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Original Article

Robust Stability of Implicit Dynamic Equations with Nabla Derivative on Time Scales

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Abstract: In this work we studied the robust stability for implicit integro-dynamic equations on time scales with nabla derivative, which is considered as a generation of differential algebraic equations and implicit difference equations. We showed the reservation of exponential stability of these equations under small Lipschitz perturbations.

Keywords: Implicit integro-dynamic equations, index 1, uniformly stability, time scale, Lipschitz perturbations.

1. Introduction

Implicit integro-dynamic equations have been extensively utilized in various disciplines, including demography, materials science, and actuarial science, with the renewal equation playing a prominent role in these applications [1-3]. Despite this, only a small fraction of such equations and systems can be solved in an explicit manner. As a result, much of the academic focus has shifted towards devising approaches to study the qualitative properties of solutions without directly solving them. One of the primary difficulties in this type of analysis is evaluating the robust stability of these systems.

Previous research has addressed robust stability in singular difference equations and dynamic equations on time scales [4, 5]. However, most of this work has been confined to systems that either lack memory or have only finite memory. This highlights the necessity of further investigating the robust stability of implicit integro-dynamic systems

$$A(t)x^{\nabla}(t) = B(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\nabla s + f(t)$$

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with $t \geq t_0$ and $A(\cdot), B(\cdot)$, $K(\cdot,\cdot)$, $f(\cdot)$ are specified later.

We address the issue of stability preservation in this dynamic equation when subjected to small perturbations. Given that the derivative of the state process $x(t)$ at any given time t depends on its entire path $x(s)$, $t_0 \leq s \leq t$, a more generalized version of the Gronwall-Bellman inequality is required to establish an upper bound for the perturbations.

This paper is structured as follows: The next section delves into the solvability of implicit integro-dynamic equations. In Section 4, we examine the conditions under which the uniform or exponential stability of solutions to implicit integro-dynamic equations remains intact under small Lipschitz perturbations. Finally, in the Appendixes, we provide a brief overview of fundamental concepts and preliminary results related to time scales.

2. Solutions of Implicit Dynamic Equations with Nabla Derivative

Let ***T*** be a time scale (see A1, Appendixes) and f^{∇} denote the nabla derivative for the function $f(\cdot)$. Consider the linear implicit dynamic equations (IDE) on time scales

$$A(t)x^{\nabla}(t) = B(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\nabla s + f(t), \tag{1}$$

where $A(\cdot)$, $B(\cdot)$ are two continuous functions defined on $\mathbb{T}_{t_0}$, valued in the set of $n \times n$-matrices, $f \in L_p^{loc}(\mathbb{T}_{t_0}, \mathbb{R}^n)$ and $K(.,.)$ be a two-variable continuous function defined on $\{(t,s): t_0 \leq s \leq t < \infty\}$, valued in $\mathbb{R}^{n\times n}$.

Suppose that $\ker A(\cdot)$ is smooth, i.e., there exists a continuously ∇-differentiable projector $Q(t)$ onto $\ker A(t)$ and $Q^2 = Q$, $\operatorname{Im} Q(t) = \ker A(t)$ for all $t \in \mathbb{T}_{t_0}$. By setting P=I-Q we can rewrite the equation (1) as

$$A(t)\left(Px\right)^{\nabla}(t) = \bar{B}(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\nabla s + f(t), \tag{2}$$

where $\bar{B} := B + A_{\sigma}P^{\Delta}$. It is seen that the solution $x(\cdot)$ of the equation (2), if it exists, is not necessarily differentiable but it is required that the component $Px(\cdot)$ is ∇-differentiable almost everywhere on $\mathbb{T}_{t_0}$. Consider the space $C_A^1(\mathbb{T}_{t_0}; \mathbb{R}^n)$ being the set of $y \in C(\mathbb{T}_{t_0}; \mathbb{R}^n)$ such that $Py(\cdot)$ is almost everywhere-differentiable on $\mathbb{T}_{t_0}$. Define the linear operators $G := A - \bar{B}Q$. It is clear that $G \in L_{\infty}^{loc}(\mathbb{T}_{t_0}; \mathbb{R}^{n\times n})$.

Definition 2.1 The IDE (1) is said to be index-1 if $G(t)$ is invertible for all $t \in \mathbb{T}_{t_0}$. For any $T > t_0$, consider two subspaces:

$$C_{QG^{-1}}([t_0,T); \mathbb{R}^n) = \{v \in C([t_0,T); \mathbb{R}^n) : v(t) \in \operatorname{Im} QG^{-1}(t)\},$$
$$C_P([t_0,T); \mathbb{R}^n) = \{u \in C([t_0,T); \mathbb{R}^n) : u(t) \in \operatorname{Im} P(t)\}.$$

Lemma 2.2 Let S be a function defined on $[t_0,T] \times C_P\big([t_0,T]; \mathbb{R}^n\big)$, valued in $\mathbb{R}^n$, such that $S(t,y)$ depends only the values of u on $[t_0,t]$ for every $y \in C_P([t_0,T]; \mathbb{R}^n)$ and satisfies the Lipschitz condition, i.e., there is a constant $k > 0$ such that

$$\|S(t,y_1) - S(t,y_2)\| \leq k \sup_{t_0 \leq s \leq t} \|y_1(s) - y_2(s)\|, \quad t \in [t_0,T],\ y_1, y_2 \in C_P([t_0,T]; \mathbb{R}^n).$$

Then, the equation $y^{\nabla} = (P^{\nabla} + PG^{-1}\bar{B})y + PG^{-1}S(t,y),$ with the initial condition $y(t_0) = P(t_0)x_0$ has a unique solution in $C_P([t_0,T];\mathbb{R}^n)$. Moreover, there exists a constant c such that if *y(t)* and *z(t)* are two solutions of above equation then $\|y(t)-z(t)\| \le c\|y(t_0)-z(t_0)\|.$

The proof of this lemma can be easily obtained by using Picard's approximation method and usual procedures. In the next step, we will use above lemma for the proof of next theorem.

Theorem 2.3 For any $t_0 \ge 0$ and $x_0 \in \mathbb{R}^n$, the equation (2) has a unique solution $x(\cdot) \in \mathcal{C}_A^1(\mathrm{T}_{t_0};\mathbb{R}^n)$, with the initial condition

$$P(t_0)\big(x(t_0)-x_0\big) = 0\,. \tag{3}$$

Proof i) Let $u(\cdot) = Px(\cdot)$ and $v(\cdot) = Qx(\cdot)$. Multiplying both sides of (2) with PG^{-1}, QG^{-1} and using Lemma A.3.2 we obtain

$$u^{\nabla}(t) = \big(P^{\nabla} + PG^{-1}\bar{B}\big)(t)u(t) + PG^{-1}f(t) + PG^{-1}\int_{t_0}^{t} K(t,s)\big(u(s)+v(s)\big)\nabla s, \tag{4}$$

$$v(t) = QG^{-1}\bar{B}u(t) + QG^{-1}f(t) + QG^{-1}\int_{t_0}^{t} K(t,s)\big(u(s)+v(s)\big)\nabla s,\ \ for\ t \ge t_0. \tag{5}$$

ii) Consider the operator $H : C\,\big([t_0,\infty);\mathbb{R}^n\big) \to C\,\big([t_0,\infty);\mathbb{R}^n\big)$ defined by

$$(Hv)(t) = v(t) - QG^{-1}\int_{t_0}^{t} K(t,s)v(s)\nabla s.$$

Follows from Theorem 3.1[1] and the continuity of $QG^{-1}(\cdot)K(.,.)$, it implies the invertibility of H because $(Hv)(t) = y(t),\ t \ge t_0$ is a Volterra integral equation of second kind. Precisely,

$$(H^{-1}y)(t) = y(t) + \sum_{n=1}^{\infty}\int_{t_0}^{t} U_n(t,s)y(s)\nabla s$$

where, U_n is defined by induction

$$U_1(t,s) = QG^{-1}(t)K(t,s), \qquad U_{n+1}(t,s) = \int_{s}^{t} U_n(t,\tau)QG^{-1}(\tau)K(\tau,s)\nabla\tau,$$

for $t \ge s \ge t_0, n \ge 1$. On the other hand, for any $T \ge t_0$ we have

$$\sup_{t_0 \le s \le t \le T}\|U_n(t,s)\| \le \left(\sup_{t_0 \le s \le t \le T}\|QG^{-1}(t)K(t,s)\|\right)^n \frac{(T-t_0)^n}{n!}. \tag{6}$$

This implies that the series $I + \sum_{n=1}^{\infty} U_n(t,s)$ is uniformly convergent on the set $\{(t,s):\ t_0 \le s \le t \le T\}$ and $R(t,s) = I + \sum_{n=1}^{\infty} U_n(t,s)$ is a continuous funtion. Thus, H^{-1} is also a second kind linear Volterra operator with the kernel $R(.,.)$. This means that H is a continuous bijection on $C\big([t_0,T];\mathbb{R}^n\big)$. Then, the equation (5) can be rewritten

$$v(t) = H^{-1}QG^{-1}\Big[\bar{B}u + \int_{t_0}^{\cdot} K(\cdot,s)u(s)\nabla s\Big](t) + (H^{-1}QG^{-1}f)(t)$$
$$= H^{-1}QG^{-1}\bar{B}u(t) + (H^{-1}u)(t) - u(t) + (H^{-1}QG^{-1}f)(t),$$

or

$$v(t) = (H^{-1}\hat{P}u)(t) - u(t) + (H^{-1}Q_{\sigma}G^{-1}f)(t), \tag{7}$$

where $\hat{Q} = I - \hat{P} = -QG^{-1}\bar{B}$ is the canonical projector onto $\ker A$.

iii) Combining (7) with (4), we obtain

$$u^{\nabla}(t)=(P^{\nabla}+PG^{-1}\overline{B})u(t)+PG^{-1}f(t)+PG^{-1}\int_{t_0}^{t}K(t,s)H^{-1}(\hat{P}u+QG^{-1}f)(s)\nabla s,\quad t\geq t_0. \tag{8}$$

By using Lemma 2.2, we see that the equation (8) has a unique solution $u(\cdot)$ with initial condition $u(t_0)=P(t_0)x_0$. Then, we use the formula (7) to obtain the solution of (2) as

$$x(t)=u(t)+v(t)=(H^{-1}\hat{P}u)(t)+(H^{-1}QG^{-1}f)(t), \tag{9}$$

for $t\geq t_0$. The proof is complete. W

Remark 2.4 i) Follow the above decoupling procedure, we state the initial condition $u(t_0)=P(t_0)x_0$, or equivalent to

$$P(t_0)\big(x(t_0)-x_0\big)=0,\ \ x_0\in \mathbb{R}^n. \tag{10}$$

We note that the above condition does not depend on the chosen projector opertor $Q(t_0)$.

ii) Let u(t) be the solution of the equation (8). Then we have $Q_\rho u^{\Delta}=Q_\rho P^{\Delta}u$. Furthermore, $Q^{\nabla}=(Q^2)^{\nabla}=Q_\rho Q^{\nabla}+Q^{\nabla}Q$ which implies $(Qu)^{\nabla}=Q_\rho u^{\nabla}+Q^{\nabla}u=Q_\rho P^{\nabla}u+Q_\rho Q^{\nabla}u+Q^{\nabla}Qu=Q^{\nabla}Qu.$ Thus, if $Q(t_0)u(t_0)=0$ then $Q(t)u(t)=0$, for all $t\geq t_0$. This means that (8) has the invariant property: every solution starting in $\operatorname{Im}P(t_0)$ remains in $\operatorname{Im}P(t_0)$ for all $x(t_0)\in\operatorname{Im}P(t_0)$ then $x(t)\in\operatorname{Im}P(t)$, for all $t\in\mathbb{T}_{t_0}$.

iii) Since QG^{-1} is independent of the choice of Q, so is the operator H and $C_{QG^{-1}}([t_0,T];\mathbb{R}^n)$ is independent of the choice Q and the space $C_{QG^{-1}}([t_0,T];\mathbb{R}^n)$ is invariant under the the operator H.

Next, we try to give the variation of constants formula for the solution of equation (2). Consider the homogeneous equation

$$A\big(Px\big)^{\nabla}(t)=\overline{B}x(t)+\int_{t_0}^{t}K(t,s)x(s)\nabla s. \tag{11}$$

Define by $\Phi(t,s),\ t\geq s\geq t_0$ the Cauchy matrix generated by homogeneous system (11) as the solution of the equation

$$A(t)\Phi^{\nabla}(t,s)=B(t)\Phi(t,s)+\int_{s}^{t}K(t,\tau)\Phi(\tau,s)\nabla\tau, \tag{12}$$

and $P(s)\big(\Phi(s,s)-I\big)=0$. Then, we have the variation of constants formula for the solution of (2).

Theorem 2.5 The solution $x(\cdot)$ of the equation (2) with the initial condition $P(t_0)\big(x(t_0)-x_0\big)=0$ can be expressed as

$$x(t)=\Phi(t,t_0)P(t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)PG^{-1}(\tau)\Big(f(\tau)+\int_{t_0}^{\tau}K(\tau,h)(H^{-1}QG^{-1}f)(h)\nabla h\Big)\nabla\tau+(H^{-1}QG^{-1}f)(t), \tag{13}$$

for all $t\geq t_0$.

Proof A similar procedure to split the solution of the equation (10) into $y(\cdot)=\overline{u}(\cdot)+\overline{v}(\cdot)$ obtains

$$\overline{u}^{\nabla}(t)=\big(P^{\nabla}+PG^{-1}\overline{B}\big)\overline{u}(t)+\int_{t_{0t}}^{t}PG^{-1}K(t,s)\big(H^{-1}\hat{P}\,\overline{u}\big)(s)\nabla s,\ \text{and}\ \ y(t)=(H^{-1}\hat{P}\,\overline{u})(t). \tag{14}$$

Denote by $\Phi_0(.,.)$ the Cauchy operator of (13), i.e., it is the solution of the matrix equation

$$\Phi_0^{\nabla}(t,s)=\big(P^{\nabla}+PG^{-1}\overline{B}\big)\Phi_0(t,s)+PG^{-1}(t)\int_{s}^{t}K(t,\tau)\big(H^{-1}\hat{P}\Phi_0(\cdot,s)\big)(\tau)\nabla\tau$$

and $\Phi_0(s,s)=I, t\ge s\ge t_0$. Then, by directly differentiating both sides we obtain the variation constants formula for the solution $u(\cdot)$ of (8) with the initial condition $u(t_0)=P(t_0)x_0$,

$$\bar{u}(t)=\Phi_0(t,t_0)P(t_0)x_0+\int_{t_0}^{t}\Phi_0(t,\tau)PG^{-1}\int_{t_0}^{\tau}\Big(f(\tau)+K(\tau,h)H^{-1}QG^{-1}f(h)\nabla h\Big)\nabla\tau. \qquad (15)$$

On the other hand, since $\bar{u}(t)=\Phi_0(t,t_0)P(t_0)x_0$ and by (14) we have the relation between $\Phi(t,s)$ and

$$\Phi(t,s)=\Big(H^{-1}\hat{P}\Phi_0(\cdot,s)P(s)\Big)(t). \qquad (16)$$

By acting $H^{-1}\hat{P}$ to both sides of (15) and paying attention to the expression (9) it is seen that the unique solution $x(\cdot)$ of (2) with the condition $P(t_0)\big(x(t_0)-x_0\big)=0$ can be given by the formula (13). The proof is complete. W

Assumption 1 There exists a differentiable projector $Q(\cdot)$ onto $\ker A(\cdot)$ such that QG^{-1} and P are bounded on $\mathbb{T}_{t_0}$.

Definition 2.6 i) The IDE (11) is uniformly stable if there exists a constant $M_0>0$ such that

$$\|\Phi(t,s)\|\le M_0,\ t\ge s.$$

ii) Let $\omega\in\Re^+$. The integro - equation (11) is said to be $\omega-$exponentially stable if there exists a constant $M>0$ such that

$$\|\Phi(t,s)\|\le Me_{\ominus\omega}(t,s), t\ge s.$$

In the next section, we consider the effect of small nonlinear perturbations to the stability of IDE (11).

3. Robust Stability of Implicit Dynamic Equation with Nabla Derivative

Consider the perturbed equation of the form

$$A(t)x^{\nabla}(t)=B(t)x(t)+\int_{t_0}^{t}K(t,s)x(s)\nabla s+F(t,x(t)),\ t\in\mathbb{T}_{t_0}. \qquad (17)$$

Assume that $F(t,0)=0$ for all $t\ge t_0$, which follows that the equation (17) has the trivial solution $x(\cdot)\equiv 0$. First at all, we consider the solvability of (17).

Assumption 2 *For all $t\ge t_0$, the functions $PG^{-1}(t)F(t,x)$ and $QG^{-1}(t)F(t,x)$ are Lipschitz in x with Lipschitz coefficient l_t and γ_t, respectively. Suppose further that $l_.$ and $\gamma_.$ are continuous functions.*

We endow $C_{QG^{-1}}([t_0,T];\mathbb{R}^n)$ with the norm inherited from $C([t_0,T];\mathbb{R}^n)$ and understand that $\|H^{-1}\|$ mean that the norm of operator H^{-1} in $C_{QG^{-1}}([t_0,T];\mathbb{R}^n)$. By denoting $\bar{\gamma}_t=\sup_{t_0\le s\le t}\gamma_s$ for $t\ge t_0$, we have

Lemma 3.1 Let $T>t_0$. If $\bar{\gamma}_T\ \|H^{-1}\|<1$, then the equation (17) with the initial condition $P(t_0)\big(x(t_0)-x_0\big)=0$ is solvable on $[t_0,T]$. Further, there exists a constant M_T such that

$$\|x(t)\|\le M_T\|P(t_0)x(t_0)\|,\ \text{ for all } t_0\le t\le T.$$

Proof Put $u(\cdot)=Px(\cdot)$ and $v(\cdot)=Qx(\cdot)$, for $T\ge t\ge t_0$ we have

$$u^{\nabla}(t)=(P^{\nabla}+PG^{-1}\bar{B})u(t)+PG^{-1}\int_{t_0}^{t}K(t,s)H^{-1}(\hat{P}u+QG^{-1}f)(s)\nabla s+PG^{-1}F\big(t,x(t)\big). \qquad (18)$$

And $x(t)=u(t)+v(t)=(H^{-1}\hat{P}u)(t)+(H^{-1}QG^{-1}F\left(\cdot,x(\cdot)\right)(t)$, for $T\ge t\ge t_0$. Fix $u(\cdot)\in C_P([t_0,T];\mathbb{R}^n)$ and consider the mapping $\Gamma_u:C([t_0,T];\mathbb{R}^n)\to C([t_0,T];\mathbb{R}^n)$ defined by

$$\Gamma_u(x)(t)=(H^{-1}\hat{P}u)(t)+H^{-1}QG^{-1}F\left(\cdot,x(\cdot)\right)(t) \text{ for } T\ge t\ge t_0.$$

It is easy to see that $\sup\limits_{t_0\le t\le T}\|\Gamma_u(x)(t)-\Gamma_u(x')(t)\|\le\bar{\gamma}_T\left\|H^{-1}\right\|\sup\limits_{t_0\le t\le T}\|x(t)-x'(t)\|$, for any $x,x'\in C([t_0,T];\mathbb{R}^n)$. Since $\bar{\gamma}_T\left\|H^{-1}\right\|<1$, Γ_u is a contractive mapping. Hence, by the fixed point theorem, there exists uniquely an $x^*\in C([t_0,T];\mathbb{R}^n)$ such that $x^*=\Gamma_u(x^*)$. Denote $x^*=g(u)$ we have $g(u)(t)=(H^{-1}\hat{P}u)(t)+H^{-1}QG^{-1}F\left(\cdot,g(u(\cdot))\right)(t)$. Further,

$$\sup_{[t_0,T]}\|g(u)(t)-g(u')(t)\|\le\beta_T\sup_{[t_0,T]}\|u(t)-u'(t)\|+\bar{\gamma}_T\left\|H^{-1}\right\|\sup_{[t_0,T]}\|g(u)(t)-g(u')(t)\|,$$

with $\beta_T=\left\|H^{-1}\hat{P}\right\|$. Letting $L_T=\dfrac{\beta_T}{1-\bar{\gamma}_T}\left\|H^{-1}\right\|$ deduces $\sup\limits_{t_0\le t\le T}\|g(u)(t)-g(u')(t)\|\le L_T\sup\limits_{t_0\le t\le T}\|u(t)-u'(t)\|$.
This means that g is Lipschitz continuous with the Lipschitz coefficient L_T. In particular,

$$\sup_{t_0\le t\le T}\|g(u)(t)\|\le L_T\sup_{t_0\le t\le T}\|u(t)\|. \tag{19}$$

Substituting $x=g(u)$ into (18) obtains

$$u^\nabla(t)=\left(P^\nabla+PG^{-1}\bar{B}\right)u(t)+PG^{-1}F\big(t,g(u)\big)(t)+PG^{-1}\int_{t_0}^{t}K(t,s)g(u)(s)\nabla s.$$

Note that for any $T\ge t\ge t_0$, the function $PG^{-1}F\left(t,g(u)(t)\right)$ is Lipschitz in u. By applying again Lemma 2.3, we can solve $u(\cdot)$ from above equation with the initial condition $u(t_0)=P(t_0)x_0$. Then the solution of (17) is given by $x(t)=g(u)(t), T\ge t\ge t_0$. Further, by Lemma 2.2

$$\|u(t)\|\le c\|u(t_0)\|,\quad T\ge t\ge t_0.$$

Combining two inequalities with (19), we obtain

$$\|x(t)\|\le M_T\left\|P(t_0)x(0)\right\|,\quad T\ge t\ge t_0,$$

where $M_T=cL_T$. The proof is complete. $\square$

From Lemma 3.1, it follows that the solution $x(\cdot)$ of the equation (17) with the initial condition $P(t_0)\left(x(t_0)-x_0\right)=0$ exists on $[t_0,\infty)$ if $\bar{\gamma}_T\left\|H^{-1}\right\|<1$ for all $T>t_0$ and the constant-variation formulas (13) can be written as

$$x(t)=\Phi(t,t_0)P(t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)PG^{-1}(\tau)\Big(F(\tau,x)+\int_{t_0}^{\tau}K(\tau,s)H^{-1}QG^{-1}F(\cdot,x)(s)\nabla s\Big)\nabla\tau+H^{-1}QG^{-1}F(t,x(t)), \tag{20}$$

with $t\ge t_0$.

To proceed, firstly, we consider the boundedness of solutions of the equation (11) under small nonlinear perturbations.

Theorem 3.2 Assume that the assumptions 1, 2 hold, the solutions of (11) is uniformly stable and H^{-1} is a bounded operator acting on $C_{QG^{-1}}\left([0,\infty),\mathbb{R}^n\right)$ with $\left\|H^{-1}\right\|=K_1$. Then, if $L=1-K_1\bar{\gamma}_\infty>0$, we can find a constant $M_2>0$ such that the solution x(.) of (11) with the initial condition (10) satisfies

$$\|x(t)\|\le M_2e^{M_2N(t)}\left\|P(t_0)x_0\right\|, \text{ for all } t\ge t_0,$$

where $N(t)=\int_{t_0}^{t}\Big(l_\tau+K_1\int_{t_0}^{\tau}\overline{\gamma}_s\left\|PG^{-1}K(\tau,s)Q(s)\right\|\nabla s\Big)\nabla\tau.$

Proof By Lemma 3.1 the condition $L=1-K_1\overline{\gamma}_\infty>0$ implies that the solution $x(.)$ of (17) with the initial condition (10) exists on $[t_0,\infty)$ and the uniform stability of solutions of (11) says that

$$\|\Phi(t,s)\|\le M_0,\quad t\ge s\ge t_0.$$

Therefore, from the formula (20), it follows that for all $t\ge t_0$

$$\|x(t)\|\le M_0\|P(t_0)x_0\|+\left\|H^{-1}QG^{-1}F(\cdot,x(\cdot))(t)\right\|$$
$$+\int_{t_0}^{t}\Big(M_0\left\|PG^{-1}(\tau)F(\tau,x(\tau))\right\|+\int_{t_0}^{\tau}\left\|PG^{-1}(\tau)K(\tau,s)H^{-1}QG^{-1}(s)F(\cdot,x(\cdot))(s)\right\|\nabla s\Big)\nabla\tau.$$

By virtue of the Lipschitz conditions of $PG^{-1}F(\cdot,x(\cdot))$ and $QG^{-1}F(\cdot,x(\cdot))$, we get

$$\|x(t)\|\le M_0\|P(t_0)x_0\|+K_1\overline{\gamma}_\infty\sup_{t_0\le s\le t}\|x(s)\|$$
$$+M_0\int_{t_0}^{t}\Big(l_\tau\sup_{t_0\le s\le\tau}\|x(s)\|+K_1\int_{t_0}^{\tau}\overline{\gamma}_s\times\left\|PG^{-1}K(\tau,s)Q(s)\right\|\sup_{t_0\le s_1\le s}\|x(s)\|\nabla s\Big)\nabla\tau.$$

Letting $M_2=\dfrac{M_0}{L}$, we have

$$\sup_{t_0\le s\le t}\|x(s)\|\le M_2\|P(t_0)x_0\|+M_2\int_{t_0}^{t}\left(l_\tau\sup_{t_0\le s\le\tau}\|x(s)\|+K_1\int_{t_0}^{\tau}\overline{\gamma}_s\times\left\|PG^{-1}(\tau)K(\tau,s)Q(s)\right\|\sup_{t_0\le s_1\le s}\|x(s)\|\right)\nabla\tau.$$

Following the generalized Gronwall-Bellman inequality in Lemma A.2.2

$$\|x(t)\|\le\sup_{t_0\le s\le t}\|x(s)\|\le M_2\|P(t_0)x_0\|e_{N_1(\cdot)}(t,t_0),$$

for all $t\ge t_0$, where $N_1(\tau)=M_2\left(l_\tau+\int_{t_0}^{\tau}K_1\overline{\gamma}_s\left\|PG^{-1}(\tau)K(\tau,s)Q(s)\right\|\Delta s\right)$. Since $N_1(\cdot)$ is positive,

$$e_{N_1(\cdot)}(t,t_0)\le\exp\left(\int_{t_0}^{t}N_1(\tau)\Delta\tau\right)\le\exp\left(\int_{t_0}^{t}M_2\Big(l_\tau+K_1\int_{t_0}^{\tau}\overline{\gamma}_s\left\|PG^{-1}(\tau)\times K(\tau,s)Q(s)\right\|\nabla s\Big)\nabla\tau\right).$$

Thus, $\|x(t)\|\le M_2e^{N(t)}\|P(t_0)x_0\|$, for all $t\ge t_0$. The proof is complete. ∎

From Theorem 3.2, we obtain the corollary

Corollary 3.3 Assume that Assumptions 1, 2 hold, the solutions of (11) is uniformly stable and H^{-1} is a bounded operator acting on $C_{QG^{-1}}([0,\infty),\mathbb{R}^n)$ with $\|H^{-1}\|=K_1$. If $L=1-K_1\overline{\gamma}_\infty>0$ and

$$N=\int_{t_0}^{\infty}\Big(l_\tau+\int_{t_0}^{\tau}K_1\overline{\gamma}_s\,\|PG^{-1}K(\tau,s)Q(s)\|\,\nabla s\Big)\nabla\tau<\infty,$$

then, the solution of the equation (17) is uniformly stable, this mean that there exists a a certain constant M_3 such that $\|x(t)\|\le M_3\|P(t_0)x_0\|,\quad t\ge t_0.$

Next, we consider the robust exponential stability of (11). For any $\lambda>0$, let

$$\tilde{G}=G(I-\lambda\nu Q),\qquad \tilde{K}(t,h)=e_\lambda(\rho(t),s)K(t,h)e_{\ominus\lambda}(h,s).$$

In a similar way to Theorem 2.3, we define the operators

$$(\tilde{H}v)(t)=v(t)-Q\tilde{G}^{-1}\int_{t_0}^{t}\tilde{K}(t,s)v(s)\nabla s.$$

Then we have the following theorem about exponential stability of solutions of the equation (11) under small nonlinear perturbations.

Theorem 3.4 If the equation (11) is ω-exponentially stable and there exists an $\lambda \in (0,\omega)$, with $\ominus\lambda \in \mathbb{R}^{+}$ such that $\tilde{H}^{-1}$ acts continuously on $C_{QG^{-1}}\left([0,\infty),\mathbb{R}^{n}\right)$ with $\|\tilde{H}^{-1}\| = \tilde{K}_1$ satisfying $\tilde{L} = 1-\tilde{K}_1\gamma_\infty > 0$. Suppose further that

$$\limsup_{\tau\to\infty}\left(l_\tau + \tilde{K}_1\int_{t_0}^{\tau}\overline{\gamma}_h e_\lambda(\tau,h)\,\|\,PG^{-1}(\tau)\times K(\tau,h)Q(h)\,\|\,\nabla h\right) = \delta < \frac{\lambda\tilde{L}}{2M}.$$

Then, there is a positive number ω_1 such that the perturbed equation (17) is ω_1-exponentially stable.

Proof Let ε_0 be a positive number such that $\delta+\varepsilon_0 \le \dfrac{\lambda\tilde{L}}{2M}$. Then, from the assumption of theorem, there is a positive number $T_0 > 0$ such that

$$l_t + \tilde{K}_1\int_{t_0}^{t}\overline{\gamma}_h e_\lambda(t,h)\left\|PG^{-1}(t)K(t,h)Q(h)\right\|\nabla h < \delta + \epsilon_0 \le \frac{\lambda\tilde{L}}{2M}, \qquad t \ge T_0.$$

Let $z(t) = e_\lambda(t,s)x(t),\ t\ge s\ge t_0.$, where $x(.)$ is the solution of (11) with the initial condition (10). Since

$$z^{\nabla}(t) = e_\lambda(\rho(t),s)x^{\nabla}(t) + \lambda e_\lambda(t,s)x(t).$$

It is easy to see that y satisfies the equation

$$\begin{aligned} A(t)(Pz)^{\nabla}(t) &= A(t)(e_\lambda(t,s)Px)^{\nabla}(t) = A(t)e_\lambda(\rho(t),s)(Px)^{\nabla}(t) + \lambda e_\lambda(t,s)A(t)Px(t)\\ &= \left[\left(1-\lambda\nu(t)\right)\overline{B}(t) + \lambda A(t)P(t)\right]z(t) + \int_s^t e_\lambda(\rho(t),s)K(t,h)e_{\ominus\nu\lambda}(h,s)z(h)\nabla h\\ &\quad + e_\lambda(\rho(t),s)F\left(t, e_{\ominus\nu\lambda}(t,s)z(t)\right) = \overline{\tilde{B}}y(t) + \int_s^t \tilde{K}(t,h)y(h)\nabla h + \tilde{F}\left(t,z(t)\right), \end{aligned} \tag{21}$$

where $t \ge s$ and

$$\overline{\tilde{B}}(t) = (1-\lambda\nu(t))\overline{B}(t) + \lambda A(t)P(t),\ \ \tilde{K}(t,h) = e_\lambda(\rho(t),s)K(t,h)e_{\ominus\nu\lambda}(h,s),$$

$$\tilde{F}\big(t,z(t)\big) = e_\lambda(\rho(t),s)F\big(t,e_{\ominus\nu\lambda}(t,s)z(t)\big).$$

and $\tilde{G} = A - \left[(1-\lambda\nu)\overline{B} + \lambda AP\right]Q = G + \lambda\nu\overline{B}Q = G\big(I + \lambda\nu G^{-1}\overline{B}Q\big) = G(I - \lambda\nu Q).$

We see that $(I-\lambda\nu Q)^{-1} = \big(P + (1-\lambda\nu)Q\big)^{-1} = P + \dfrac{1}{1-\lambda\nu}Q$, which implies $\tilde{G}$ is invertible, and $\tilde{G}^{-1} = \Big(P + \dfrac{1}{1-\lambda\nu}Q\Big)G^{-1}$, it is seen that the equation (21) is index-1.

Furthermore, $P\tilde{G}^{-1} = PG^{-1}$, $\qquad (1-\lambda\nu)Q\tilde{G}^{-1} = QG^{-1}$, and

$$P\tilde{G}^{-1}\tilde{F}(t,z(t)) = e_\lambda\left(\rho(t),s\right)\times PG^{-1}F\big(t,e_{\ominus\nu\lambda}(t,s)z(t)\big),\quad Q\tilde{G}^{-1}\tilde{F}(t,\cdot) = e_\lambda(t,s)QG^{-1}F\left(t,e_{\ominus\nu\lambda}(t,s)z(t)\right).$$

Further, $P\tilde{G}^{-1}\tilde{F}(t,.)$ and $Q\tilde{G}^{-1}\tilde{F}(t,.)$ are $\left(1-\lambda\nu(t)\right)l_t$ and γ_t-Lipschitz, respectively. Consider the corresponding homogeneous equation to (21)

$$A(t)(Pz)^{\nabla}(t) = \overline{\tilde{B}}(t)z(t) + \int_s^t \tilde{K}(t,h)z(h)\nabla h. \tag{22}$$

By definition, the Cauchy operator $\tilde{\Phi}(t,h),\ t\ge h\ge s$ of (22) and $\Phi(t,h)$ of (11) have a relation

$$\tilde{\Phi}(t,h) = e_\lambda(t,h)\Phi(t,h),\ \ t \ge h \ge s.$$

Therefore, for all $t \ge h \ge s$, $\|\tilde{\Phi}(t,h)\| = e_{\lambda}(t,h)\|\Phi(t,h)\| \le Me_{\lambda\ominus\omega}(t,h) \le M.$

This means that (22) is uniformly stable, and the solution of (21) is expressed by

$$z(t) = \tilde{\Phi}(t,s)P(s)z_0 + \int_s^t \tilde{\Phi}(t,\tau)P\tilde{G}^{-1}(\tau)\left(\tilde{F}(\tau,z(\tau)) + \int_s^{\tau} \tilde{K}(\tau,h)\tilde{H}^{-1}Q\tilde{G}^{-1}\tilde{F}(\cdot,z(\cdot))(h)\nabla h\right)\nabla\tau$$
$$+\tilde{H}^{-1}Q\tilde{G}^{-1}\tilde{F}(t,z(t)).$$

We have

$$\|z(t)\| \le M\|P(s)z_0\| + M\int_s^t\left(\left(1-\lambda\nu(\tau)\right)l_\tau\|z(\tau)\| + \int_s^{\tau}\|P\tilde{G}^{-1}(\tau)\tilde{K}(\tau,h)Q(h)\| \times \tilde{K}_1\bar{\gamma}_h \sup_{t_0\le h_1\le h}\|z(h_1)\|\nabla h\right)\nabla\tau$$
$$+\tilde{K}_1\bar{\gamma}_\infty \sup_{t_0\le h\le t}\|z(h)\| \le M\|P(s)z_0\| + M\int_s^t\left(l_\tau\|z(\tau)\| + \int_s^{\tau}\|P\tilde{G}^{-1}(\tau)\tilde{K}(\tau,h)Q(h)\| \times \tilde{K}_1\bar{\gamma}_h \sup_{t_0\le h_1\le h}\|z(h_1)\|\nabla h\right)$$
$$+\tilde{K}_1\bar{\gamma}_\infty \sup_{t_0\le h\le t}\|z(h)\|.$$

By using Theorem 3.2, with $\tilde{M}_2 = \dfrac{M}{\tilde{L}}$ we have $\|z(t)\| \le \tilde{M}e_{\tilde{N}(\cdot,s)}(t,s)\|P(s)x_0\|$, $t \ge s$, where

$$\tilde{N}(\tau,s) = \tilde{M}_2\left(l_\tau + \tilde{K}_1\int_s^{\tau}\bar{\gamma}_h e_\lambda(\tau,h)\times\|PG^{-1}(\tau)K(\tau,h)Q(h)\|\Delta h\right).$$

Consider the cases, when $t \ge T_0 \ge s \ge t_0$. We see that

$$\|x(t)\| = e_{\ominus\lambda}(t,s)\|z(t)\| \le \tilde{M}e_{\ominus\lambda}(t,s)e_{\tilde{N}}(t,s)\|P(s)x_0\| = \tilde{M}e_{\ominus\lambda}(t,s)e_{\tilde{N}}(t,T_0)e_{\tilde{N}}(T_0,s)\|P(s)x_0\|$$
$$\le \tilde{M}e_{\ominus\lambda}(t,s)e_{\tilde{N}}(t,s)e_{\tilde{N}}(T_0,t_0)\|P(s)x_0\| = \tilde{M}e_{\tilde{N}\ominus\lambda}(t,s)e_{\tilde{N}}(T_0,t_0)\|P(s)x_0\|,$$

where

$$\tilde{N}\ominus\lambda = \tilde{M}_2\left[l_\tau + \tilde{K}_1\int_s^{\tau}\bar{\gamma}_h e_\lambda(\tau,h)\times\|PG^{-1}(\tau)K(\tau,h)Q(h)\|\Delta h\right]\ominus\lambda$$
$$\le \tilde{M}_2(\delta+\epsilon_0)\ominus\lambda = \frac{(\delta+\varepsilon_0)\tilde{M}_2-\lambda}{1-\lambda\nu(\tau)} \le \frac{-\lambda/2}{1-\lambda\nu(\tau)} \le \frac{-\lambda/2}{1-\nu(\tau)\lambda/2} \le \ominus\frac{\lambda}{2}.$$

Thus, $\|x(t)\| \le K_1e_{\ominus\lambda/2}(t,s)\|P(s)x_0\|$, where $K_1 = \tilde{M}e_{\tilde{N}(\cdot)}(T_0,t_0)$.

In case $t > s > T \ge t_0$, using a similar argument as above we get $\|x(t)\| \le M\|x(s)\|e_{\ominus\lambda/2}(t,s).$

Consider the remaining case $t_0 \le s \le t \le T_0$. By virtue of the positivity of $\tilde{L}$ and Lemma 3.1 we get

$$\|x(t)\| \le M_{T_0}\|x(s)\| \le M_{T_0}e_{\lambda/2}(t,s)e_{\ominus\lambda/2}(t,s)\|x(s)\|.$$

Combining the above estimates yields $\|x(t)\| \le Ke_{\ominus\omega_1}(t,s)\|x(s)\|$ for all $t \ge s \ge t_0$, where $\omega_1 = \lambda/2$, $K = \max\{M, K_1, M_{T_0}e_{\lambda/2}(T_0,t_0)\}$. The proof is complete. ■

4. Conclusion

In this work, we have explored the robust stability of linear time-varying IDEs on time scales with Nabla derivative. Several characterizations of robust stability for IDEs under Lipschitz perturbations have been derived. This work unifies and extends many previous results on the robust stability of time-varying ordinary differential and difference equations, time-varying differential-algebraic equations, and time-varying implicit difference equations.

Appendixes

A.1. Time Scales

A time scale $\boldsymbol{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$, enclosed with the topology inherited from the standard topology on $\mathbb{R}$. We define the *backward operator* is defined as $\rho(t) = \sup\{s \in \mathrm{T} : s < t\}$ and the backward graininess is $\nu(t) = t - \rho(t).$ A point $t \in \mathbf{T}$ is said to be *right-dense* if $\sigma(t) = t,$ *right*-scattered if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered.

A function f is *called ld-continuous* if it is continuous at every left-dense point and its right-sided limits exist. The set of ld-continuous functions defined on the interval J valued in X will be denoted by $C_{ld}\left(J, X\right).$ A function *f* from $\mathbf{T}$ to $\mathbb{R}$ is *regressive* (resp., *positively regressive*) if for every $t \in \mathbf{T},$ then $1 - \nu(t) f(t) \neq 0$ (resp., $1 - \nu(t) f(t) > 0$).

We denote by $\Re$ (resp., $\Re^+$) the set regressive (resp., positively regressive) functions, and $C_{ld}\Re$ (resp., $C_{ld}\Re^+$) the set of ld-continuous (resp., positively regressive) regressive functions from $\mathbf{T}$ to $\mathbb{R}$. For all $p, q \in \Re$, we define $p \oplus q = p + q - \nu pq,\ p \ominus q = \dfrac{p-q}{1-\nu q}.$ It is easy to verify that $p \oplus q,\ p\Theta q \in \Re$. Hence, $\Re$ with the calculation $\oplus$ forms an Abelian group.

Definition A.1.1 (see [5]). (Nabla Derivative)**.** A function $f : \mathbf{T} \to \mathbb{R}^d$ is called nabla differentiable at t if there exists a vector $f^{\nabla}(t)$: for all ε >0, there exists a neighborhood U of *t such that*

$$f(\rho(t)) - f(s)\left(\rho(t) - s\right) \leq \varepsilon\left|\rho(t) - s\right|,$$

for all $s \in U$ and for some δ >0. The vector $f^{\nabla}(t)$ is called the deltaderivative of f at t.

A.2. Exponential Functions

For any regressive *ld*-continuous functions $p(.)$ from $\mathbf{T}$ to $\mathbb{R}$, the solution of the dynamic equation $x^{\nabla} = p(t)x,$ with the initial condition $x(s) = 1$, defines a so-called exponential function. We denote this exponential function by $\hat{e}_p(t, s)$. For the properties of exponential function $\hat{e}_p(t, s)$ the interested reader can see [5].

Lemma A.2.1 (see [5]). If $p,\ q$ are regressive, rd-continuous functions and $t, r, s \in \mathrm{T}$ then the following hold:

$$e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s), \qquad e_p(\rho(t), s) = (1 - \nu(t)p(t))e_p(t,s),$$

$$\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s), \qquad \frac{e_p(t,s)}{e_p(s,r)} = e_{p\ominus p}(t,s), \qquad e_p(t,s)e_p(s,r) = e_p(t,r).$$

It is known that for any positively regressive number α , we have the estimate $0 < \hat{e}_{\alpha}(t, t_0) \leq e^{C_0\alpha(t-t_0)}$, where C_0 is a constant depending on the bounds of ν (see [5]).

To consider the robust stability we need the Gronwall-Bellman's inequality. It will be introduced and applied in the following lemma.

Lemma A.2.2 (Extended Gronwall-Bellman's lemma, see [6]) Let the functions $u(t), v(t), w(t,r)$ be nonnegative and continuous for $a \le \tau \le r \le t$, and let c_1 and c_2 be nonnegative. If for $t \in \mathrm{T}_a$

$$u(t) \le \varphi(t)\left(c_1 + c_2 \int_\tau^t \left[v(s)u(s) + \int_\tau^s w(s,r)u(r)\nabla r \right] \nabla s\right),$$

then with $p(\cdot) = c_2 v(\cdot) + c_2 \int_\tau^{\cdot} w(\cdot, r)\nabla r$, we have $u(t) \le c_1\varphi(t)\hat{e}_{p(\cdot)}(t,\tau),\ t \ge \tau.$

We denote by ∇_T the Caratheodory extention of the set funtion m_2 associated with the family $\Im_2 = \{(a,b] \subset \mathrm{T}\}$, where $m_2(a,b] = b - a$. The Lebesgue integral of a measurable function f with respect to ∇_T is denoted by $\int_a^b f(s)\nabla_T s$ (see [7]).

A.3. Some result for Linear Algebra

Let A and B be given $n \times n$ matrices, Q be a projector onto $\ker A$. Denote $S = \{x : Bx \in \operatorname{Im} A\}$. Then, we have some results on linear algebra that have been proven in [8] as follows.

Lemma A.3.1 The following assertions are equivalent

a) $S \cap \mathrm{Ker} A = \{0\}$.

b) The matrix $G = A - \overline{B}Q$ is nonsingular.

c) $\mathbb{R}^n = S \oplus \mathrm{Ker}\, A$.

Lemma A.3.2 Suppose that the matrix G is nonsingular. Then, there hold the following relations:

a) $P = G^{-1}A,\ G^{-1}\overline{B}Q = -Q$ and $\tilde{Q} = -QG^{-1}\overline{B}$ is the projector onto KerA along S.

c) If $\hat{Q}$ is a projector onto KerA then $PG^{-1}\overline{B} = PG^{-1}\overline{B}\hat{P}$, and $QG^{-1}\overline{B} = QG^{-1}\overline{B}\hat{P} - H^{-1}\hat{Q}.$

d) $PG^{-1},\ QG^{-1}$ do not depend on the choice of Q.

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