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A Final Value Problem for the Homogeneous Space Fractional Damped Wave Equation with Gaussian White Noise

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Abstract: In this work, we consider the problem for the homogeneous space fractional damped wave equation with Gaussian white noise. As commonly acknowledged, the problem is severely ill-posed according to Hadamard's sense. Consequently, we propose the Fourier truncation method to regularize the problem. With different assumptions on the exact solution, the estimation of the expectation of the error between the regularized solution and the exact solution in L^2 - norm is obtained. Finally, we provide an example to illustrate our theoretically obtained results.

Keywords: Space fractional equation, damped wave equation, truncation method, regularization, Gaussian white noise. MSC: 35L05, 47J06, 47H10, 60G15.

1. Introduction

Damped wave equations appear across many branches of mathematics and physics, with a range of significant applications in both science and engineering. For instance, in physics these equations are used to describe wave phenomena such as sound, light, and electromagnetic waves. In engineering, they help analyze the stress and strain experienced by elastic materials under different loading conditions. In control theory, they are applied to model and manage dynamic systems where wave propagation and diffusion effects interact. Space fractional damped wave equations, which include fractional derivatives, provide a more detailed and accurate representation of physical phenomena than traditional integer-order equations. These equations have a wide array of applications. In biology, for example, they help model the diffusion of molecules within cells, accounting for the intricate geometry and varying conditions of the environment [1-7].

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Motivated by various applications of space fractional damped wave equation, in this paper, we study the problem of finding the function $u(x,t)$ satisfying the homogeneous space fractional damped wave equation

$$u_{tt} + (-\Delta)^{\gamma} u + u_t + (-\Delta)^{\gamma} u_t = 0, (x,t) \in (0,1) \times [0,T], \quad (1)$$

with the following conditions

$$u(0,t) = u(1,t) = 0, t \in [0,T], \quad (2)$$
$$u(x,T) = g(x), x \in (0,1), \quad (3)$$
$$u_t(x,T) = h(x), x \in (0,1), \quad (4)$$

where $\gamma \in (0,1)$ and $(-\Delta)^{\gamma}$ is the fractional Laplacian which will be defined later. The functions $g, h \in L^2(0,1)$ are given final value data.

The widely recognized fact is that the problem outlined in Eqs. (1) - (4) is severely ill-posed as the solution lacks continuity with respect to the input data. In other words, even a minor alteration in the input data can lead to a significant variation in the solution. Consequently, to attain a stable solution, it is necessary to implement an appropriate regularization process.

In addition, there is the error in the measurement, so we need to assume the presence of the approximation g_{ε} and h_{ε}. If the error comes from controllable sources, it is assumed to be bounded by a fixed $\varepsilon > 0$ and has been studied much in previous papers. However, evaluating the error of the solution becomes more complex because the solution itself is a random variable. White noise is a commonly used random process, valued for its wide range of applications in fields like engineering, science, and business. It plays a key role in areas such as electronic systems, signal processing, econometric modeling, and acoustics, among others [8 - 11].

As we know, the problem (1) - (4) with Gaussian white noise has not been explored and this is the motivation of our paper. Hence, in this work, we study the problems (1) - (4) with the following random model

$$\begin{aligned} g_{\varepsilon}(x) &= g(x) + \varepsilon \xi(x), \\ h_{\varepsilon}(x) &= h(x) + \varepsilon \xi(x), \end{aligned} \quad (5)$$

where $\varepsilon > 0$ represents the magnitude of the noise and ξ is a Gaussian white noise process. For regularizing the problem, we will employ the truncation method. Subject to various conditions on the exact solution, we will establish the convergence rate of Hölder or logarithmic type of the expectation of the error between the regularized solution and the exact solution in L^2 - norm.

The structure of the remaining sections of this work is organized as follows: In Section 2, we introduce some definitions and derive the solution of the problem. In Section 3, we prove the ill-posedness of the problem. In Section 4, we present the regularization method and provide an estimate for the expectation of the error between the regularized solution and the exact solution. Section 5 includes a numerical example to demonstrate the effectiveness of the theory. Finally, in Section 6, we conclude our findings.

2. Preliminaries and Fundamental Solution

Definition 2.1. (see [12]) Let us consider

$$L^2(0,1) = \left\{ v: (0,1) \to \mathbb{R} \mid v \text{ is Lebesgue measurable and } \int_0^1 |v(x)|^2 dx < \infty \right\}$$

with the inner product

$$\langle v_1, v_2\rangle = \int_0^1 v_1(x)v_2(x)dx, \text{ for } v_1, v_2 \in L^2(0,1),$$

and

$$\|v\| = \left(\int_0^1 |v(x)|^2 dx\right)^{\frac{1}{2}}.$$

Definition 2.2. (see [7]) Let $v \in L^2(0,1)$ and $\gamma \in (0,1)$. The fractional Laplacian operator $(-\Delta)^\gamma : L^2(0,1) \to L^2(0,1)$ is defined as follows

$$(-\Delta)^\gamma v(x) = \sum_{n=1}^{\infty} (n^2\pi^2)^\gamma \langle v, \psi_n\rangle \psi_n(x),$$

where $\psi_n(x) = \sqrt{2}\sin(n\pi x)$.

Definition 2.3. (see [7]) For $s > 0$, let us consider

$$H^s(0,1) = \left\{v \in L^2(0,1): \sum_{n=1}^{\infty} (n^2\pi^2)^s |\langle v, \psi_n\rangle|^2 < \infty\right\},$$

and

$$\|v\|_{H^s(0,1)} = \left(\sum_{n=1}^{\infty} (n^2\pi^2)^s |\langle v, \psi_n\rangle|^2\right)^{\frac{1}{2}}.$$

Notify that $H^s(0,1)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{H^s(0,1)} = \sum_{n=1}^{\infty} (n^2\pi^2)^s \langle f, \psi_n\rangle\langle g, \psi_n\rangle.$$

Definition 2.4. (see [12]) For a Hilbert space X , we consider

$$C([0,T];X) = \left\{v: [0,T] \to X \text{ is measurable and } \sup_{0\le t\le T} \|v(.,t)\|_X < \infty\right\},$$

and

$$\|v\|_{C([0,T];X)} = \sup_{0\le t\le T} \|v(.,t)\|_X.$$

Definition 2.5. (see [13]) For a measure probability space Ω, let us consider the Bochner space

$$L^2(\Omega, L^2(0,1)) = \{v: \Omega \to L^2(0,1) \text{ is measurable and } \mathbb{E}\|v\|^2 < \infty\},$$

and

$$\|v\|_{L^2(\Omega, L^2(0,1))} = \sqrt{\mathbb{E}\|v\|^2}.$$

Definition 2.6. (see [13]) Let us consider the normed space

$$V = \left\{v: [0,T] \to L^2(\Omega, L^2(0,1)) \text{ is measurable and } \sup_{0\le t\le T} \sqrt{\mathbb{E}\|v(.,t)\|^2} < \infty\right\},$$

and

$$\|v\|_V = \sup_{0\le t\le T} \sqrt{\mathbb{E}\|v(.,t)\|^2}.$$

Definition 2.7. (see [14]) ξ is a white noise process if $Cov_\xi = I$ and the random variables are Gaussian: for all functions $g_1, g_2 \in L^2(0,1)$, the random variables $\langle \xi, g_j\rangle$ have normal distributions $\mathcal{N}\left(0, \|g_j\|^2\right)$ and $\mathrm{Cov}(\langle \xi, g_1\rangle, \langle \xi, g_2\rangle) = \langle g_1, g_2\rangle$.

Lemma 2.1. (see [14]) Let ξ be a white noise process in Hilbert space $L^2(0,1)$ and $\{\psi_n\}$ be an orthonormal basis in $L^2(0,1)$. Define ξ_n by $\xi_n = \langle \xi, \psi_n \rangle, n \in \mathbb{N}$. Then $\{\xi_n\}$ are independent and identically distributed standard Gaussian random variables.

Lemma 2.2. (see [6]) Let $a, b, T > 0, t \in [0, T]$, we have some following inequalities

i) $\left|e^{-a} - e^{-b}\right| \leq |a - b|$,

ii) $\left|\frac{e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1}\right| \leq (T - t)e^{\left(1+(n\pi)^{2\gamma}\right)(T-t)}$,

iii) $\left|\frac{(n\pi)^{2\gamma}e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1}\right| \leq \sqrt{2 + 4T^2}\, e^{\left(1+(n\pi)^{2\gamma}\right)(T-t)}$.

Theorem 2.1. If the problem (1) - (4) has a solution in $C([0,T]; L^2(0,1))$ then the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} g_n + \frac{e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} h_n\right]\psi_n(x),$$

where

$$g_n = \langle g, \psi_n \rangle, h_n = \langle h, \psi_n \rangle.$$

Proof. Suppose the solution of the problem (1) - (4) has the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\psi_n(x),$$

where $u_n(t) = \langle u(\cdot, t), \psi_n \rangle$.

By multiplying both sides of (1) with $\psi_n(x)$ and integrating with respect to x over the domain (0,1), we obtain

$$u_n''(t) + (1 + (n\pi)^{2\gamma})u_n'(t) + (n\pi)^{2\gamma}u_n(t) = 0. \tag{6}$$

Solving the characteristic equation

$$\lambda^2 + (1 + (n\pi)^{2\gamma})\lambda + (n\pi)^{2\gamma} = 0\,, \tag{7}$$

we get

$$\lambda_1 = -1, \lambda_2 = -(n\pi)^{2\gamma}\,. \tag{8}$$

It implies from (6) that

$$u_n(t) = C_1 e^{-t} + C_2 e^{-(n\pi)^{2\gamma}t}\,. \tag{9}$$

Using the conditions $u_n(T) = g_n$ and $u_n'(T) = h_n$, we get

$$\begin{cases} C_1 e^{-T} + C_2 e^{-(n\pi)^{2\gamma}T} = g_n \\ -C_1 e^{-T} - (n\pi)^{2\gamma} C_2 e^{-(n\pi)^{2\gamma}T} = h_n \end{cases}. \tag{10}$$

Solving the system (10), we obtain

$$\begin{cases} C_1 = \dfrac{(g_n(n\pi)^{2\gamma} + h_n)e^T}{(n\pi)^{2\gamma} - 1} \\ C_2 = \dfrac{-(g_n + h_n)e^{(n\pi)^{2\gamma}T}}{(n\pi)^{2\gamma} - 1} \end{cases}. \tag{11}$$

Substituting (11) into (9), we get the solution of the equation (2.1) as follows

$$u_n(t) = \frac{(n\pi)^{2\gamma}e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} g_n + \frac{e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} h_n.$$

It implies that

$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}g_n+\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}h_n\right]\psi_n(x). \quad (12)$$

This completes the proof of Theorem 2.1.

In the next section, we will give an example to prove the ill - posedness of the problem (1) - (4).

3. Example for the Ill-posedness of the Problems (1) - (4) with Gaussian White Noise

We give an example which shows that the problem (1) - (4) has a solution and its solution is not stable. Let us consider the problem of finding the function $u(x,t)$ that satisfies

$$\begin{cases}u_{tt}+(-\Delta)^{\gamma}u+u_t+(-\Delta)^{\gamma}u_t=0,(x,t)\in(0,1)\times[0,T],\\ u(0,t)=u(1,t)=0,t\in[0,T],\\ u(x,T)=g(x),x\in(0,1),\\ u_t(x,T)=h(x),x\in(0,1),\end{cases} \quad (13)$$

where

$$g,h\in L^2(0,1),\psi_n(x)=\sqrt{2}\sin(n\pi x).$$

Let $g_{ex},h_{ex}\in L^2(0,1)$ and $u_{ex}\in C([0,T];L^2(0,1))$. The exact solution of the problem (13) corresponding to the exact data g_{ex} and h_{ex} is

$$u_{ex}(x,t)=\sum_{n=1}^{\infty}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}(g_{ex})_n+\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}(h_{ex})_n\right]\psi_n(x). \quad (14)$$

Let ξ be as in Lemma 2.1, $g_p,h_p\in L^2(0,1)(p\in\mathbb{N})$ be the measured data as follows

$$g_p(x)=g_{ex}(x)+\frac{1}{p^{\frac{3}{4}}}\sum_{n=1}^{p}\langle\xi,\psi_n\rangle\psi_n(x),$$

$$h_p(x)=h_{ex}(x)+\frac{1}{4p^{\frac{3}{4}}}\sum_{n=1}^{p}\langle\xi,\psi_n\rangle\psi_n(x),$$

where $\psi_n(x)=\sqrt{2}\sin(n\pi x)$.

We get

$$\mathbb{E}\|g_p-g_{ex}\|^2=\frac{1}{p^{\frac{3}{2}}}\mathbb{E}\left(\sum_{n=1}^{p}\xi_n^2\right),$$

$$\mathbb{E}\|h_p-h_{ex}\|^2=\frac{1}{p^{\frac{3}{2}}}\mathbb{E}\left(\sum_{n=1}^{p}\xi_n^2\right),$$

where $\xi_n=\langle\xi,\psi_n\rangle$.

From Lemma (2.2), we get $\mathbb{E}(\xi_n^2)=1$. It leads to

$$\mathbb{E}\|g_p-g_{ex}\|^2=\frac{1}{p^{\frac{3}{2}}}\sum_{n=1}^{p}\mathbb{E}(\xi_n^2)=\frac{1}{p^{\frac{3}{2}}}\sum_{n=1}^{p}1=\frac{p}{p^{\frac{3}{2}}}=\frac{1}{p^{\frac{1}{2}}},$$

$$\mathbb{E}\|h_p-h_{ex}\|^2=\frac{1}{p^{\frac{3}{2}}}\sum_{n=1}^{p}\mathbb{E}(\xi_n^2)=\frac{1}{p^{\frac{3}{2}}}\sum_{n=1}^{p}1=\frac{p}{p^{\frac{3}{2}}}=\frac{1}{p^{\frac{1}{2}}}. \quad (15)$$

Let $g_p, h_p \in L^2(0,1)$ and $u_p \in C([0,T]; L^2(0,1))$. The exact solution of the problem (13) corresponding to the measured data g_p and h_p is

$$u_p(x,t) = \sum_{n=1}^{\infty} \left[\frac{(n\pi)^{2\gamma} e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} \left(g_p\right)_n + \frac{e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} \left(h_p\right)_n \right] \psi_n(x). \quad (16)$$

From (14) and (16), we get

$$\begin{aligned}
&\mathbb{E}\left\|u_p(.,t) - u_{ex}(.,t)\right\|^2 \\
&= \mathbb{E}\left(\sum_{n=1}^{\infty} \left[\frac{(n\pi)^{2\gamma} e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} \left(\left(g_p\right)_n - (g_{ex})_n \right) + \frac{e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} \left(\left(h_p\right)_n - (h_{ex})_n \right) \right]^2 \right) \\
&\geq \mathbb{E}\left(\left[\frac{e^{(T-t)} - e^{(p\pi)^{2\gamma}(T-t)}}{(p\pi)^{2\gamma} - 1} \left(\left(h_p\right)_p - (h_{ex})_p \right) + \frac{(p\pi)^{2\gamma} e^{(T-t)} - e^{(p\pi)^{2\gamma}(T-t)}}{(p\pi)^{2\gamma} - 1} \left(\left(g_p\right)_p - (g_{ex})_p \right) \right]^2 \right).
\end{aligned}$$

Using the inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2, a, b \in \mathbb{R}$, we have the estimate

$$\begin{aligned}
\mathbb{E}\left\|u_p(.,t) - u_{ex}(.,t)\right\|^2 \;&\geq \frac{1}{2}\mathbb{E}\left[\frac{e^{(T-t)} - e^{(p\pi)^{2\gamma}(T-t)}}{(p\pi)^{2\gamma} - 1} \left(\left(h_p\right)_p - (h_{ex})_p \right) \right]^2 \\
&\quad - \mathbb{E}\left[\frac{(p\pi)^{2\gamma} e^{(T-t)} - e^{(p\pi)^{2\gamma}(T-t)}}{(p\pi)^{2\gamma} - 1} \left(\left(g_p\right)_p - (g_{ex})_p \right) \right]^2. \quad (17)
\end{aligned}$$

We put

$$I_1 = \frac{1}{2}\mathbb{E}\left[\frac{e^{(T-t)} - e^{(p\pi)^{2\gamma}(T-t)}}{(p\pi)^{2\gamma} - 1} \left(\left(h_p\right)_p - (h_{ex})_p \right) \right]^2,$$

$$I_2 = \mathbb{E}\left[\frac{(p\pi)^{2\gamma} e^{(T-t)} - e^{(p\pi)^{2\gamma}(T-t)}}{(p\pi)^{2\gamma} - 1} \left(\left(g_p\right)_p - (g_{ex})_p \right) \right]^2. \quad (18)$$

Firstly, we have

$$\begin{aligned}
I_1 \;&= \frac{1}{2}\left[\frac{e^{(p\pi)^{2\gamma}(T-t)} \left(e^{-\left((p\pi)^{2\gamma} - 1\right)(T-t)} - 1 \right)}{\left((p\pi)^{2\gamma} - 1\right) p^{\frac{3}{4}}} \right]^2 \mathbb{E}\left(\xi_p^2\right) \\
&= \frac{1}{2}\left[\frac{e^{(p\pi)^{2\gamma}(T-t)} \left(e^{-\left((p\pi)^{2\gamma} - 1\right)(T-t)} - 1 \right)}{\left((p\pi)^{2\gamma} - 1\right) p^{\frac{3}{4}}} \right]^2. \quad (19)
\end{aligned}$$

Secondly, we get

$$\begin{aligned}
I_2 \;&= \frac{1}{p^{\frac{3}{2}}}\left[\frac{(p\pi)^{2\gamma} e^{(T-t)} - e^{(p\pi)^{2\gamma}(T-t)}}{(p\pi)^{2\gamma} - 1} \right]^2 \mathbb{E}\left(\xi_p^2\right) \\
&\leq \frac{(2 + 4T^2) e^{2\left(1 + (p\pi)^{2\gamma}\right)(T-t)}}{p^{\frac{3}{2}}}. \quad (20)
\end{aligned}$$

Combining (17), (18) , (19) and (20) one can yield

$$\mathbb{E}\left\|u_p(.,t)-u_{ex}(.,t)\right\|^2 \geq \frac{1}{2}\left[\frac{e^{(p\pi)^{2\gamma}(T-t)}\left(1-e^{-\left((p\pi)^{2\gamma}-1\right)(T-t)}\right)}{\left((p\pi)^{2\gamma}-1\right)p^{\frac{3}{4}}}\right]^2-\frac{(2+4T^2)e^{2\left(1+(p\pi)^{2\gamma}\right)(T-t)}}{p^{\frac{3}{2}}}. \quad (21)$$

This leads to

$$\sup_{0\leq t\leq 1}\mathbb{E}\left\|u_p(.,t)-u_{ex}(.,t)\right\|^2 \geq \frac{1}{2}\left[\frac{e^{(p\pi)^{2\gamma}T}\left(1-e^{-\left((p\pi)^{2\gamma}-1\right)T}\right)}{\left((p\pi)^{2\gamma}-1\right)p^{\frac{3}{4}}}\right]^2-\frac{2+4T^2}{p^{\frac{3}{2}}}\to\infty, \quad (22)$$

when $p\to\infty$.

From (15), we notice that

$$\lim_{p\to\infty}\mathbb{E}\left\|g_p-g_{ex}\right\|^2=0,$$

$$\lim_{p\to\infty}\mathbb{E}\left\|h_p-h_{ex}\right\|^2=0. \quad (23)$$

From (22) and (23), we deduce that the problems (1) – (4) fails the stability condition. Hence, the problem (1) - (4) is ill - posed.

Next, we will give a regularization method for the problems (1) - (4).

4. Regularization and Error Estimate

The following lemma is necessary to prove our main results.

Lemma 4.1. Given $\varepsilon\in(0,1)$ and $s>0$. Let $g,h\in H^s(0,1)$. Suppose that $N(\varepsilon)$ is a positive integer such that $\lim_{\varepsilon\to 0}N(\varepsilon)=+\infty$ and $\lim_{\varepsilon\to 0}\varepsilon^2N(\varepsilon)=0$. Let $g_{N(\varepsilon)},h_{N(\varepsilon)}\in L^2(0,1)$ be as follows

$$g_{N(\varepsilon)}(x)=\sum_{n=1}^{N(\varepsilon)}\langle g_\varepsilon,\psi_n\rangle\psi_n(x),$$
$$h_{N(\varepsilon)}(x)=\sum_{n=1}^{N(\varepsilon)}\langle h_\varepsilon,\psi_n\rangle\psi_n(x).$$

Then we have the following estimate

$$\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2\leq\varepsilon^2N(\varepsilon)+\frac{1}{(N(\varepsilon))^{2s}}\|g\|^2_{H^s(0,1)},$$

$$\mathbb{E}\left\|h_{N(\varepsilon)}-h\right\|^2\leq\varepsilon^2N(\varepsilon)+\frac{1}{(N(\varepsilon))^{2s}}\|h\|^2_{H^s(0,1)}. \quad (24)$$

Proof. We have

$$\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2=\mathbb{E}\left(\sum_{n=1}^{N(\varepsilon)}\langle g_\varepsilon-g,\psi_n\rangle^2\right)+\mathbb{E}\left(\sum_{n>N(\varepsilon)}\langle g,\psi_n\rangle^2\right)$$
$$=\varepsilon^2\mathbb{E}\left(\sum_{n=1}^{N(\varepsilon)}\xi_n^2\right)+\sum_{n>N(\varepsilon)}n^{-2s}n^{2s}\langle g,\psi_n\rangle^2.$$

Then we get

$$\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2 \leq \varepsilon^2 N(\varepsilon)+\frac{1}{(N(\varepsilon))^{2s}}\|g\|_{H^s(0,1)}^2$$

Similarly, we obtain

$$\mathbb{E}\left\|h_{N(\varepsilon)}-h\right\|^2 \leq \varepsilon^2 N(\varepsilon)+\frac{1}{(N(\varepsilon))^{2s}}\|h\|_{H^s(0,1)}^2.$$

This completes the proof of Lemma 4.1. □

We know that the terms $\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}$ and $\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}$ are unbounded , so they are the instability causes. Hence, to obtain the stability of the solution, we apply the Fourier truncation method to cut-off the high frequency term in the solution and establish a regularized solution as follows

$$u_{N(\varepsilon)}^{\varepsilon}(x,t)=\sum_{n=1}^{B_{N(\varepsilon)}}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}\left(g_{N(\varepsilon)}\right)_n+\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}\left(h_{N(\varepsilon)}\right)_n\right]\psi_n(x), \quad (25)$$

where $B_{N(\varepsilon)}$ is a positive integer satisfying $\lim_{\varepsilon\to 0} B_{N(\varepsilon)}=+\infty$ and will be chosen later.

Next, we will give the expectation of the error estimate between the regularized solution and the exact solution under different conditions.

Theorem 4.1. Let $N(\varepsilon), g_{N(\varepsilon)}, h_{N(\varepsilon)}$ be as in Lemma 4.1. Suppose there exist $s>0, M_1>0$ such that $\|g\|_{H^s(0,1)} \leq M_1$ and $\|h\|_{H^s(0,1)} \leq M_1$. Let u be the exact solution of the problem (1) – (4) and $u_{N(\varepsilon)}^{\varepsilon}$ be the regularized solution corresponding to the random data $g_{N(\varepsilon)}$ and $h_{N(\varepsilon)}$.

i) If for $\beta>0$, there exists $Q_1>0$ such that

$$\sum_{n=1}^{\infty} n^{2\beta}e^{2t(n\pi)^{2\gamma}}|u_n(t)|^2 \leq Q_1, \forall t\in[0,T], \quad (26)$$

then the following estimate holds

$$\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-u(.,t)\right\|^2 \leq M_2\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-\beta}{\gamma}}\varepsilon^{\frac{2st}{(2s+1)T}}\right), t\in[0,T], \quad (27)$$

where $M_2=\max\left\{4(2+5T^2)(1+M_1^2), 2Q_1\left(\frac{s}{(2\pi)^{2\gamma}(2s+1)T}\right)^{\frac{-q}{\gamma}}\right\}$.

ii) If for $r>0$, there exists $Q_2>0$ such that

$$\sum_{n=1}^{\infty} e^{2r(n\pi)^{2\gamma}}|u_n(t)|^2 \leq Q_2, \forall t\in[0,T], \quad (28)$$

then the following estimate holds

$$\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-u(.,t)\right\|^2 \leq M_3\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+\varepsilon^{\frac{2rs}{(2s+1)T}}\right), t\in[0,T], \quad (29)$$

where $M_3=\max\{4(2+5T^2)(1+M_1^2), 2Q_2\}$

Proof. We put

$$v_{N(\varepsilon)}^{\varepsilon}(x,t)=\sum_{n=1}^{B_{N(\varepsilon)}}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}g_n+\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}h_n\right]\psi_n(x), \quad (30)$$

Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we have

$$\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-u(.,t)\right\|^2 \leq 2\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2+2\left\|u(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2. \quad (31)$$

Firstly, we estimate $\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2$.
Using the inequality $(a+b)^2\le 2(a^2+b^2)$, we get

$$\begin{aligned}&\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2\\&\le 2\mathbb{E}\left(\sum_{n=1}^{B_{N(\varepsilon)}}\left|\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}\left(\left(g_{N(\varepsilon)}\right)_n-g_n\right)\right|^2\right)\\&+2\mathbb{E}\left(\sum_{n=1}^{B_{N(\varepsilon)}}\left|\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}\left(\left(h_{N(\varepsilon)}\right)_n-h_n\right)\right|^2\right).\end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned}&\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2\\&\le 2(2+4T^2)e^{2\left(1+\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}\right)(T-t)}\mathbb{E}\left(\sum_{n=1}^{\infty}\left|\left(g_{N(\varepsilon)}\right)_n-g_n\right|^2\right)\\&+2(T-t)^2e^{2\left(1+\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}\right)(T-t)}\mathbb{E}\left(\sum_{n=1}^{\infty}\left|\left(h_{N(\varepsilon)}\right)_n-h_n\right|^2\right)\\&\le 2(2+4T^2)e^{2\left(1+\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}\right)(T-t)}\mathbb{E}\left\|g_{N(\varepsilon)}-g\right\|^2\\&+2T^2e^{2\left(1+\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}\right)(T-t)}\mathbb{E}\left\|h_{N(\varepsilon)}-h\right\|^2.\end{aligned}$$

From Lemma 4.1, we obtain

$$\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2\le 2(2+5T^2)e^{2(T-t)\left(1+\left(B_{N(\varepsilon)}\pi^2\gamma\right)\right.}\left(\varepsilon^2N(\varepsilon)+\frac{1}{(N(\varepsilon))^{2s}}\|g\|_{H^s(D)}^2\right)$$

$$\le 2(2+5T^2)e^{2(T-t)\left(1+\left(B_{N(\varepsilon)}\pi^2\gamma\right)\right.}\left(\varepsilon^2N(\varepsilon)+\frac{M_1^2}{(N(\varepsilon))^{2s}}\right). \quad (32)$$

Secondly, we estimate $\left\|u(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2$.
Case 1: When the condition (26) holds.
For $\beta>0$, we obtain

$$\begin{aligned}&\left\|u(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2\\&\le\sum_{n>B_{N(\varepsilon)}}n^{-2\beta}e^{-2t(n\pi)^{2\gamma}}n^{2\beta}e^{2t(n\pi)^{2\gamma}}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}g_n+\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}h_n\right]^2.\end{aligned}$$

We get

$$\begin{aligned}&\left\|u(.,t)-v_{N(\varepsilon)}^{\varepsilon}(.,t)\right\|^2\\&\le\left(B_{N(\varepsilon)}\right)^{-2\beta}e^{-2t\left(B_{N(\varepsilon)}\pi^{2\gamma}\right.}\sum_{n=1}^{\infty}n^{2\beta}e^{2t(n\pi)^{2\gamma}}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}g_n+\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}h_n\right]^2\\&\le\left(B_{N(\varepsilon)}\right)^{-2\beta}e^{-2t\left(B_{N(\varepsilon)}\pi^{2\gamma}\right.}\sum_{n=1}^{\infty}n^{2\beta}e^{2t(n\pi)^{2\gamma}}|u_n(t)|^2\end{aligned}$$

So, we have

$$\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 \le \left(B_{N(\varepsilon)}\right)^{-2\beta} e^{-2t\left(B_{N(\varepsilon)}\pi^{2\gamma}\right.} Q_1. \quad (33)$$

Combining (31), (32) and (33) one can have

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2$$
$$\le 4(2+5T^2)e^{2(T-t)\left(1+\left(B_{N(\varepsilon)}\right)^{2\gamma}\right)}\left(\varepsilon^2 N(\varepsilon)+\frac{M_1^2}{(N(\varepsilon))^{2s}}\right)+2Q_1\left(B_{N(\varepsilon)}\right)^{-2\beta} e^{-2t\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}}.$$

We choose $N(\varepsilon)=\varepsilon^{-\frac{2}{2s+1}}$ and $B_{N(\varepsilon)}=\left(\frac{s}{(2\pi)^{2\gamma}(2s+1)T}\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{2\gamma}}$.

Then we have

$$\mathrm{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2$$
$$\le 4(2+5T^2)(1+M_1^2)\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+2Q_1\left(\frac{s}{(\pi)^{2\gamma}(2s+1)T}\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-\beta}{\gamma}}\varepsilon^{\frac{2st}{(2s+1)T}}.$$

Putting $M_2=\max\left\{4(2+5T^2)(1+M_1^2),2Q_1\left(\frac{s}{(2\pi)^{2\gamma}(2s+1)T}\right)^{\frac{-q}{\gamma}}\right\}$, we get the estimate

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2 \le M_2\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}}+\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-\beta}{\gamma}}\varepsilon^{\frac{2st}{(2s+1)T}}\right). \quad (34)$$

Case 2: When the condition (28) holds.

For $r>0$, we obtain

$$\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$
$$\le \sum_{n>B_{N(\varepsilon)}} e^{-2r(n\pi)^{2\gamma}}e^{2r(n\pi)^{2\gamma}}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}g_n+\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}h_n\right]^2.$$

Then we get

$$\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2$$
$$\le e^{-2r\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}}\sum_{n=1}^{\infty} e^{2r(n\pi)^{2\gamma}}\left[\frac{(n\pi)^{2\gamma}e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}g_n+\frac{e^{(T-t)}-e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma}-1}h_n\right]^2$$
$$\le e^{-2r\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}}\sum_{n=1}^{\infty} e^{2r(n\pi)^{2\gamma}}|u_n(t)|^2$$

So, we have

$$\left\|u(.,t)-v^{\varepsilon}_{N(\varepsilon)}(.,t)\right\|^2 \le e^{-2r\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}}Q_2\,. \quad (35)$$

Combining (31), (32) and (35) one can have

$$\mathbb{E}\left\|u^{\varepsilon}_{N(\varepsilon)}(.,t)-u(.,t)\right\|^2$$

$$\le 4(2+5T^2)e^{2(T-t)\left(1+(n\pi)^{2\gamma}\right)}\left(\varepsilon^2 N(\varepsilon)+\frac{M_1^2}{(N(\varepsilon))^{2s}}\right)+2Q_2 e^{-2r\left(B_{N(\varepsilon)}\pi\right)^{2\gamma}}. \quad (36)$$

We choose $N(\varepsilon)=\varepsilon^{-\frac{2}{2s+1}}$ and $B_{N(\varepsilon)}=\left(\frac{s}{(2\pi)^{2\gamma}(2s+1)T}\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{1}{2\gamma}}$.

Then we have

$$\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-u(.,t)\right\|^2 \le 4(2+5T^2)(1+M_1^2)\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + 2Q_2\varepsilon^{\frac{2rs}{(2s+1)T}}.$$

Putting $M_3 = \max\{4(2+5T^2)(1+M_1^2), 2Q_2\}$, we get the estimate

$$\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,t)-u(.,t)\right\|^2 \le M_3\left(\varepsilon^{\frac{2s(T+t)}{(2s+1)T}} + \varepsilon^{\frac{2rs}{(2s+1)T}}\right). \quad (37)$$

This completes the proof of Theorem 4.1.

Remark 4.1.

i) The error estimate derived from (27) at the initial time $t = 0$ is given by

$$\mathbb{E}\left\|u_{N(\varepsilon)}^{\varepsilon}(.,0)-u(.,0)\right\|^2 \le M_2\left(\varepsilon^{\frac{2s}{2s+1}} + \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-q}{\gamma}}\right).$$

This error estimate exhibits a logarithmic-type convergence due to the insufficiently strong condition (26) on the exact solution.

ii) We notice that the error estimate provided in (29) demonstrates Hölder-type convergence for all $t \in [0,T]$. This convergence rate is better than the logarithmic-type rate described in (27). However, it's important to note that the error estimate (29) requires a strong condition on the exact solution $u(x,t)$, which can be considered a disadvantage.

5. Numerical Example

In this section, we construct an illustrate example for our regularization method. We consider the following problem

$$\begin{cases} u_{tt} + (-\Delta)^{\gamma}u + u_t + (-\Delta)^{\gamma}u_t = 0, (x,t) \in (0,1)\times[0,1], \\ u(0,t) = u(1,t) = 0, t \in [0,1], \\ u(x,1) = g(x), x \in (0,1), \\ u_t(x,1) = h(x), x \in (0,1), \end{cases} \quad (38)$$

where $\gamma = \frac{1}{30}$ and

$$g(x) = e^{-1}\sin(\pi x),$$
$$h(x) = -e^{-1}\sin(\pi x).$$

The exact solution of the problem (38) is

$$u_{\text{exact}}(x,t) = e^{-t}\sin(\pi x).$$

We get the regularization parameters

$$N = [N(\varepsilon)] = \left[\varepsilon^{\frac{-2}{3}}\right] \text{ and } B_N = \left[B_{N(\varepsilon)}\right] = \left[\left(\frac{2}{3}\ln\left(\frac{1}{\varepsilon}\right)\right)\right]^{\frac{1}{2\gamma}}.$$

Consider the random data

$$g_N(x) = e^{-1}\sin(\pi x) + \varepsilon\sum_{n=1}^{N}\langle\xi,\psi_n\rangle\psi_n(x),$$
$$h_N(x) = -e^{-1}\sin(\pi x) + \varepsilon\sum_{n=1}^{N}\langle\xi,\psi_n\rangle\psi_n(x),$$

where $\psi_n(x) = \sqrt{2}\sin(n\pi x)$ and $\langle\xi,\psi_n\rangle$ are random variables with mean 0 and variance 1 .

From (25), we get the regularized solution at the point (x,t)

$$u_N^\varepsilon(x,t) = \sum_{n=1}^{B_N} \left[\frac{(n\pi)^{2\gamma} e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} (g_N)_n + \frac{e^{(T-t)} - e^{(n\pi)^{2\gamma}(T-t)}}{(n\pi)^{2\gamma} - 1} (h_N)_n \right] \psi_n(x),$$

where

$$(g_N)_n = \langle g_N, \psi_n \rangle,$$
$$(h_N)_n = \langle h_N, \psi_n \rangle.$$

Next, we divide the time interval [0,1] into 10 subintervals by 11 points

$$t_j = \frac{j-1}{10}, j = 1,2, \ldots 11.$$

Put $\varepsilon = 0.1, \varepsilon = 0.01, \varepsilon = 0.001$, respectively. The results of our computational method are shown in Figs.1 1-2 and listed in Table 1.

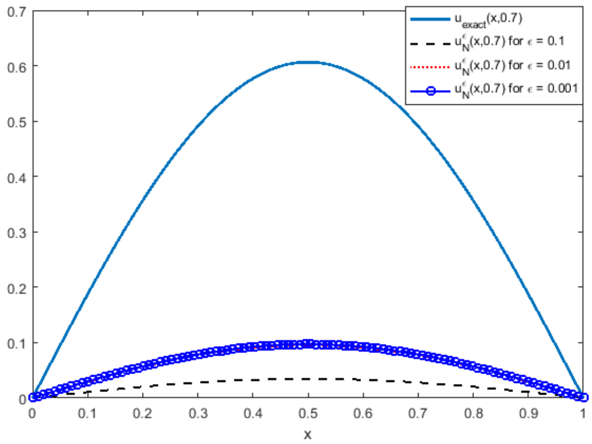

Figure 1. The graph of the exact solution $u_{\text{exact}}(x, 0.7)$ and the regularized solution $u_N^\varepsilon(x, 0.7)$ corresponding to $\varepsilon = 0.1, 0.01, 0.001$.

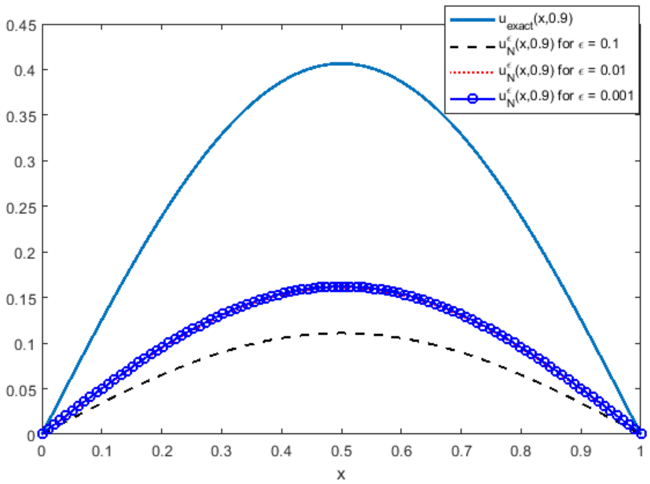

Figure 2. The graph of the exact solution $u_{\text{exact}}(x, 0.9)$ and the regularized solution $u_N^\varepsilon(x, 0.9)$ corresponding to $\varepsilon = 0.1, 0.01, 0.001$.

Table 1. The expectation of the error between the regularized solution $u_N^\varepsilon(.,t)$ and the exact solution $u_{\text{exact}}(.,t)$ at different values of time corresponding to $\varepsilon = 0.1, 0.01, 0.001$.

	$\mathbb{E}\|u_N^\varepsilon(.,t) - u_{exact}(.,t)\|^2$		
t,ε	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$
$t = 0$	*1.2353*	*1.1895*	*1.1869*
$t = 0.1$	$8.7819e-01$	$8.4252e-01$	$8.4048e-01$
$t = 0.2$	$6.1855e-01$	$5.9077e-01$	$5.8922e-01$
$t = 0.3$	$4.3104e-01$	$4.0945e-01$	$4.0827e-01$
$t = 0.4$	$2.9667e-01$	$2.7993e-01$	$2.7905e-01$
$t = 0.5$	$2.0125e-01$	$1.8831e-01$	$1.8765e-01$
$t = 0.6$	$1.3419e-01$	$1.2421e-01$	$1.2372e-01$
$t = 0.7$	$8.7644e-02$	$7.9977e-02$	$7.9618e-02$
$t = 0.8$	$5.5816e-02$	$4.9947e-02$	$4.9687e-02$
$t = 0.9$	$3.4450e-02$	$2.9977e-02$	$2.9791e-02$
$t = 1$	$2.0438e-02$	$1.7045e-02$	$1.6916e-02$

6. Conclusion

In this work, by Fourier truncation method, we regularized the homogeneous space fractional damped wave equation with Gaussian white noise. With some conditions on the exact solution, we obtained the error estimate between the regularized solution and the exact solution in different norms. We also gave a numerical experiment to illustrate our method. In further work, we will consider the problems with locally Lipschitz condition on the source term.

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