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Original Article

A Backward Problem for the Homogeneous Diffusion Equation with Coupling Operator and Random Noise

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Abstract: In this work, we consider the problem of recovering the heat distribution for a homogeneous diffusion equation with white noise. As commonly acknowledged, the problem is severely ill-posed according to Hadamard's definition. Consequently, we propose the Fourier truncation method to regularize this problem. With different assumptions on the exact solution, the estimation of the expectation of the error between the regularized solution and the exact solution was obtained. Finally, we provided an example to illustrate our theoretically obtained results.

Keywords: Homogeneous diffusion equation, truncation method, regularized solution, heat distribution, white noise.

1. Introduction

The coupling operator of local and nonlocal type of $-\alpha\Delta(\cdot) + \beta(-\Delta)^\gamma(\cdot)$ where $\alpha, \beta > 0, \gamma \in (0,1)$ arises in various real-world applications. This operator has been used to describe diffusion processes involving particles that exhibit both Lévy and Brownian motion simultaneously. From a practical standpoint, it plays a role in modeling the dynamics of biological populations, where individuals may alternate between short- and long-range random movements. This behavior can, for example, represent a combination of local environmental exploration and long-distance foraging or hunting strategies. Another concrete application of such coupling operators was found in plasma physics. In astrophysical plasmas, for instance, magnetic fields are employed to confine high-temperature plasma [1, 2].

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From a theoretical perspective, diffusion operators combining local and nonlocal terms have been investigated in several contexts, including boundary value problems, the logistic equation, and shape optimization problems involving mixed operators [3-5].

Inspired by the various applications of nonlinear diffusion with coupling operator, let $D = (0, \pi)$, we investigate the problem of determining the temperature distribution $u(x,t)$ for $t \in [0,T)$ which satisfies the following problem

$$u_t(x,t) - \alpha\Delta u(x,t) + \beta(-\Delta)^\gamma u(x,t) = 0, (x,t) \in D \times [0,T], \quad (1)$$

$$u(0,t) = u(\pi,t) = 0, t \in [0,T], \quad (2)$$

$$u(x,T) = g(x), x \in D \quad (3)$$

where $\alpha, \beta > 0, \gamma \in (0,1), T > 0$, the final data $g \in L^2(D)$, and $(-\Delta)^\gamma$ is the fractional Laplacian operator which will be defined in section 2.

The problem (1) - (3) is widely acknowledged as severely ill-posed, indicating that the solution does not exhibit continuous dependence on the input data. In other words, even minor perturbations in the input data can lead to significant changes in the solution. Therefore, implementing an appropriate regularization process is essential to obtain a stable solution.

The problem (1) - (3) for the case $\alpha > 0, \beta = 0$ become the backward problem for the classical parabolic equation, which has been extensively investigated in [8-10]. For instance, in [8], Denche and Bessila used a quasi-boundary value method to regularize the problem. When $\alpha = 0, \beta > 0$, the problem (1) – (3) will become the backward problem for the space-fractional diffusion equation has been studied by many mathematicians (f.i. see in [11-13]). For example, in [11], Zheng applied the fractional Tikhonov regularization method to tackle the problem.

As far as known, the problem (1) – (3) with white noise has not been explored and this is the motivation of our work. Hence, in this work, we study the problem (1) - (3) with the following random model:

$$g_\varepsilon(x) = g(x) + \varepsilon\xi(x), \quad (4)$$

where $\varepsilon > 0$ represents the magnitude of the noise and ξ is a Gaussian white noise process. To address the regularization of the problem, we will employ the Fourier truncation method. Considering various conditions on the exact solution, we aim to determine the convergence rate of Hölder or logarithmic type of the expectation of the error between the regularized solution and the exact solution.

The remainder of this work is organized as follows: In Section 2, we introduce relevant definitions and derive the solution to the problem (1) - (3). Section 3 is devoted to proving the illposedness of the problem (1) - (3). In Section 4, we propose a regularization method and estimate the expected error between the regularized solution and the exact solution. Section 5 presents a numerical example to demonstrate the effectiveness of the proposed theory. Finally, Section 6 provides concluding remarks.

2. Preliminaries and Fundamental Solution

Throughout this work, we denote $D = (0, \pi)$.

Definition 2.1 (see [13]) Let us consider

$$L^2(D) = \left\{ v: D \to \mathbb{R} \mid v \text{ is Lebesgue measurable and } \int_0^\pi |v(x)|^2 dx < \infty \right\},$$

with the inner product

$$\langle v_1, v_2 \rangle = \int_0^\pi v_1(x) v_2(x) dx, \text{ for } v_1, v_2 \in L^2(D)$$

and

$$\|v\| = \left(\int_0^{\pi} |v(x)|^2 dx\right)^{1/2}$$

Lemma 2.1 (see [13]) Let $\{\lambda_n\}_{n\in\mathbb{N}}$ are all the eigenvalues of the operator $-\Delta$, and $\{\phi_n(x)\}_{n\in\mathbb{N}}$ are the corresponding eigenfunctions satisfy

$$\begin{cases} -\Delta\phi_n(x) = \lambda_n\phi_n(x), x \in D \\ \phi_n(x) = 0, x \in \partial D \end{cases}$$

where $\Delta = \frac{d^2}{dx^2}$ is the one-dimensional Laplace operator. Then

$$\lambda_n = n^2 \text{ and } \phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx).$$

Note that $\{\phi_n(x)\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(D)$.

Definition 2.2 (see [12]) Let $v \in L^2(D)$. For every $\alpha > 0$, the fractional Laplacian operator is defined as follows

$$(-\Delta)^{\alpha} v(x) = \sum_{n=1}^{\infty} n^{2\alpha}\langle v, \phi_n\rangle \phi_n(x)$$

where $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$.

Definition 2.3 (see [12]) For $s > 0$, let us consider

$$H^s(D) = \left\{v \in L^2(D): \sum_{n=1}^{\infty} n^{2s}|\langle v, \phi_n\rangle|^2 < \infty\right\},$$

and

$$\|v\|_{H^s(D)} = \left(\sum_{n=1}^{\infty} n^{2s}|\langle v, \phi_n\rangle|^2\right)^{1/2}$$

where $\phi_n(x)$ is given by (2.1).

Notify that $H^s(D)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{H^s(D)} = \sum_{n=1}^{\infty} n^{2s}\langle f, \phi_n\rangle\langle g, \phi_n\rangle$$

Definition 2.4 (see [13]) Let us consider

$$C([0,T]; L^2(D)) = \left\{v: [0,T] \to L^2(D) \text{ is measurable and } \sup_{0\le t\le T}\|v(.,t)\| < \infty\right\}$$

and

$$\|v\|_{C([0,T];L^2(D))} = \sup_{0\le t\le T}\|v(.,t)\|.$$

Definition 2.5 (see [14]) Given a measure probability space Ω. Let us consider the Bochner space

$$L^2(\Omega, L^2(D)) = \{v: \Omega \to L^2(D) \text{ is measurable and } \mathbb{E}\|v\|^2 < \infty\}$$

and

$$\|v\|_{L^2(\Omega, L^2(D))} = \sqrt{\mathbb{E}\|v\|^2}$$

Definition 2.6 (see [14]) Let us consider the normed space

$$V_T = \left\{v: [0,T] \to L^2(\Omega, L^2(D)) \text{ is measurable and } \sup_{0\le t\le T}\sqrt{\mathbb{E}\|v(.,t)\|^2} < \infty\right\}$$

and

$$\|v\|_{V_T} = \sup_{0\le t\le T} \sqrt{\mathbb{E}\|v(.,t)\|^2}$$

Definition 2.7 (see [15]) Let H be a Hilbert space. We say that ξ is a white noise process if $\text{Cov}_\xi = I$ and the random variables are Gaussian: for all functions $g_1, g_2 \in H$, the random variables $\langle \xi, g_j \rangle$ have normal distributions $\mathcal{N}\left(0, \|g_j\|^2\right)$ and $\text{Cov}(\langle \xi, g_1\rangle, \langle \xi, g_2\rangle) = \langle g_1, g_2\rangle$.

Lemma 2.2 (see [15]) Let ξ be a white noise process in a Hilbert space H and $\{\phi_n\}$ be an orthonormal basis in H. Define $\xi_n = \langle \xi, \phi_n\rangle$. Then $\{\xi_n\}$ are independent and identically distributed standard Gaussian random variables.

Theorem 2.8 Let $g \in L^2(D)$. If the problem (1) - (3) has a solution in $C([0,T]; L^2(D))$ then the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} e^{(\alpha n^2 + \beta n^{2\gamma})(T-t)} g_n \phi_n(x), \tag{5}$$

where

$$\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx),$$

$$g_n = \langle g, \phi_n \rangle,$$

$$f_n(u)(s) = \langle f(.,s,u(.,s)), \phi_n\rangle.$$

Proof. If $u(x,t)$ represents a solution of the problem (1) - (3), and it takes the form $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x)$, where $u_n(t) = \langle u(.,t), \phi_n\rangle$, then by multiplying both sides of (1) by $\phi_n(x)$ and integrating over the domain D with respect to x, we obtain:

$$\frac{d}{dt} u_n(t) + (\alpha n^2 + \beta n^{2\gamma}) u_n(t) = 0. \tag{6}$$

Multiplying both sides of (6) by $e^{(\alpha n^2+\beta n^{2\gamma})t}$ and taking the integral from t to T, one obtain:

$$\int_t^T \left(e^{(\alpha n^2+\beta n^{2\gamma})s} u_n(s)\right)' ds = 0.$$

Then

$$e^{(\alpha n^2+\beta n^{2\gamma})T} u_n(T) - e^{(\alpha n^2+\beta n^{2\gamma})t} u_n(t) = 0.$$

It implies that

$$u_n(t) = e^{(\alpha n^2+\beta n^{2\gamma})(T-t)} g_n.$$

So, one can get

$$u(x,t) = \sum_{n=1}^{\infty} e^{(\alpha n^2+\beta n^{2\gamma})(T-t)} g_n \phi_n(x).$$

This completes the proof of Theorem 2.8.

In the next section, we will give an example to prove the ill - posedness of the problem (1) – (3).

3. Example for the Ill-posedness of the Problem (1) - (3) with Gaussian White Noise

We give an example which shows that the problem (1) - (3) has a solution and its solution is not stable. Let us consider the problem

$$\begin{cases} u_t(x,t) - 0.1\Delta u(x,t) + 0.2(-\Delta)^\gamma u(x,t) = 0, (x,t) \in (0,\pi) \times [0,1], \\ u(0,t) = u(\pi,t) = 0, t \in [0,1], \\ u(x,1) = g(x), x \in (0,\pi). \end{cases} \quad (7)$$

Let $g_{ex} \in L^2(D), u_{ex} \in C([0,T]; L^2(D))$. The exact solution of the problem (7) corresponding to the exact data g_{ex} is

$$u_{ex}(x,t) = \sum_{n=1}^{\infty} e^{(\alpha n^2 + \beta n^{2\gamma})(T-t)} (g_{ex})_n \phi_n(x).$$

Choose the measured data

$$g_n(x) = g_{ex}(x) + \frac{1}{n^{\frac{3}{4}}} \sum_{p=1}^{n} \langle \xi, \phi_p \rangle \phi_p(x), n \geq 1,$$

where $\phi_p(x) = \sqrt{\frac{2}{\pi}} \sin(px)$.

We get

$$\mathbb{E}\|g_n - g_{ex}\|^2 = \frac{1}{n^{\frac{3}{2}}} \mathbb{E}\left(\sum_{p=1}^{n} \xi_p^2 \right),$$

where $\xi_p = \langle \xi, \phi_p \rangle$.

It follows from Lemma 2.2 that $\mathbb{E}(\xi_p^2) = 1$. It leads to

$$\mathbb{E}\|g_n - g\|^2 = \frac{1}{n^{\frac{1}{2}}}. \quad (8)$$

The exact solution of the problem (7) corresponding to the measured data g_n is

$$u_n(x,t) = \sum_{p=1}^{\infty} e^{(p^2 + 2p^{2\gamma})(T-t)} (g_n)_p \phi_p(x), \quad (9)$$

where $(g_n)_p = \langle g_n, \phi_p \rangle$.

We get

$$\begin{aligned} &\mathbb{E}\|u_n(.,t) - u_{ex}(.,t)\|^2 \\ &= \mathbb{E}\left(\textstyle\sum_{p=1}^{\infty} \left[e^{(0.1p^2 + 0.2p^{2\gamma})(T-t)} ((g_n)_p - g_p) \right]^2 \right) \\ &\geq \mathbb{E}\left(\left[e^{(0.1n^2 + 0.2n^{2\gamma})(T-t)} ((g_n)_n - g_n) \right]^2 \right) \\ &\geq \frac{1}{n^{\frac{3}{2}}} e^{2(0.1n^2 + 0.2n^{2\gamma})(T-t)} \mathbb{E}(\xi_n^2) \\ &\geq \frac{e^{2(0.1n^2 + 0.2n^{2\gamma})(T-t)}}{n^{\frac{3}{2}}}. \end{aligned}$$

It implies that

$$\sup_{0 \leq t \leq T} \mathbb{E}\|u_n(.,t) - u(.,t)\|^2 \to \infty, \quad (10)$$

when $n \to \infty$.

From (8), we notice that

$$\mathbb{E}\|g_n - g\|^2 \to 0, \tag{11}$$

when $n \to \infty$. From (10) and (11), we deduce that the problem (1) - (3) violates the stability. Hence, the problem (1) - (3) is ill - posed.

Next, we will give a regularization method for the problem (1) - (3).

4. Regularization and Error Estimate

Lemma 4.1 Given $\varepsilon \in (0,1)$ and $s > 0$. Let $g \in H^s(D)$. Suppose that $N(\varepsilon)$ be a positive integer such that $\lim_{\varepsilon \to 0} N(\varepsilon) = +\infty$ and $\lim_{\varepsilon \to 0} \varepsilon^2 N(\varepsilon) = 0$. Put $g_{N(\varepsilon)}$ such that

$$g_{N(\varepsilon)}(x) = \sum_{n=1}^{N(\varepsilon)} \langle g_\varepsilon, \phi_n \rangle \phi_n(x).$$

Then we have the following estimate

$$\mathbb{E}\left\|g_{N(\varepsilon)} - g\right\|^2 \le \varepsilon^2 N(\varepsilon) + \frac{1}{(N(\varepsilon))^{2s}} \|g\|_{H^s(D)}^2. \tag{12}$$

Proof. We have

$$\mathbb{E}\left\|g_{N(\varepsilon)} - g\right\|^2 = \mathbb{E}\left(\sum_{n=1}^{N(\varepsilon)} \langle g_\varepsilon - g, \phi_n \rangle^2\right) + \mathbb{E}\left(\sum_{n > N(\varepsilon)} \langle g, \phi_n \rangle^2\right)$$

$$= \varepsilon^2 \mathbb{E}\left(\sum_{n=1}^{N(\varepsilon)} \xi_n^2\right) + \sum_{n > N(\varepsilon)} n^{-2s} n^{2s} \langle g, \phi_n \rangle^2.$$

Then we get

$$\mathbb{E}\left\|g_{N(\varepsilon)} - g\right\|^2 \le \varepsilon^2 N(\varepsilon) + \frac{1}{(N(\varepsilon))^{2s}} \|g\|_{H^s(D)}^2.$$

This completes the proof of Lemma 4.1.

We know that the terms $e^{(\alpha n^2 + \beta n^{2\gamma})(T-t)}$ (with large *n*) is the instability cause. Hence, to obtain the stability of the solution, we apply the Fourier truncation method to establish a regularized solution as follows:

$$u_{N(\varepsilon)}^\varepsilon(x,t) = \sum_{n=1}^{B_{N(\varepsilon)}} e^{(\alpha n^2 + \beta n^{2\gamma})(T-t)} \left(g_{N(\varepsilon)}\right)_n \phi_n(x), \tag{13}$$

where $B_{N(\varepsilon)}$ is a positive integer satisfying $\lim_{\varepsilon \to 0} B_{N(\varepsilon)} = +\infty$ and will be chosen later. Next we will give the expectation of the error estimate between the regularized solution and the exact solution under different conditions.

Theorem 4.1. Given $s > 0$. Suppose there exists $M_1 > 0$ such that $\|g\|_{H^s(D)} \le M_1$. Let u be the exact solution of the problem (1) - (3) corresponding to the exact data g and $u_{N(\varepsilon)}^\varepsilon$ be the regularized solution corresponding to the random data $g_{N(\varepsilon)}$.

Suppose there exist $q > 0$ and $Q_1 > 0$ such that

$$\sum_{n=1}^{\infty} n^{2q} e^{2tn^2} |u_n(t)|^2 \le Q_1, \forall t \in [0,T]. \tag{14}$$

Then the following estimate holds

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2 \le M_2\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-q}\varepsilon^{\frac{2st}{(2s+1)(\alpha+\beta)T}}\right), t\in[0,T], \tag{15}$$

where $M_2 = 2\max\left\{1+M_1^2, Q_1\left(\frac{s}{(2s+1)(\alpha+\beta)T}\right)^{-q}\right\}$.

Proof:

We put

$$v^{\varepsilon}_{N(\varepsilon)}(x,t) = \sum_{n=1}^{B_{N(\varepsilon)}} e^{(\alpha n^2+\beta n^{2\gamma})(T-t)} g_n\phi_n(x).$$

We have

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2 \le 2\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2+2\mathbb{E}\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2. \tag{16}$$

Firstly, we estimate $\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$. We get

$$\begin{aligned}
&\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2\\
&= \mathbb{E}\left(\sum_{n=1}^{B_{N(\varepsilon)}}\left|e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}\left(\left(g_{N(\varepsilon)}\right)_n-g_n\right)\right|^2\right)\\
&\le e^{2(\alpha+\beta)\left(B_{N(\varepsilon)}\right)^2(T-t)}\mathbb{E}\left(\sum_{n=1}^{\infty}\left|\left(g_{N(\varepsilon)}\right)_n-g_n\right|^2\right)\\
&\le e^{2(\alpha+\beta)\left(B_{N(\varepsilon)}\right)^2(T-t)}\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2.
\end{aligned}$$

From Lemma 4.1, we obtain

$$\begin{aligned}
\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 &\le e^{2(\alpha+\beta)\left(B_{N(\varepsilon)}\right)^2(T-t)}\left(\varepsilon^2N(\varepsilon)+\frac{1}{(N(\varepsilon))^{2s}}\|g\|^2_{H^s(D)}\right)\\
&\le e^{2(\alpha+\beta)\left(B_{N(\varepsilon)}\right)^2(T-t)}\left(\varepsilon^2N(\varepsilon)+\frac{M_1^2}{(N(\varepsilon))^{2s}}\right).
\end{aligned} \tag{17}$$

Secondly, we estimate $\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$. We obtain

$$\begin{aligned}
\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 &= \sum_{n>B_{N(\varepsilon)}} n^{-2q}e^{-2tn^2}n^{2q}e^{2tn^2}\left[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n\right]^2\\
&\le \left(B_{N(\varepsilon)}\right)^{-2q}e^{-2t\left(B_{N(\varepsilon)}\right)^2}Q_1 .
\end{aligned} \tag{18}$$

Combining (16), (17) and (18) gives

$$\begin{aligned}
&\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2\\
&\le 2\left[e^{2(\alpha+\beta)\left(B_{N(\varepsilon)}\right)^2(T-t)}\left(\varepsilon^2N(\varepsilon)+\frac{M_1^2}{(N(\varepsilon))^{2s}}\right)+Q_1\left(B_{N(\varepsilon)}\right)^{-2q}e^{-2t\left(B_{N(\varepsilon)}\right)^2}\right].
\end{aligned} \tag{19}$$

We choose $N(\varepsilon)=\varepsilon^{-\frac{2}{2s+1}}$ and $B_{N(\varepsilon)}=\left(\frac{s}{(2s+1)(\alpha+\beta)T}\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{2}}$. Then we have

$$\begin{aligned}
&\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2\\
&\le 2\left[(1+M_1^2)\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+Q_1\left(\frac{s}{(2s+1)(\alpha+\beta)T}\ln\left(\frac{1}{\varepsilon}\right)\right)^{-q}\varepsilon^{\frac{2st}{(2s+1)(\alpha+\beta)T}}\right].
\end{aligned}$$

Putting $M_2 = 2\max\left\{1 + M_1^2, Q_1\left(\frac{s}{(2s+1)(\alpha+\beta)T}\right)^{-q}\right\}$, we get the estimate

$$\mathbb{E}\left\|u^\varepsilon_{N(\varepsilon)}(.,t) - u(.,t)\right\|^2 \le M_2\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-q}\varepsilon^{\frac{2st}{(2s+1)(\alpha+\beta)T}}\right).$$

This completes the proof of Theorem 4.1.

Theorem 4.2. Let g be as in Theorem 4.1. Let u be the exact solution of the problem (1) - (3) and $u^\varepsilon_{N(\varepsilon)}$ be the regularized solution corresponding to the random data $g_{N(\varepsilon)}$. Suppose there exist $r > 0$ and $Q_2 > 0$ such that

$$\sum_{n=1}^{\infty} e^{2rn^{2\alpha}}|u_n(t)|^2 \le Q_2, \forall t \in [0,T]. \tag{20}$$

Then the following estimate holds

$$\mathbb{E}\left\|u^\varepsilon_{N(\varepsilon)}(.,t) - u(.,t)\right\|^2 \le M_3\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + \varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right), t \in [0,T], \tag{21}$$

where $M_3 = 2\max\{1 + M_1^2, Q_2\}$.

Proof:

Now we estimate $\left\|u(.,t) - v^\varepsilon_{N(\varepsilon)}(.,t)\right\|^2$. We obtain

$$\begin{aligned}\left\|u(.,t) - v^\varepsilon_{N(\varepsilon)}(.,t)\right\|^2 &= \sum_{n > B_{N(\varepsilon)}} e^{-2rn^2}e^{2rn^2}\left[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n\right]^2 \\ &\le e^{-2r(B_{N(\varepsilon)})^2}\sum_{n=1}^{\infty} e^{2rn^2}\left[e^{(\alpha n^2+\beta n^{2\gamma})(T-t)}g_n\right]^2 \\ &\le e^{-2r(B_{N(\varepsilon)})^2}Q_2.\end{aligned} \tag{22}$$

Combining (16), (17) and (22) gives

$$\begin{aligned}&\mathbb{E}\left\|u^\varepsilon_{N(\varepsilon)}(.,t) - u(.,t)\right\|^2 \\ &\le 2\left[e^{2(\alpha+\beta)(B_{N(\varepsilon)})^2(T-t)}\left(\varepsilon^2 N(\varepsilon) + \frac{M_1^2}{(N(\varepsilon))^{2s}}\right) + Q_2 e^{-2r(B_{N(\varepsilon)})^2}\right].\end{aligned} \tag{23}$$

We choose $N(\varepsilon) = \varepsilon^{-\frac{2}{2s+1}}$ and $B_{N(\varepsilon)} = \left(\frac{s}{(2s+1)(\alpha+\beta)T}\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{2}}$.

Then we have

$$\mathbb{E}\left\|u^\varepsilon_{N(\varepsilon)}(.,t) - u(.,t)\right\|^2 \le 2\left[(1 + M_1^2)\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + Q_2\varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right].$$

Putting $M_3 = 2\max\{1 + M_1^2, Q_2\}$, we obtain

$$\mathbb{E}\left\|u^\varepsilon_{N(\varepsilon)}(.,t) - u(.,t)\right\|^2 \le M_3\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + \varepsilon^{\frac{2sr}{(2s+1)(\alpha+\beta)T}}\right).$$

This completes the proof of Theorem 4.2.

5. Numerical Example

In this section, we construct an illustrate example for our regularization method. We consider the following problem

$$\begin{cases} u_t(x,t) - 0.1\Delta u(x,t) + 0.2(-\Delta)^\gamma u(x,t) = 0, (x,t) \in (0,\pi) \times [0,1], \\ u(0,t) = u(\pi,t) = 0, t \in [0,1], \\ u(x,1) = g(x), x \in (0,\pi), \end{cases} \quad (24)$$

where $\gamma = 0.7$ and

$$g(x) = e^{-0.3}\sin(x).$$

The exact solution of the problem (24) is

$$u_{\text{exact}}(x,t) = e^{-0.3t}\sin(x).$$

We get the regularization parameters

$$N = [N(\varepsilon)] = \left[\varepsilon^{\frac{-2}{3}}\right] \text{ and } B_N = \left[B_{N(\varepsilon)}\right] = \left[\left(\frac{10}{9}\ln\left(\frac{1}{\varepsilon}\right)\right)\right]^{\frac{1}{2}}.$$

Consider the random data

$$g_N(x) = e^{-0.3}\sin(x) + \varepsilon \sum_{n=1}^{N} \langle \xi, \phi_n \rangle \phi_n(x),$$

where $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ and $\langle \xi, \phi_n \rangle$ is Gaussian random variable with mean 0 and variance 1.

From (13), we get the regularized solution at the point (x,t)

$$u_N^\varepsilon(x,t) = \sum_{n=1}^{B_N} e^{(0.1n^2+0.2n^{2\gamma})(1-t)}(g_N)_n \phi_n(x),$$

where

$$(g_N)_n = \langle g_N, \phi_n \rangle.$$

The results of our computational method are shown in Figs. 1-2, and listed in Table 1.

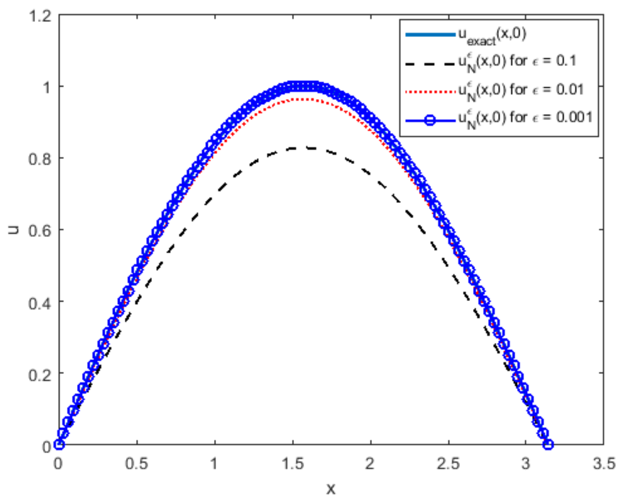

Figure 1. The graph of the exact solution $u_{\text{exact}}(.,0)$ and the regularized solution $u_N^\varepsilon(.,0)$ corresponding to $\varepsilon = 0.1, 0.01, 0.001$.

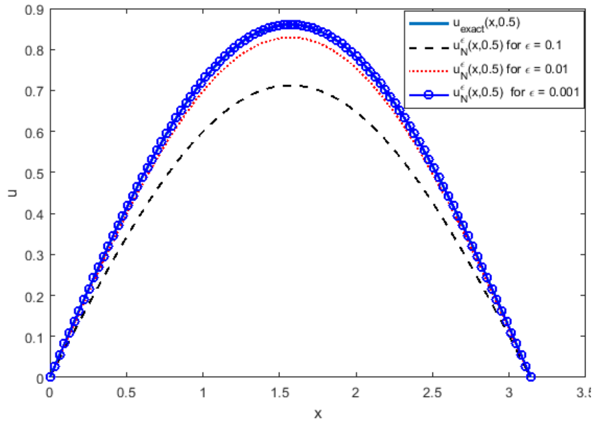

Figure 2. The graph of the exact solution $u_{\text{exact}}(.,0.5)$ and the regularized solution $u_N^\varepsilon(.,0.5)$ corresponding to $\varepsilon = 0.1, 0.01, 0.001$.

Table 1. The expectation of the error between the regularized solution $u_N^\varepsilon(.,t)$ and the exact solution $u_{\text{exact}}(.,t)$ at different values of time corresponding to $\varepsilon = 0.1, 0.01, 0.001$.

$\mathbb{E}\|u_N^\varepsilon(.,t) - u_{\text{exact}}(.,t)\|^2$			
t, ε	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$
$t = 0$	$4.6283e-02$	$5.0135e-04$	$4.5053e-06$
$t = 0.1$	$4.3588e-02$	$4.7215e-04$	$4.2430e-06$
$t = 0.2$	$4.1049e-02$	$4.4465e-04$	$3.9959e-06$
$t = 0.3$	$3.8659e-02$	$4.1876e-04$	$3.7632e-06$
$t = 0.4$	$3.6408e-02$	$3.9437e-04$	$3.5440e-06$
$t = 0.5$	$3.4287e-02$	$3.7141e-04$	$3.3376e-06$
$t = 0.6$	$3.2291e-02$	$3.4978e-04$	$3.1433e-06$
$t = 0.7$	$3.0410e-02$	$3.2941e-04$	$2.9602e-06$
$t = 0.8$	$2.8639e-02$	$3.1022e-04$	$2.7878e-06$
$t = 0.9$	$2.6971e-02$	$2.9216e-04$	$2.6255e-06$
$t = 1$	$2.5401e-02$	$2.7514e-04$	$2.4726e-06$

6. Conclusion

In this work, by Fourier truncation method, we regularized the nonlinear diffusion equation with coupling operator and Gaussian white noise. With some conditions on the exact solution, we obtained the error estimate between the regularized solution and the exact solution. We also gave a numerical experiment results to illustrate our theoretical method. In further work, we will consider the problem in the nonhomogeneous or the nonlinear case of the source term.

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