{23} + E_{26}A^*_{33},$

$I_{54} = E_{12}A^*_{14} + E_{22}A^*_{24} + E_{26}A^*_{34} + F_{12}, I_{55} = E_{12}A^*_{15} + E_{22}A^*_{25} + E_{26}A^*_{35} + H_{12},$

$I_{56} = E_{12}A^*_{16} + E_{22}A^*_{26} + E_{26}A^*_{36} + F_{22}, I_{57} = E_{12}A^*_{17} + E_{22}A^*_{27} + E_{26}A^*_{37} + H_{22},$

$I_{58} = E_{12}A^*_{18} + E_{22}A^*_{28} + E_{26}A^*_{38} + F_{26}, I_{59} = E_{12}A^*_{19} + E_{22}A^*_{29} + E_{26}A^*_{39} + H_{26},$

$I_{61} = E_{16}A^*_{11} + E_{26}A^*_{21} + E_{66}A^*_{31}, I_{62} = E_{16}A^*_{12} + E_{26}A^*_{22} + E_{66}A^*_{32}, I_{63} = E_{16}A^*_{13} + E_{26}A^*_{23} + E_{66}A^*_{33},$

$I_{64} = E_{16}A^*_{14} + E_{26}A^*_{24} + E_{66}A^*_{34} + F_{16}, I_{65} = E_{16}A^*_{15} + E_{26}A^*_{25} + E_{66}A^*_{35} + H_{16},$

$I_{66} = E_{16}A^*_{16} + E_{26}A^*_{26} + E_{66}A^*_{36} + F_{26}, I_{67} = E_{16}A^*_{17} + E_{26}A^*_{27} + E_{66}A^*_{37} + H_{26},$

$I_{68} = E_{16}A^*_{18} + E_{26}A^*_{28} + E_{66}A^*_{38} + F_{66}, I_{69} = E_{16}A^*_{19} + E_{26}A^*_{29} + E_{66}A^*_{39} + H_{66},$

$I_{71} = A_{54} - 3c_1D_{54}, I_{72} = A_{55} - 3c_1D_{55}, I_{73} = A_{44} - 3c_1D_{44}, I_{74} = A_{45} - 3c_1D_{45},$

$I_{81} = D_{54} - 3c_1F_{54}, I_{82} = D_{55} - 3c_1F_{55}, I_{83} = D_{44} - 3c_1F_{44}, I_{84} = D_{45} - 3c_1F_{45}.$

$I^*_{11} = -3I_{82}c_1 + I_{72}, I^*_{12} = \left(-3I_{81} - 3I_{84}\right)c_1 + I_{71} + I_{74},$

$I^*_{13} = -3I_{83}c_1 + I_{73}, I^*_{14} = -c_1^2 I_{45}, I^*_{15} = -2c_1^2(I_{49} + I_{65}),$

$I^*_{16} = -c_1^2(I_{47} + I_{55} + 4I_{69}), I^*_{17} = -2c_1^2(I_{59} + I_{67}), I^*_{18} = -c_1^2 I_{57},$

$$I_{21}^{*}=-3I_{82}c_1+I_{72},I_{22}^{*}=-3I_{84}c_1+I_{74},I_{23}^{*}=c_1(-c_1I_{45}+I_{44}),$$
$$I_{24}^{*}=c_1\left[(-I_{49}-2I_{65})c_1+I_{48}+2I_{64}\right],I_{25}^{*}=c_1\left[(-I_{55}-2I_{69})c_1+I_{54}+2I_{68}\right],I_{26}^{*}=c_1(-c_1I_{59}+I_{58}),$$
$$I_{31}^{*}=-3I_{81}c_1+I_{71},I_{32}^{*}=-3I_{83}c_1+I_{73},I_{33}^{*}=c_1(-c_1I_{49}+I_{48}),$$
$$I_{34}^{*}=c_1\left[(-I_{47}-2I_{69})c_1+I_{46}+2I_{68}\right],I_{35}^{*}=c_1\left[(-I_{59}-2I_{67})c_1+I_{56}+2I_{66}\right],I_{36}^{*}=c_1(-c_1I_{57}+I_{56}),$$
$$I_{41}^{*}=c_1I_{42},I_{42}^{*}=c_1(-I_{43}+2I_{62}),I_{43}^{*}=c_1(I_{41}+I_{52}-2I_{63}),I_{44}^{*}=c_1(-I_{53}+2I_{61}),I_{45}^{*}=c_1I_{51}$$
$$I_{51}^{*}=3I_{82}c_1-I_{72},I_{52}^{*}=3I_{81}c_1-I_{71},I_{53}^{*}=c_1^2I_{45}-c_1I_{15},$$
$$I_{54}^{*}=-c_1\left[\left(-2I_{49}-I_{65}\right)c_1+2I_{19}+I_{35}\right],I_{55}^{*}=-c_1\left[\left(-I_{47}-2I_{69}\right)c_1+I_{17}+2I_{39}\right],I_{56}^{*}=c_1^2I_{67}-c_1I_{37},$$
$$I_{61}^{*}=3I_{82}c_1-I_{72},I_{62}^{*}=c_1^2I_{45}+\left(-I_{15}-I_{44}\right)c_1+I_{14},$$
$$I_{63}^{*}=\left(I_{49}+I_{65}\right)c_1^2+\left(-I_{19}-I_{35}-I_{48}-I_{64}\right)c_1+I_{18}+I_{34},I_{64}^{*}=c_1^2I_{69}+\left(-I_{39}-I_{68}\right)c_1+I_{38},$$
$$I_{71}^{*}=3I_{81}c_1-I_{71},I_{72}^{*}=c_1^2I_{49}+\left(-I_{19}-I_{48}\right)c_1+I_{18},$$
$$I_{73}^{*}=(I_{47}+I_{69})c_1^2+\left(-I_{17}-I_{39}-I_{46}-I_{68}\right)c_1+I_{16}+I_{38},I_{74}^{*}=c_1^2I_{67}+\left(-I_{37}-I_{66}\right)c_1+I_{36},$$
$$I_{81}^{*}=-I_{42}c_1+I_{12},I_{82}^{*}=\left(I_{43}-I_{62}\right)c_1-I_{13}+I_{32},I_{83}^{*}=\left(-I_{41}+I_{63}\right)c_1+I_{11}-I_{33},I_{84}^{*}=-I_{61}c_1+I_{31},$$
$$I_{91}^{*}=3I_{84}c_1-I_{74},I_{92}^{*}=3I_{83}c_1-I_{73},I_{93}^{*}=c_1^2I_{65}-c_1I_{35},$$
$$I_{94}^{*}=-c_1\left[\left(-I_{55}-2I_{69}\right)c_1+I_{25}+2I_{39}\right],I_{95}^{*}=-c_1\left[\left(-2I_{59}-I_{67}\right)c_1+2I_{29}+I_{37}\right],I_{96}^{*}=c_1^2I_{57}-c_1I_{27},$$
$$I_{101}^{*}=3I_{84}c_1-I_{74},I_{102}^{*}=c_1^2I_{65}+\left(-I_{35}-I_{64}\right)c_1+I_{34},$$
$$I_{103}^{*}=\left(I_{55}+I_{69}\right)c_1^2+\left(-I_{25}-I_{39}-I_{54}-I_{68}\right)c_1+I_{24}+I_{38},I_{104}^{*}=c_1^2I_{59}+\left(-I_{29}-I_{58}\right)c_1+I_{28},$$
$$I_{111}^{*}=3I_{83}c_1-I_{73},I_{112}^{*}=c_1^2I_{69}+\left(-I_{39}-I_{68}\right)c_1+I_{38},$$
$$I_{113}^{*}=\left(I_{59}+I_{67}\right)c_1^2+\left(-I_{29}-I_{37}-I_{58}-I_{66}\right)c_1+I_{28}+I_{36},I_{114}^{*}=c_1^2I_{57}+\left(-I_{27}-I_{56}\right)c_1+I_{26},$$
$$I_{121}^{*}=-I_{62}c_1+I_{32},I_{122}^{*}=\left(-I_{52}+I_{63}\right)c_1+I_{22}-I_{33},I_{123}^{*}=\left(I_{53}-I_{61}\right)c_1-I_{23}+I_{31},I_{124}^{*}=-I_{51}c_1+I_{21}.$$

Appendix C

$$\Delta_k=R\left(k_{01}^2\lambda_m^8+k_{05}^2\delta_n^8+\left(2k_{01}k_{03}-k_{02}^2\right)\lambda_m^6\delta_n^2+\left(2k_{01}k_{05}+k_{03}^2-2k_{02}k_{04}\right)\lambda_m^4\delta_n^4+\left(2k_{03}k_{05}-k_{04}^2\right)\lambda_m^2\delta_n^6\right)$$

$$K_{01}=-\begin{pmatrix}+\left(k_{01}k_{06}\lambda_m-k_{02}k_{09}\delta_n\right)\lambda_m^6+\left(k_{03}k_{06}+k_{01}k_{08}-k_{02}k_{07}\right)\lambda_m^5\delta_n^2\\+\left(k_{03}k_{08}+k_{05}k_{06}-k_{04}k_{07}\right)\lambda_m^3\delta_n^4+\left(k_{05}k_{08}\delta_n^3-k_{04}k_{09}\lambda_m^3\right)\lambda_m\delta_n^3\end{pmatrix}R$$

$$K_{02}=-\begin{pmatrix}\left(k_{05}k_{13}\delta_n^6+\left(k_{01}k_{11}-k_{02}k_{10}\right)\lambda_m^6\right)\delta_n\\+\left(k_{03}k_{11}+k_{01}k_{13}-k_{02}k_{12}-k_{04}k_{10}\right)\lambda_m^4\delta_n^3+\left(k_{03}k_{13}+k_{05}k_{11}-k_{04}k_{12}\right)\lambda_m^2\delta_n^5\end{pmatrix}R$$

$$K_{03}=-\begin{pmatrix}+\begin{pmatrix}k_{01}k_{14}\lambda_m^8+k_{05}k_{18}\delta_n^8+\left(k_{01}k_{16}+k_{03}k_{14}-k_{02}k_{15}\right)\lambda_m^6\delta_n^2\\+\left(k_{05}k_{14}+k_{03}k_{16}+k_{01}k_{18}-k_{02}k_{17}-k_{04}k_{15}\right)\lambda_m^4\delta_n^4+\left(k_{03}k_{18}+k_{05}k_{16}-k_{04}k_{17}\right)\lambda_m^2\delta_n^6\end{pmatrix}R\\-\left(k_{01}\lambda_m^4+k_{03}\lambda_m^2\delta_n^2+k_{05}\delta_n^4\right)\delta_n^2\end{pmatrix}$$

$$K_{04}=\begin{pmatrix}+\left(k_{01}k_{09}\lambda_m^5+\left(k_{05}k_{07}-k_{04}k_{08}\right)\delta_n^5\right)\lambda_m+\left(k_{01}k_{07}-k_{02}k_{06}\right)\lambda_m^5\delta_n\\+\left(k_{03}k_{07}-k_{02}k_{08}-k_{04}k_{06}\right)\lambda_m^3\delta_n^3+\left(k_{03}k_{09}\lambda_m^2+k_{05}k_{09}\delta_n^2\right)\lambda_m^2\delta_n^2\end{pmatrix}\lambda_m R$$

$$K_{05}=\begin{pmatrix}+k_{01}k_{10}\lambda_m^6+\left(k_{05}k_{12}-k_{04}k_{13}\right)\delta_n^6+\left(k_{01}k_{12}+k_{03}k_{10}-k_{02}k_{11}\right)\lambda_m^4\delta_n^2\\+\left(k_{03}k_{12}+k_{05}k_{10}-k_{02}k_{13}-k_{04}k_{11}\right)\lambda_m^2\delta_n^4\end{pmatrix}\lambda_m R$$

$$K_{06}=\left(\begin{pmatrix}\left(k_{01}k_{15}\lambda_m^2-k_{02}k_{14}\lambda_m^2-k_{02}k_{16}\delta_n^2\right)\lambda_m^4\delta_n+\left(k_{01}k_{17}+k_{03}k_{15}-k_{04}k_{14}\right)\lambda_m^4\delta_n^3\\+\left(k_{03}k_{17}+k_{05}k_{15}-k_{02}k_{18}-k_{04}k_{16}\right)\lambda_m^2\delta_n^5+\left(k_{05}k_{17}-k_{04}k_{18}\right)\delta_n^7\end{pmatrix}R+\left(k_{02}\lambda_m^2+k_{04}\delta_n^2\right)\delta_n^3\right)\lambda_m$$

Appendix D

$$l_{11}=I_{41}^*\lambda_m^4\frac{K_{03}}{\Delta_k}-I_{42}^*\lambda_m^3\delta_n\frac{K_{06}}{\Delta_k}+I_{43}^*\lambda_m^2\delta_n^2\frac{K_{03}}{\Delta_k}-I_{44}^*\lambda_m\delta_n^3\frac{K_{06}}{\Delta_k}+I_{45}^*\delta_n^4\frac{K_{03}}{\Delta_k}-k_1-k_2\lambda_m^2-k_2\delta_m^2-\frac{1}{R}\delta_n^2\frac{K_{03}}{\Delta_k},$$

$$l_{12}=I_{41}^*\lambda_m^4\frac{K_{01}}{\Delta_k}-I_{42}^*\lambda_m^3\delta_n\frac{K_{04}}{\Delta_k}+I_{43}^*\lambda_m^2\delta_n^2\frac{K_{01}}{\Delta_k}-I_{44}^*\lambda_m\delta_n^3\frac{K_{04}}{\Delta_k}+I_{45}^*\delta_n^4\frac{K_{01}}{\Delta_k}-I_{21}^*\lambda_m+I_{23}^*\lambda_m^3+I_{25}^*\lambda_m\delta_n^2-\frac{1}{R}\delta_n^2\frac{K_{01}}{\Delta_k},$$

$$l_{13}=I_{41}^*\lambda_m^4\frac{K_{02}}{\Delta_k}-I_{42}^*\lambda_m^3\delta_n\frac{K_{05}}{\Delta_k}+I_{43}^*\lambda_m^2\delta_n^2\frac{K_{02}}{\Delta_k}-I_{44}^*\lambda_m\delta_n^3\frac{K_{05}}{\Delta_k}+I_{45}^*\delta_n^4\frac{K_{02}}{\Delta_k}-I_{32}^*\delta_n+I_{34}^*\lambda_m^2\delta_n+I_{36}^*\delta_n^3-\frac{1}{R}\delta_n^2\frac{K_{02}}{\Delta_k},$$

$$l_{14}=\frac{32\lambda_m\delta_n}{3ab}\frac{K_{01}}{\Delta_k},l_{15}=\frac{32\lambda_m\delta_n}{3ab}\frac{K_{02}}{\Delta_k},n_1=-I_{11}^*\lambda_m^2-I_{13}^*\delta_m^2+I_{14}^*\lambda_m^4+I_{16}^*\lambda_m^2\delta_m^2+I_{18}^*\delta_m^4,n_2=\frac{32\lambda_m\delta_n}{3ab}\frac{K_{03}}{\Delta_k},$$

$$n_3=-\frac{I_{41}^*}{A_{22}^*}\frac{8\lambda_m\delta_n}{3ab}-\frac{I_{45}^*}{A_{11}^*}\frac{8\lambda_m\delta_n}{3ab}+\frac{1}{RA_{11}^*}\frac{2\lambda_m}{3\delta_n ab},n_4=-\frac{\lambda_m^4}{16A_{11}^*}-\frac{\delta_m^4}{16A_{22}^*},n_5=\frac{16}{\lambda_m\delta_n ab}$$

$$l_{21}=-I_{81}^*\frac{K_{03}}{\Delta_k}\lambda_m^3+I_{82}^*\frac{K_{06}}{\Delta_k}\lambda_m^2\delta_n-I_{83}^*\frac{K_{03}}{\Delta_k}\lambda_m\delta_n^2+I_{84}^*\frac{K_{06}}{\Delta_k}\delta_n^3,l_{22}=-I_{81}^*\frac{K_{01}}{\Delta_k}\lambda_m^3-I_{62}^*\lambda_m^2+I_{82}^*\frac{K_{04}}{\Delta_k}\lambda_m^2\delta_n$$

$$-I_{83}^*\frac{K_{01}}{\Delta_k}\lambda_m\delta_n^2-I_{64}^*\delta_n^2+I_{84}^*\frac{K_{04}}{\Delta_k}\delta_n^3+I_{61}^*,l_{23}=-I_{81}^*\frac{K_{02}}{\Delta_k}\lambda_m^3+I_{82}^*\frac{K_{05}}{\Delta_k}\lambda_m^2\delta_n-I_{73}^*\lambda_m\delta_n-I_{83}^*\frac{K_{02}}{\Delta_k}\lambda_m\delta_n^2$$

$$+I_{84}^*\frac{K_{05}}{\Delta_k}\delta_n^3,n_6=-I_{53}^*\lambda_m^3+I_{51}^*\lambda_m-I_{55}^*\lambda_m\delta_m^2,n_7=\frac{I_{81}^*}{A_{22}^*}\frac{8\delta_n}{3ab},$$

$$l_{31}=I_{121}^*\frac{K_{06}}{\Delta_k}\lambda_m^3-I_{122}^*\frac{K_{03}}{\Delta_k}\lambda_m^2\delta_n+I_{123}^*\frac{K_{06}}{\Delta_k}\lambda_m\delta_n^2-I_{124}^*\frac{K_{03}}{\Delta_k}\delta_n^3,l_{32}=I_{121}^*\frac{K_{04}}{\Delta_k}\lambda_m^3-I_{122}^*\frac{K_{01}}{\Delta_k}\lambda_m^2\delta_n$$

$$-I_{103}^*\lambda_m\delta_n+I_{123}^*\frac{K_{04}}{\Delta_k}\lambda_m\delta_n^2-I_{124}^*\frac{K_{01}}{\Delta_k}\delta_n^3,l_{33}=I_{121}^*\frac{K_{05}}{\Delta_k}\lambda_m^3-I_{112}^*\lambda_m^2-I_{122}^*\frac{K_{02}}{\Delta_k}\lambda_m^2\delta_n+I_{123}^*\frac{K_{05}}{\Delta_k}\lambda_m\delta_n^2$$

$$-I_{114}^*\delta_n^2-I_{124}^*\frac{K_{02}}{\Delta_k}\delta_n^3+I_{111}^*,n_8=I_{92}^*\delta_n-I_{94}^*\lambda_m^2\delta_n-I_{96}^*\delta_n^3,n_9=\frac{I_{124}^*}{A_{11}^*}\frac{8}{3}\frac{\lambda_m}{ab}$$