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Tail Distribution, Smoothness and Gaussian Density Estimates of Stochastic Differential Delay Equations

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Abstract: In this paper, we study the tail distribution, smoothness and density estimates of the solution of a fundamental class stochastic differential delay equations. Base on the techniques of the Malliavin calculus we obtain an explicit estimate for tail distributions and upper and lower Gaussian estimates for density.

Keywords: Stochastic differential delay equation, Tail distribution, Density estimates, Malliavin calculus.

1. Introduction

Stochastic delay differential equations are crucial in the modeling of scientific and engineering systems. The significance of stochastic differential delay equations (SDDEs) lies in the fact that many phenomena observed in our surroundings do not have an immediate impact from the moment they occur. For instance, in numerous physical phenomena with a random nature, the future state of a system is not only determined by its current state, but also by its entire past history over a finite time interval. Similarly, a patient may exhibit symptoms of an illness days or even weeks after being infected.

In this paper, we consider the basic stochastic differential delay equations of the form

$$\begin{cases} X_t = \varphi(0) + \int_0^t b(s, X_s, X_{s-\tau})ds + \int_0^t \sigma(s, X_s, X_{s-\tau})dB_s, & t \in [0,T] \\ X_t = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1}$$

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where $\tau > 0$ and the initial data $\varphi : [-\tau, 0] \to \mathbb{R}$ is continuous deterministic function, $(B_t)_{t\in[0,T]}$ is a standard Brownian motion and b, σ are deterministic functions on $\mathbb{R}^2$. This model has been applied in various contexts, as seen in [1]. However, it is important to note that the majority of SDDEs, including those in the form of equation (1), cannot be solved explicitly. Naturally, one would like to investigate the distribution function and properties of the solutions of (1). There have been numerous studies on solutions for SDDEs. For instance, in 2000, E. Buckwar conducted research on the numerical analysis of SDDEs with discrete time lags [1]. In 2003, Mao and his colleagues studied the numerical solutions of SDDEs under the local Lipschitz condition [2]. More recently, in 2019, A. Bahar investigated the numerical solution of Equation (1.1) under sufficient conditions when the delay term takes on random values [3].

Additionally, the convergence rate is also a significant concern. E. Buckwar studied the weak convergence of the Euler scheme (see [4]). In 2022, N.T.Dung and his colleagues provided an interesting result on the weak convergence with respect to delay parameter of the solutions. In this paper, explicit estimates for the rate of convergence can be found, along with their application to the Carathéodory approximation scheme [5]. Our focus here is on the tail distribution and density of the solution of (1). Because of their importance, the distribution function and density estimates have garnered considerable attention in research. Notably, N.T.Dung has studied the tail distributions and densities of various random processes using the techniques of Malliavin calculus (see [6-10]). More recently, we have applied the same method to provide tail distribution estimates for the mixed-fractional CEV model and fractional CIR model [11, 12]. In this paper, we continue to use the techniques of Malliavin calculus to obtain explicit estimates for the tail distribution of solution of (1) as in Theorem 3.1 and some estimates for density of the solution of (1) as in Theorem 4.1 and Theorem 4.2.

The rest of the paper is organized as follows: In section 2, we recall some fundamental concepts of Malliavin calculus and some general estimates for tail distributions and densities. The main results of the paper are presented and proven in Sections 3 and 4. Section 3 discusses tail distribution estimates, while Section 4 focuses on density estimates.

2. Preliminaries

This paper is strongly based on techniques of Malliavin calculus. For the reader's convenience, let us recall some elements of Malliavin calculus (for more details see [13]). We suppose that $(B_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a natural filtration generated by the Brownian motion B. For $h \in L^2[0,T]$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int_0^T h(t)dB_t.$$

Let $\mathcal{S}$ denote a dense subset of $L^2(\Omega, \mathcal{F}, P)$ that consists of smooth random variables of the form

$$F = f\big(B(h_1), B(h_2), \ldots, B(h_n)\big), \tag{2}$$

where $n \in \mathbb{N}, f \in C_0^\infty(\mathbb{R}^n), h_1, h_2, \ldots, h_n \in L^2[0,T]$. If F has the form (2), we define its Malliavin derivative as the process $DF := D_t F, t \in [0,T]$ given by

$$D_t F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(B(h_1), B(h_2), \ldots, B(h_n)).h_k(t)$$

More generally, for each $k \geq 1$, we can define the iterated derivative operator on a cylindrical random variable by setting

$$D^k_{t_1,\dots,t_k} F = D_{t_1} \dots D_{t_k} F.$$

For any $1 \leq p,k < \infty$, we denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\| F \|^p_{k,p} := E|F|^p + E\left[\left(\int_0^T |D_u F|^2\, du \right)^{\frac{p}{2}} \right] + \dots + E\left[\left(\int_0^T \dots \int_0^T |D^k_{t_1,\dots,t_k} F|^2\, dt_1 \dots dt_k \right)^{\frac{p}{2}} \right].$$

A randon variable F is said Malliavin differentable if it belongs to $\mathbb{D}^{1,2}$. For any $F \in \mathbb{D}^{1,2}$, the Clark-Ocone formula says that

$$F - E[F] = \int_0^T E[D_s F \mid \mathcal{F}_s] dB_s.$$

To get the result on tail estimates, we use the following general estimate for tail probabilities:

Lemma 2.1. Let $F \in \mathbb{D}^{1,2}$ is a centered random variable. Assume there exists a non-random constant M such that

$$\int_0^T E[D_r F \mid \mathcal{F}_r]^2 dr \leq M^2 \quad \text{a.s.}$$

Then following estimate for tail probabilities holds

$$P(F \geq x) \leq e^{-\frac{x^2}{2M^2}}, \quad x > 0.$$

Proof. For the reader's convenience, we recall here the proof provided in [7]. By Clark-Ocone formula, we have

$$F = E[F] + \int_0^T E[D_r F \mid \mathcal{F}_r \,|] dB_r.$$

When $E[F] = 0$ we have

$$F = \int_0^T E[D_r F \mid \mathcal{F}_r \,|] dB_r.$$

Hence, for any $\lambda \in \mathbb{R}$, it holds that

$$\begin{aligned} Ee^{\lambda F} &= E\exp\left(\lambda \int_0^T E[D_r F \mid \mathcal{F}_r \,|] dB_r - \frac{\lambda^2}{2} \int_0^T (E[D_r F \mid \mathcal{F}_r \,|])^2 dr + \frac{\lambda^2}{2} \int_0^T (E[D_r F \mid \mathcal{F}_r \,|])^2 dr \right) \\ &\leq e^{\frac{\lambda^2}{2} M^2} E\exp\left(\lambda \int_0^T E[D_r F \mid \mathcal{F}_r \,|] dB_r - \frac{\lambda^2}{2} \int_0^T (E[D_r F \mid \mathcal{F}_r \,|])^2 dr \right). \end{aligned}$$

By Itô formula, the stochastic process

$$N_T = \exp\left(\lambda \int_0^T E[D_r F \mid \mathcal{F}_r \,|] B_r - \frac{\lambda^2}{2} \int_0^T (E[D_r F \mid \mathcal{F}_r \,|])^2 dr \right)$$

satisfies

$$N_T = 1 + \lambda \int_0^T N_r E[D_r F \mid \mathcal{F}_r \,|] dB_r.$$

Thus, $EN_T = 1$. We can get

$$Ee^{\lambda F} \le e^{\frac{\lambda^2}{2}M^2} EN_T = e^{\frac{\lambda^2}{2}M^2}.$$

This, together with Markov's inequality, gives us

$$P(F \le x) = P(e^{\lambda F} \le e^{\lambda x}) \le e^{\frac{\lambda^2}{2}M^2 - \lambda x}, \quad \lambda > 0, x \in \mathbb{R}.$$

When $x > 0$, the function $\lambda \to e^{\frac{\lambda^2}{2}M^2 - \lambda x}$ attains its minimum value at $\lambda_0 = \frac{x}{M^2}$. Choosing $\lambda = \lambda_0$ we obtain

$$P(F \ge x) \le e^{-\frac{x^2}{2M^2}}, \quad x > 0.$$

Similarly, we also have

$$P(F \le -x) = P(-F \ge -x) \le e^{-\frac{x^2}{2M^2}}, x > 0.$$

So, we deduce

$$P(|F| \ge x) \le 2e^{-\frac{x^2}{2M^2}}, \quad x > 0.$$

It know that any random variable F in $L_2(\Omega, \mathcal{F}, P)$ can be expanded into an orthogonal sum of its Wiener chaos:

$$F = \sum_{n=0}^{\infty} J_n F,$$

where $J_0 F = E(F)$ and J_n denotes the projection onto the n th Wiener chaos. From this chaos expansion, one may define the Ornstein-Uhlenbeck operator L by

$$LF = \sum_{n=0}^{\infty} -nJ_n F$$

and its pseudo-inverse by

$$L^{-1}F = \sum_{n=1}^{\infty} -\frac{1}{n} J_n F.$$

Let F be in $\mathbb{D}^{1,2}$ with mean zero, and we define the function g_F by

$$g_F(x) := E\left(\left\langle DF, -DL^{-1}F \right\rangle_{L^2[0,T]} | F = x\right), \quad x \in \mathbb{R}.$$

Nourdin and Viens in [14] used this function to obtain a new density formula and to provide upper and lower Gaussian estimates for this density. Theorem 3.1 and Corollary 3.5 of [14] can be ummarised as follows.

Lemma 2.2. Let F be in $\mathbb{D}^{1,2}$ with mean zero.

i) The law of F has a density ρ_F with respect to the Lebesgue measure if and only if $g\_F(x) > 0$ a.s. In this case, $\mathrm{Supp}(\rho_F)$ is a closed interval of $\mathbb{R}$ containing 0 and we have

$$\rho_F(x) = \frac{E|F|}{2g_F(x)} \exp\left(-\int_0^x \frac{u}{g_F(u)} du\right), \quad a.e. \ x \in \mathrm{Supp}(\rho_F).$$

ii) If there exist positive constants c, C such that, for all $x \in \mathbb{R}$, almost surely

$$c \leq \int_0^T D_r F E[D_r F \mid \mathcal{F}_r] dr \leq C,$$

then the density ρ_F of F exists and satisfies, for almost all $x \in \mathbb{R}$

$$\frac{E|F|}{2C} \exp\left(-\frac{x^2}{2c}\right) \leq \rho_F(x) \leq \frac{E|F|}{2c} \exp\left(-\frac{x^2}{2C}\right).$$

In the whole this section, we always consider the stochastic differential delay equation (1) with the following basic assumptions:

($\mathbf{A_1}$) The coefficients $b, \sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Lipschitz and have linear growth, that is, there exists $K, L > 0$ such that

$$|b(x_1, y_1) - b(x_2, y_2)| + |\sigma(x_1, y_1) - \sigma(x_2, y_2)| \leq K\left(|x_1 - x_2| + |y_1 - y_2|\right)$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and

$$|b(x, y)| + |\sigma(x, y)| \leq L(1 + |x| + |y|)$$

for all $x, y \in \mathbb{R}.$

($\mathbf{A_2}$) σ is bounded, that is

$$\|\sigma\|_\infty := \sup_{(x,y)\in\mathbb{R}^2} |\sigma(x, y)| < \infty.$$

Hereafter, we denote by C a generic constant which may vary at each appearance.

For any $a, b \in \mathbb{R},$ we denote $a \vee b = \max\{a, b\}$. In our proofs, we sometime use the fundamental inequality

$$(a_1 + ... + a_n)^p \leq n^{p-1}(a_1^p + ... + a_n^p),$$

for all $a_1, ..., a_n \geq 0$ and $p \geq 2.$

For any function h of n variables, we denote

$$h_i'(x_1, ..., x_n) := \frac{\partial h}{\partial x_i}(x_1, ..., x_n).$$

3. Tail Distribution Estimates

Proposition 3.1. Suppose Assumption A_1. Then, the equation (1) has a unique solution $(X_t)_{t\in[-\tau,T]}$ satisfies for every $p \geq 1,$ we have

$$\sup_{0\leq t\leq T} E|X_t|^p \leq C, \tag{3}$$

Where C is a positive constant not depending on t.

Proof. As we know, under the conditions of Lipschitz and linear growth properties of coefficients, the equation (1) have a unique solution. We refer the reader to [3] for more details.

Next, we prove (3). For any $p \geq 2,$ it follows from (1) that

$$|X_t|^p \leq 3^{p-1}\left(|\varphi(0)|^p + \left|\int_0^t b(s,X_s,X_{s-\tau})ds\right|^p + \left|\int_0^t \sigma(s,X_s,X_{s-\tau})dB_s\right|^p\right),\ \ t\in[0,T].$$

By using Holder and Burkholder-Davis-Gundy inequalities we deduce

$$E|X_t|^p \leq 3^{p-1}|\varphi(0)|^p + 3^{p-1}t^{p-1}\int_0^t E|b(s,X_s,X_{s-\tau})|^p\, ds + 3^{p-1}Ct^{\frac{p}{2}-1}\int_0^t E|\sigma(s,X_s,X_{s-\tau})|^p\, ds.$$

By Assumption (A_1) we get

$$|b(s,X_s,X_{s-\tau})|^p \leq L^p\left(1+|X_s|+|X_{s-\tau}|\right)^p \leq L^p 3^{p-1}\left(1+|X_s|^p+|X_{s-\tau}|^p\right),$$
$$|\sigma(s,X_s,X_{s-\tau})|^p \leq L^p\left(1+|X_s|+|X_{s-\tau}|\right)^p \leq L^p 3^{p-1}\left(1+|X_s|^p+|X_{s-\tau}|^p\right).$$

Consequently

$$\begin{aligned} E|X_t|^p &\leq C + CL^p\int_0^t E\left(1+|X_s|^p+|X_{s-\tau}|^p\right)ds \\ &\leq C + C\int_0^t E|X_s|^p\, ds + C\int_0^t E|X_{s-\tau}|^p\, ds \\ &\leq C + C\int_0^t E|X_s|^p\, ds + C\int_{-\tau}^{0\wedge(t-\tau)}\varphi(s)ds + \int_0^{(t-\tau)\vee 0} E|X_s|^p\, ds. \end{aligned}$$

Note that φ is a continuous function on $[-\tau,0]$ then φ is bounded on $[-\tau,0]$. Consequently,

$$E|X_t|^p \leq C + C\int_0^t E|X_s|^p\, ds,\ \ 0\leq t\leq T,$$

where C is positive constant depending on $L,p,T.$ By Gronwall's lemma we hence

$$E|X_s|^p \leq Ce^{Ct} \leq C,\ \ 0\leq t\leq T, p\geq 2.$$

Because $L^q[0,T]\subset L^p[0,T]$ for all $0<q<p,$ so we imply $E|X_s|^p \leq C,\ \ 0\leq t\leq T, p\geq 1.$
This finishes the proof of the proposition.

Proposition 3.2. Let the assumption A_1 holds. Then, the unique solution $(X_t)_{t\in[-\tau,T]}$ of (1) is Malliavin differentable. Moreover, the derivative $D_\theta X_t$ satisfies

i) When $t\in[-\tau,0], D_\theta X_t = 0,$ for all $0\leq\theta\leq T,$

ii) When $t\in(0,T], D_\theta X_t = 0$ for $\theta > t$

$$\begin{aligned} D_\theta X_t &= \sigma(\theta,X_\theta,X_{\theta-\tau}) + \int_\theta^t \bar{b}_2(s)D_\theta X_s ds + \int_{\theta+\tau}^t \bar{b}_3(s)D_\theta X_{s-\tau}ds \\ &\quad + \int_\theta^t \bar{\sigma}_2(s)D_\theta X_s dB_s + \int_{\theta+\tau}^t \bar{\sigma}_3(s)D_\theta X_{s-\tau}dB_s,\ \ 0\leq\theta\leq t-\tau \end{aligned} \tag{4}$$

$$D_\theta X_t = \sigma(\theta,X_\theta,X_{\theta-\tau}) + \int_\theta^t \bar{b}_2(s)D_\theta X_s ds + \int_\theta^t \bar{\sigma}_2(s)D_\theta X_s dB_s,\ \ (t-\tau)\vee 0 < \theta \leq t. \tag{5}$$

where $\bar{b}_2(s), \bar{\sigma}_2(s), \bar{b}_3(s), \bar{\sigma}_3(s)$ are adapted stochastic processes and bounded by the Lipschitz constant K. Here, we use the convention $[0,t-\tau]=\varnothing$ if $t<\tau.$

Proof. For $t\in[-\tau,0],$ we have $X_t=\varphi(t)$ is deterministic. Hence, the Malliavin derivative of X_t are vanished, that is, $D_\theta X_t = 0$ for all $0\leq\theta\leq T.$ On the other hand, since the solution $(X_t)_{t\in[0,T]}$ is $\mathbb{F}$,-

adapted, we have $D_\theta X_t = 0$ for $\theta > t$. When $\theta \le t$, by using the same argument as in the proof of Theorem 2.2.1 in [13] we can show that the solution $(X_t)_{t\in[0,T]}$ is Malliavin differentable. Moreover, appling the derivative operator D to the solution of the equation (1), we obtain

$$D_\theta X_t = \sigma(\theta, X_\theta, X_{\theta-\tau}) + \int_\theta^t D_\theta[b(s, X_s, X_{s-\tau})]ds + \int_\theta^t D_\theta[\sigma(s, X_s, X_{s-\tau})]dB_s, \quad t \in (0,T]. \tag{6}$$

By Lipschitz property of b and σ, using Proposition 1.2.4 in [13] we imply that there exists $\bar{b}_2(s), \bar{\sigma}_2(s), \bar{b}_3(s), \bar{\sigma}_3(s)$ that are adapted processes and bounded by K such that

$$D_\theta[b(s, X_s, X_{s-\tau})] = \bar{b}_2(s) D_\theta X_s + \bar{b}_3(s) D_\theta X_{s-\tau}, \tag{7}$$

$$D_\theta[\sigma(s, X_s, X_{s-\tau})] = \bar{\sigma}_2(s) D_\theta X_s + \bar{\sigma}_3(s) D_\theta X_{s-\tau}. \tag{8}$$

Inserting (7) and (8) into (6) we have

$$\begin{aligned} D_\theta X_t = \sigma(\theta, X_\theta, X_{\theta-\tau}) &+ \int_\theta^t \bar{b}_2(s) D_\theta X_s ds + \int_\theta^t \bar{b}_3(s) D_\theta X_{s-\tau} ds \\ &+ \int_\theta^t \bar{\sigma}_2(s) D_\theta X_s dB_s + \int_\theta^t \bar{\sigma}_3(s) D_\theta X_{s-\tau} dB_s. \end{aligned}$$

Because $D_\theta X_{s-\tau} = 0$ for $\theta > s-\tau$, we obtain the equations (4) and (5). The proof of Proposition is complete.

Proposition 3.3. Let the assumptions (A_1) and (A_2) hold. Let $(X_t)_{t\in[-\tau,T]}$ be the solution to the equation (1). Then we have

$$E[D_\theta X_t \mid \mathcal{F}_\theta]^2 \le \|\sigma\|_\infty^2 e^{(4K^2+4K)t}, \quad \theta \le t \le T.$$

Proof. From equations (4) and (5) we can rewrite $D_\theta X_t$ as follows

$$\begin{aligned} D_\theta X_t \quad &= \sigma(\theta, X_\theta, X_{\theta-\tau}) + \int_\theta^t \bar{b}_2(s) D_\theta X_s ds + \int_\theta^t \bar{b}_3(s) D_\theta X_{s-\tau} \mathbf{1}_{[\theta+\tau,t]}(s) ds \\ &+ \int_\theta^t \bar{\sigma}_2(s) D_\theta X_s dB_s + \int_\theta^t \bar{\sigma}_3(s) D_\theta X_{s-\tau} \mathbf{1}_{[\theta+\tau,t]}(s) dB_s \end{aligned}$$

for $0 \le \theta \le t \le T$. Thanks to Itô formula, we have

$$\begin{aligned} |D_\theta X_t|^2 = |\sigma(\theta, X_\theta, X_{\theta-\tau})|^2 &+ \int_\theta^t \left(2\bar{b}_2(s) + \bar{\sigma}_2^2(s)\right)|D_\theta X_s|^2 ds + \int_\theta^t \left(\bar{\sigma}_3(s)\mathbf{1}_{[\theta+\tau,t]}(s)\right)^2 |D_\theta X_{s-\tau}|^2 ds \\ &+ \int_\theta^t \left(2\bar{b}_3(s) + 2\bar{\sigma}_2(s)\bar{\sigma}_3(s)\right) D_\theta X_s D_\theta X_{s-\tau} \mathbf{1}_{[\theta+\tau,t]}(s) ds \\ &+ \int_\theta^t 2\bar{\sigma}_2(s)|D_\theta X_s|^2 dB_s + \int_\theta^t 2\bar{\sigma}_3(s) D_\theta X_s D_\theta X_{s-\tau} \mathbf{1}_{[\theta+\tau,t]}(s) dB_s. \end{aligned}$$

Using the fundamental inequality $a^2 + b^2 \ge 2ab$, we deduce

$$\begin{aligned} |D_\theta X_t|^2 \le |\sigma(\theta, X_\theta, X_{\theta-\tau})|^2 &+ \int_\theta^t \left(2\bar{b}_2(s) + \bar{\sigma}_2^2(s) + \left|\bar{b}_3(s) + \bar{\sigma}_2(s)\bar{\sigma}_3(s)\right|\mathbf{1}_{[\theta+\tau,t]}(s)\right)|D_\theta X_s|^2 ds \\ &+ \int_\theta^t \left(\left(\bar{\sigma}_3(s)\mathbf{1}_{[\theta+\tau,t]}(s)\right)^2 + \left|\bar{b}_3(s) + \bar{\sigma}_2(s)\bar{\sigma}_3(s)\right|\mathbf{1}_{[\theta+\tau,t]}(s)\right)|D_\theta X_{s-\tau}|^2 ds \\ &+ \int_\theta^t 2\bar{\sigma}_2(s)|D_\theta X_s|^2 dB_s + \int_\theta^t 2\bar{\sigma}_3(s) D_\theta X_s D_\theta X_{s-\tau} \mathbf{1}_{[\theta+\tau,t]}(s) dB_s. \end{aligned}$$

Because of Assumption (A_2) and the boundedness of $\bar{b}_2(s), \bar{b}_3(s), \bar{\sigma}_3(s), \bar{\sigma}_2(s)$, we can see

$$E[|D_\theta X_t|^2|\mathcal{F}_\theta] \le \|\sigma\|_\infty^2 + \left(2K^2+3K\right)\int_\theta^t E\left[|D_\theta X_s|^2 \mid \mathcal{F}_\theta\right]ds + \left(2K^2+K\right)\int_\theta^t E\left[|D_\theta X_{s-\tau}|^2 \mid \mathcal{F}_\theta\right]ds$$
$$\le \|\sigma\|_\infty^2 + (4K^2+4K)\int_\theta^t E\left[|D_\theta X_s|^2 \mid \mathcal{F}_\theta\right]ds.$$

Thanks to Gronwall's lemma we obtain

$$E[|D_\theta X_t|^2|\mathcal{F}_\theta] \le \|\sigma\|_\infty^2 e^{\left(4K^2+4K\right)(t-\theta)} \le \|\sigma\|_\infty^2 e^{\left(4K^2+4K\right)t}, \quad \theta \le t \le T.$$

The proof of Proposition is complete.

Theorem 3.1. Let the assumptions (A_1) and (A_2) hold. Let $(X_t)_{t\in[-\tau,T]}$ be the solution to the equation (1). Then, for each $t\in(0,T],$ the tail distribution of X_t satisfies

$$P(|X_t| \ge x) \le 2\exp\left(\frac{(x-|EX_t|)^2}{2\|\sigma\|_\infty^2 e^{\left(4K^2+4K\right)t}}\right), \ x \ge |EX_t|.$$

Proof. Proposition 3.3 holds that

$$\int_0^T E[|D_\theta X_t|^2|\mathcal{F}_\theta]d\theta = \int_0^t E[|D_\theta X_t|^2|\mathcal{F}_\theta]d\theta \le \|\sigma\|_\infty^2 e^{\left(4K^2+4K\right)t}, \quad 0\le t \le T.$$

Fixed $t\in(0,T],$ we consider the random variable $F = X_t - EX_t$. It is easy to see that $EF=0$ and $D_\theta F = D_\theta X_t$. So F fulfills the conditions of Lemma 2.1. Consequently,

$$P(|X_t| \ge x) = P(|X_t| - |EX_t| \ge x - |EX_t|)$$
$$\le P(|X_t - EX_t| \ge x - |EX_t|) = P(|F| \ge x - |EX_t|)$$
$$\le 2\exp\left(\frac{(x-|EX_t|)^2}{2\|\sigma\|_\infty^2 e^{\left(4K^2+4K\right)t}}\right), x \ge |EX_t|.$$

This finishes the proof.

4. Smoothness and Gaussian Density Estimates

We have to additionally impose the following condition:

(A$_3$) $b(x,y)$ and $\sigma(x,y)$ are twice differentable with bounded derivatives.

Remark 4.1. When assumption (A_3) holds, $D_\theta X_t$ in Proposition 3.2 becomes

i) When $t\in[-\tau,0], D_\theta X_t = 0,$ for all $0\le\theta\le T,$

ii) When $t\in(0,T], D_\theta X_t = 0$ for $\theta > t$ and

$$D_\theta X_t = \sigma(\theta, X_\theta, X_{\theta-\tau}) + \int_\theta^t b_2'(s,X_s,X_{s-\tau})D_\theta X_s ds + \int_{\theta+\tau}^t b_3'(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}ds$$
$$+\int_\theta^t \sigma_2'(s,X_s,X_{s-\tau})D_\theta X_s dB_s + \int_{\theta+\tau}^t \sigma_3'(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}dB_s, \quad 0\le\theta\le t-\tau. \tag{9}$$

$$D_\theta X_t = \sigma(\theta, X_\theta, X_{\theta-\tau}) + \int_\theta^t b_2'(s,X_s,X_{s-\tau})D_\theta X_s ds$$
$$+\int_\theta^t \sigma_2'(s,X_s,X_{s-\tau})D_\theta X_s dB_s, \quad (t-\tau)\vee 0 < \theta \le t. \tag{10}$$

Proposition 4.1. Assume that the assumptions $(A_1),(A_2)$ and (A_3) hold and let $(X_t)_{t\in[-\tau,T]}$ be the solution to the equation (1). In addition, we assume that $\|\sigma\|_0 := \inf_{(x,y)\in\mathbb{R}^2} |\sigma(x,y)| > a > 0.$ Then, exists a finite constant $C>0$ such that

$$|D_\theta X_t| \le C \ \ a.s. \text{ for } 0\le\theta\le t\le T.$$

Proof. From Remark 4.1, $D_\theta X_t$ can rewrite as follows

$$\begin{aligned}D_\theta X_t &= \sigma(\theta,X_\theta,X_{\theta-\tau}) + \int_\theta^t b_{2'}(s,X_s,X_{s-\tau})D_\theta X_s ds + \int_\theta^t b_{3'}(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)ds \\ &+ \int_\theta^t \sigma_{2'}(s,X_s,X_{s-\tau})D_\theta X_s dB_s + \int_\theta^t \sigma_{3'}(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)dB_s.\end{aligned} \tag{11}$$

We consider the process $Y_t = F(X_t)$ for $0\le t\le T$ where $F(x)=\int_0^x \frac{1}{\sigma(y,y-\tau)}dy.$ By the chain rule of Malliavin derivative, we have

$$D_\theta X_t = \sigma(t,X_t,X_{t-\tau})D_\theta Y_t, 0\le\theta\le t\le T. \tag{12}$$

By Itô's formula, we get

$$dY_t = \left(\frac{b(t,X_t,X_{t-\tau})}{\sigma(t,X_t,X_{t-\tau})} - \frac{1}{2}\big(\sigma_{2'}(t,X_t,X_{t-\tau}) + \sigma_{3'}(t,X_t,X_{t-\tau})\big)\right)dt + dB_t.$$

Equivalently

$$Y_t = Y_0 + I(t) + B_t, \ \ t\in[0,T], \tag{13}$$

where

$$I(t) = \int_0^t \left(\frac{b(s,X_s,X_{s-\tau})}{\sigma(s,X_s,X_{s-\tau})} - \frac{1}{2}\big(\sigma_{2'}(s,X_s,X_{s-\tau}) + \sigma_{3'}(s,X_s,X_{s-\tau})\big)\right)ds.$$

For $0\le\theta\le t-\tau$ we have

$$\begin{aligned}D_\theta I(t) &= \int_\theta^t \frac{b_2'(s,X_s,X_{s-\tau})\sigma(s,X_s,X_{s-\tau}) - \sigma_2'(s,X_s,X_{s-\tau})b(s,X_s,X_{s-\tau})}{\sigma^2(s,X_s,X_{s-\tau})}D_\theta X_s ds \\ &+ \int_{\theta+\tau}^t \frac{b_3'(s,X_s,X_{s-\tau})\sigma(s,X_s,X_{s-\tau}) + \sigma_3'(s,X_s,X_{s-\tau})b(s,X_s,X_{s-\tau})}{\sigma^2(s,X_s,X_{s-\tau})}D_\theta X_{s-\tau}ds \\ &+ \int_\theta^t \frac{1}{2}\big(\sigma_{22''}(s,X_s,X_{s-\tau}) + \sigma_{32''}(s,X_s,X_{s-\tau})\big)D_\theta X_s ds \\ &+ \int_{\theta+\tau}^t \frac{1}{2}\big(\sigma_{23''}(s,X_s,X_{s-\tau}) + \sigma_{33''}(s,X_s,X_{s-\tau})\big)D_\theta X_{s-\tau}ds.\end{aligned}$$

From the assumptions $(A_1),(A_2),(A_3),$ we obtain

$$|D_\theta I(t)| \le C\int_\theta^t |D_\theta X_s|ds + C\int_{\theta+\tau}^t |D_\theta X_{s-\tau}|ds \le C\int_\theta^t |D_\theta X_s|ds.$$

Combining with (12) and (13) we hence

$$|D_\theta Y_t| \le C\int_\theta^t |D_\theta Y_s|ds.$$

Using Gronwall inequality

$$|D_\theta Y_t| \le \exp C(t-\theta) \le C. \tag{14}$$

Consequently

$$|D_\theta X_t| \le C,\ 0 \le t \le T.$$

Proposition 4.2. Let $(X_t)_{t\in[-\tau,T]}$ be the unique solution to the equation (1). Assume that the assumptions $(A_1),(A_2)$ and (A_3) hold. In addition, we assume that $\|\sigma\|_0 := \inf_{(x,y)\in\mathbb{R}^2} |\sigma(x,y)| > a > 0$. Then, exists a finite constant $c>0$ such that

$$\int_0^t D_\theta X_t E[D_\theta X_t \mid \mathcal{F}_\theta] d\theta \ge ct \quad a.s. \text{ for all } t\in(0,T].$$

Proof. We have

$$D_\theta X_t = \sigma(\theta, X_\theta, X_{\theta-\tau}) + \int_\theta^t b_{2'}(s,X_s,X_{s-\tau}) D_\theta X_s ds + \int_\theta^t b_{3'}(s,X_s,X_{s-\tau}) D_\theta X_{s-\tau} \mathbf{1}_{[\theta+\tau,t]}(s) ds$$
$$+ \int_\theta^t \sigma_{2'}(s,X_s,X_{s-\tau}) D_\theta X_s dB_s + \int_\theta^t \sigma_{3'}(s,X_s,X_{s-\tau}) D_\theta X_{s-\tau} \mathbf{1}_{[\theta+\tau,t]}(s) dB_s,$$

with the initial condition $D_\theta X_t|_{t=\theta} = \sigma(\theta, X_\theta, X_{\theta-\tau})$. Hence, by the comparison theorem, $D_\theta X_t \ge 0$ when $\sigma(\theta, X_\theta, X_{\theta-\tau}) \ge 0$ and $D_\theta X_t \le 0$ when $\sigma(\theta, X_\theta, X_{\theta-\tau}) \le 0$. So, for all $\theta \le t \le T$,

$$D_\theta X_t E[D_\theta X_t \mid \mathcal{F}_\theta] \ge 0 \quad a.s.$$

For $\varepsilon\in(0,1]$, we define

$$h(t) = \int_{(1-\varepsilon)t}^t D_\theta X_t E[D_\theta X_t \mid \mathcal{F}_\theta] d\theta = \int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) D_\theta Y_t E\left[\sigma(t,X_t,X_{t-\tau}) D_\theta Y_t \mid \mathcal{F}_\theta\right] d\theta.$$

From the equation (13) we have $D_\theta Y_t = D_\theta I(t) + 1$. So, we can rewrite $h(t)$ as follows

$$h(t) = \int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) E\left[\sigma(t,X_t,X_{t-\tau}) \mid \mathcal{F}_\theta\right] d\theta$$
$$+ \int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) E\left[\sigma(t,X_t,X_{t-\tau}) D_\theta I(t) \mid \mathcal{F}_\theta\right] d\theta$$
$$+ \int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) D_\theta I(t) E\left[\sigma(t,X_t,X_{t-\tau}) \mid \mathcal{F}_\theta\right] d\theta$$
$$+ \int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) D_\theta I(t) E\left[\sigma(t,X_t,X_{t-\tau}) D_\theta I(t) \mid \mathcal{F}_\theta\right] d\theta.$$

By (14), $D_\theta I(t) \le C(t-\theta)$. Hence, we get

$$\left|\int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) E\left[\sigma(t,X_t,X_{t-\tau}) D_\theta I(t) \mid \mathcal{F}_\theta\right] d\theta\right| \le C\|\sigma\|_\infty^2 \int_{(1-\varepsilon)t}^t (t-\theta) d\theta \le Ct^2\varepsilon^2,$$
$$\left|\int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) D_\theta I(t) E\left[\sigma(t,X_t,X_{t-\tau}) \mid \mathcal{F}_\theta\right] d\theta\right| \le Ct^2\varepsilon^2,$$
$$\left|\int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) D_\theta I(t) E\left[\sigma(t,X_t,X_{t-\tau}) D_\theta I(t) \mid \mathcal{F}_\theta\right] d\theta\right| \le Ct^3\varepsilon^3.$$

Using the elementary inequatily $|a+b| \ge |a| - |b|$ yields

$$h(t) \ge \int_{(1-\varepsilon)t}^t \sigma(t,X_t,X_{t-\tau}) E\left[\sigma(t,X_t,X_{t-\tau}) \mid \mathcal{F}_\theta\right] d\theta - 2Ct^2\varepsilon^2 - Ct^3\varepsilon^3$$
$$\ge \int_{(1-\varepsilon)t}^t \|\sigma\|_0^2 d\theta - 2Ct^2\varepsilon^2 - Ct^3\varepsilon^3$$
$$\ge t\varepsilon\left(\|\sigma\|_0^2 - 2Ct\varepsilon - Ct^2\varepsilon^2\right).$$

We choose $\varepsilon \in (0,1]$ such that $2Ct\varepsilon + Ct^2\varepsilon^2 \le \frac{\|\sigma\|_0^2}{2}$, then we obtain

$$h(t) \ge \frac{\|\sigma\|_0^2}{2} t\varepsilon := ct.$$

The proof of Proposition is complete.

Theorem 4.1. The assumptions in Proposition 4.2 are satisfied and let $(X_t)_{t\in[-\tau,T]}$ be the unique solution to the equation (1). Then, for each $t \in (0,T]$, the law of randon variable X_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Moreover, the density ρ_{X_t} of X_t satisfies the bounds for all $x \in \mathbb{R}$

$$\frac{E|X_t - EX_t|}{2Ct}\exp\left(\frac{(x-EX_t)^2}{2ct}\right) \le \rho_{X_t}(x) \le \frac{E|X_t - EX_t|}{2ct}\exp\left(\frac{(x-EX_t)^2}{2Ct}\right), \tag{15}$$

where c, C are finite positive constants not depending on t.

Proof. For each $t \in (0,T]$, we consider the random variable $F := X_t - EX_t$. It is easy to show th F has mean zero and is Malliavin differentiable with $D_\theta F = D_\theta X_t$. Hence, by Proposition 4.1 and Proposition 4.2, we can get

$$ct \le \rho_F(x) \le Ct, \ \ x \in \mathbb{R},$$

where $\rho_F(x)$ is the density of the random variable F. From Lemma 2.2, we can conclude that the density ρ_F satisfies $\frac{E|F|}{2Ct}\exp\left(\frac{-x^2}{2ct}\right) \le \rho_F(x) \le \frac{E|F|}{2ct}\exp\left(\frac{-x^2}{2Ct}\right)$ $x \in \mathbb{R}$.

Moreover, $\rho_{X_t}(x) = \rho_F(x - EX_t)$. So, we obtain (15).

Theorem 4.2. Suppose the Assumptions (A_1) and (A_2). Let $(X_t)_{t\in[-\tau,T]}$ be the solution to the equation (1). In addition, we assume that $b(x,y)$ and $\sigma(x,y)$ are infinitely differentiable functions in x, y with bounded derivatives of all orders. Then, for each $t \in (0,T]$, the random variable X_t has an infinitely differentiable density with respect to Lebesgue measure on $\mathbb{R}$.

Proof. Fix $t \in (0,T]$, using Theorem 2.1.4 in [13], we have to check the following two properties:

i) $X_t \in \mathbb{D}^\infty = \bigcap_{i\ge1}\bigcap_{p\ge1} D^{i,p}$

ii) $E\left[\|DX_t\|^2\right]^{-1} \in \bigcap_{p\ge1} L^p(\Omega)$.

On the other hand, it is easy to show that the coefficients of equation (1) are infinitely differentiable in x and y with bounded partial derivatives of all orders. Therefore, we can deduce that $X_t \in \mathbb{D}^\infty$. So, property i) is checked. Now, let us check property ii)

From equation (11), using Holder and Burkholder-Davis-Gundy inequalities and Gronwall's lemma, we can verify that

$$\sup_{\theta,t\in[0,T]} E|D_\theta X_t|^p \le C, \ \ \forall p \ge 2, \tag{17}$$

where C is a positive constant. By using the fundamental inequality $(a+b+c)^2 \geq \frac{a^2}{2} - 2(b^2+c^2),$ we obtain from the equation (11) that

$$D_\theta X_t \geq \frac{1}{2}\sigma^2(X_\theta, X_{\theta-\tau}) - 2\Big(\int_\theta^t b_2'(s,X_s,X_{s-\tau})D_\theta X_s ds + \int_\theta^t b_3'(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)ds\Big)^2$$
$$-2\Big(\int_\theta^t \sigma_2(s,X_s,X_{s-\tau})D_\theta X_s dB_s + \int_\theta^t \sigma_3(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)dB_s\Big)^2, 0 \leq \theta \leq t \leq T.$$

For each $y \geq y_0 := \frac{4}{t\|\sigma_0\|^2}$, the real number $\varepsilon := \frac{4}{yt\|\sigma_0\|^2}$ belong to $(0,1]$. Hence,

$$\|DX_t\|^2_{L^2[0,T]} \geq \int_{t(1-\varepsilon)}^t |D_\theta X_t|^2 \, d\theta$$
$$\geq \int_{t(1-\varepsilon)}^t \frac{1}{2}\sigma^2(X_\theta, X_{\theta-\tau})d\theta$$
$$-2\int_{t(1-\varepsilon)}^t \Big(\int_\theta^t b_2(s,X_s,X_{s-\tau})D_\theta X_s ds + \int_\theta^t b_3(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)ds\Big)^2 d\theta$$
$$-2\int_{t(1-\varepsilon)}^t \Big(\int_\theta^t \sigma_2(s,X_s,X_{s-\tau})D_\theta X_s dB_s + \int_\theta^t \sigma_3(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)dB_s\Big)^2 d\theta$$
$$\geq \frac{\|\sigma_0\|^2 t\varepsilon}{2} - I_y(t) = \frac{2}{y} - I_y(t),$$

Where

$$I_y(t) = 2\int_{t(1-\varepsilon)}^t \Big(\int_\theta^t b_2(s,X_s,X_{s-\tau})D_\theta X_s ds + \int_\theta^t b_3(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)ds\Big)^2 d\theta$$
$$+2\int_{t(1-\varepsilon)}^t \Big(\int_\theta^t \sigma_2(s,X_s,X_{s-\tau})D_\theta X_s dB_s + \int_\theta^t \sigma_3(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)dB_s\Big)^2 d\theta.$$

By Markov inequality, we get

$$P\left(\| DX_t \|^2_{L^2[0,T]} \leq \frac{1}{y}\right) \leq P\left(\frac{2}{y} - I_y(t) \leq \frac{1}{y}\right) = P\left(I_y(t) \geq \frac{1}{y}\right) \tag{18}$$
$$\leq y^{\frac{q}{2}} E\left(|I_y(t)|^{\frac{q}{2}}\right), \forall q \geq 2. \tag{19}$$

By the inequality $(|a|+|b|)^{\frac{q}{2}} \leq 2^{\frac{q}{2}-1}\left(|a|^{\frac{q}{2}} + |b|^{\frac{q}{2}}\right)$, we obtain

$$E|I_y(t)|^{\frac{q}{2}} \leq 2^{q-1}\Bigg[E\Big(\int_{t(1-\varepsilon)}^t \Big(\int_\theta^t b_2(s,X_s,X_{s-\tau})D_\theta X_s ds + \int_\theta^t b_3(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)ds\Big)^2 d\theta\Big)^{\frac{q}{2}}$$
$$+E\Big(\int_{t(1-\varepsilon)}^t \Big(\int_\theta^t \sigma_2(s,X_s,X_{s-\tau})D_\theta X_s dB_s + \int_\theta^t \sigma_3(s,X_s,X_{s-\tau})D_\theta X_{s-\tau}\mathbf{1}_{[\theta+\tau,t]}(s)dB_s\Big)^2 d\theta\Big)^{\frac{q}{2}}\Bigg].$$

By using Holder and Burkholder-Davis-Gundy inequalities, we have

$$E|I_y(t)|^{\frac{q}{2}} \le C(t\varepsilon)^{\frac{q}{2}-1}\left(\int_{t(1-\varepsilon)}^{t}(t-\theta)^{\frac{q}{2}-1}\int_\theta^t E|D_\theta X_s|^p\, dsd\theta + \int_{t(1-\varepsilon)}^{t}\left(\int_\theta^t E|D_\theta X_s|^2\, ds\right)^{\frac{q}{2}}\right).$$

From (17), we imply that

$$E|I_y(t)|^{\frac{q}{2}} \le C(t\varepsilon)^{\frac{q}{2}-1}\left(\int_{t(1-\varepsilon)}^{t}(t-\theta)^{\frac{q}{2}}d\theta + \int_{t(1-\varepsilon)}^{t}(t-\theta)^{\frac{q}{2}}d\theta\right) \le C(t\varepsilon)^q = C\left(\frac{4}{y\|\sigma_0\|^2}\right)^q, \quad (20)$$

where C is positive constant not depending on t. Combining (19) and (20) we deduce

$$P\left(\|DX_t\|^2_{L^2[0,T]} \le \frac{1}{y}\right) \le Cy^{\frac{q}{2}}\left(\frac{4}{y\|\sigma_0\|^2}\right)^q, \quad \forall q\ge 2, y \ge y_0.$$

For any $p \ge 1$ and $q > 2p$, we have the following estimates

$$\begin{aligned} E\left[\frac{1}{\|DX_t\|^{2p}_{L^2[0,T]}}\right] &= \int_0^\infty py^{p-1}P\left(\|DX_t\|^{-2}_{L^2[0,T]} \ge y\right)dy \\ &\le \int_0^{y_0} py^{p-1}dy + \int_{y_0}^\infty py^{p-1}P\left(\|DX_t\|^2_{L^2[0,T]} < \frac{1}{y}\right)dy \\ &\le y_0^p + pC\int_{y_0}^\infty y^{p-1}y^{\frac{q}{2}}\left(\frac{4}{y\|\sigma_0\|^2}\right)^q dy \\ &= y_0^p + pC\left(\frac{4}{\|\sigma_0\|^2}\right)^p \frac{y_0^{p-\frac{q}{2}}}{p-\frac{q}{2}}. \end{aligned}$$

We recall here that $y_0 = \dfrac{4}{t\|\sigma_0\|^2}$. So we obtain

$$E\left[\frac{1}{\|DX\|^{2p}_{tL^2[0,T]}}\right] \le \infty, \ \forall p \ge 1.$$

The property ii) is proved. The proof of Theorem is completed.

5. Conclusion

In this paper, we utilized Malliavin calculus techniques to estimate the tail distribution and density of solutions for stochastic differential delay equations (SDDEs). Our contribution lies in our ability to obtain explicit estimates for tail distributions, as well as upper and lower Gaussian estimates for density. This work adds to the fundamental properties of SDDEs and enriches our understanding of them.

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