



Original Article

Static Stability of Piezoelectric Bio-inspired Sandwich Plate

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Abstract: This paper focuses on the static stability analysis of a sandwich plate structure resting on an elastic foundation and subjected to the simultaneous action of axial compressive loading and a uniform temperature rise. The sandwich plate consists of three layers, with the core made of a bio-inspired material and the two face sheets composed of piezoelectric materials, in which seven different fiber distribution patterns of the core layer are considered. The equilibrium and strain-compatibility equations are formulated based on Reddy's higher order shear deformation plate theory incorporating von Kármán geometric nonlinearity. These governing equations then are solved using the Bubnov–Galerkin method to derive the relationships between axial compressive load and deflection amplitude as well as the expressions for determining the critical axial compressive load. Subsequently, the effects of geometric parameters, elastic foundations coefficients, fiber distribution patterns and temperature increment on the static stability of the piezoelectric sandwich plate are investigated.

Keywords: Static stability; sandwich plate; bio-inspired material; piezoelectric material; temperature increment.

1. Introduction

Bio-inspired materials have emerged as a major trend in scientific research and technological development due to their ability to replicate nature's sophisticated structural principles and adaptive mechanisms. By mimicking hierarchically optimized features observed in natural systems such as bones, shells, bamboo, spider silk and crustacean exoskeletons, these materials achieve an exceptional balance of stiffness, strength, toughness and durability across multiple length scales. Such highly optimized hierarchical designs provide new pathways for developing sustainable, high-performance, and environmentally friendly materials for advanced applications including aerospace, soft robotics, biomedicine, energy systems and civil infrastructure. Garg et al., [1] investigated buckling and free

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vibration analysis of bio-inspired laminated sandwich plates with helicoidal/Bouligand face sheets containing soft core using higher-order zigzag theory. Kalhori et al., [2] presented torsional and axial buckling analysis of bio-inspired helicoidal laminated composite cylindrical shells, which are supported by a Winkler-type elastic foundation. Neubacher et al., [3] introduced experimental and numerical investigation of bio-inspired carbon fibre reinforced polymer structures with layer thickness gradients. Based on higher order quasi-3D kinematic plate theory, Karamanli et al., [4] presented comprehensive analysis of bio-inspired laminated composite plates using finite element models. Thu et al., [5] explored the static and transient responses of a bio-inspired helicoid laminated composite shell under the influence of an explosive load using an isomorphic method. Wang et al., [6] considered dynamic behavior of bio-inspired heterocyclic aramid fibre-reinforced laminates subjected to low-velocity drop-weight impact. Do and Pham [7] used iso-geometric analysis based on higher order shear deformation theory to study the dynamic response of bio-inspired helicoid laminated composite plates resting on Pasternak foundation excited by explosive loading. Lu et al., [8] carried out optimization design and secondary buckling of bio-inspired helicoidal laminated plates with linear, nonlinear recursive, Fibonacci, exponential and their combination-form arrangements under thermomechanical loads. Mohamed et al., [9] investigated the static, buckling and vibration behaviors of laminated composite plate structure with helicoidal orientation scheme using first order shear deformation plate theory.

Piezoelectric materials exhibit the unique ability to convert mechanical strain into electrical charge and vice versa, enabling direct coupling between mechanical and electrical fields. Owing to this electromechanical behavior, they are widely used in sensors, actuators, energy harvesters, and vibration control systems. These materials play a crucial role in smart structures and advanced engineering applications across aerospace, biomedical and civil engineering fields. Many studies have been published by researchers on the mechanical behaviors of various structures integrated with piezoelectric layers such as plates [10, 11], shells [12, 13], cylindrical panels [14, 15] and beams [16, 17]. Scientists have also employed a wide range of different methods to study piezoelectric materials including analytical methods [18, 19], the finite element method [20, 21] and experimental approach [22].

Bio-inspired materials are among the novel composite structures with significant potential for applications across many engineering fields. However, the mechanical behaviors of structures made from bio-inspired materials have not yet received sufficient attention. This study presents analytical approach to investigate the buckling and post-buckling of sandwich bio-inspired plate integrated with two piezoelectric layers. Seven distribution patterns of bio-inspired core are considered. The results on critical buckling loads and axial compressive load–deflection amplitude curves of sandwich plates provide a scientific basis for manufacturers and designers to select appropriate geometric dimensions and material distribution patterns, thereby ensuring the structural stability of the plates.

2. Problem Statement

Consider a sandwich plate with length a , width b and thickness h resting on elastic foundations, as shown in Figure 1. The sandwich plate consists of three layers, in which the core layer is made of a bio-inspired material and is integrated with two piezoelectric face sheets. The bio-inspired core is composed of N layers of equal thickness. Let h_c and h_p denote the thicknesses of the bio-inspired layer and the piezoelectric layer, respectively.

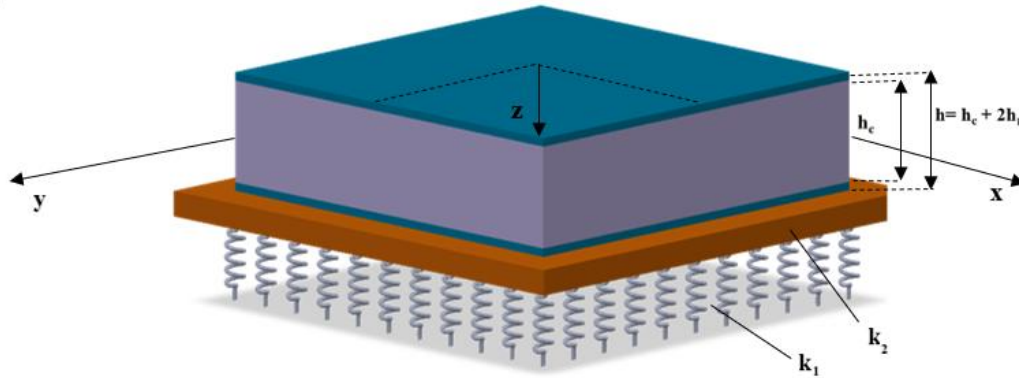


Figure 1. Modeling of piezoelectric bio-inspired composite plate.

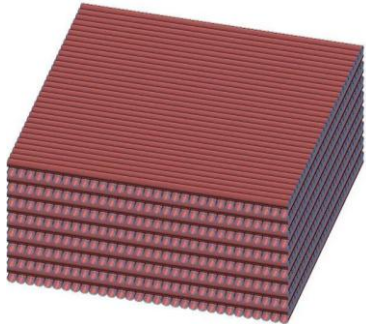
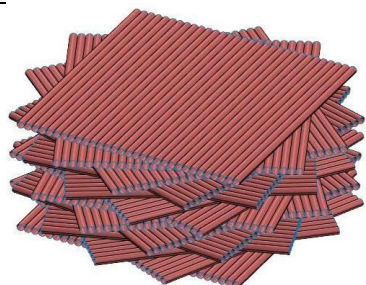
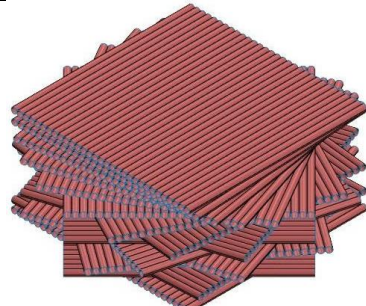
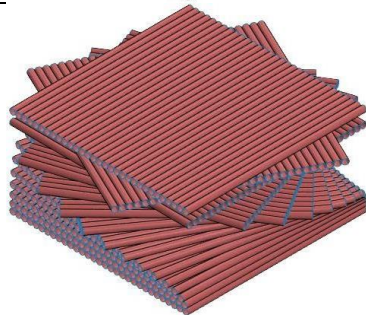
The mechanical properties of bio-inspired material are determined to depend on five values of temperature based on experiment results as shown in Table 1 [8]. In this study, seven types of carbon fiber distributions in the bio-inspired material are considered, with the fiber orientation distributions described in Table 2. G-1195N is chosen for the piezoelectric material with material properties are $E_p = 63 \text{ GPa}$, $\nu_p = 0.3$, $\alpha_p = 1.2 \times 10^{-10} \text{ } 1/^{\circ} \text{C}$, $d_{31} = d_{32} = 2.54 \times 10^{-10} \text{ m/V}$ [10].

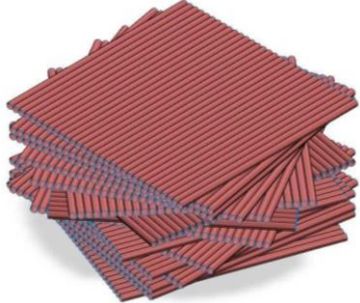
Table 1. Mechanical properties of bio-inspired materials

Temperature T_i	Mechanical properties								
	E_{11} (GPa)	E_{22} (GPa)	E_{33} (GPa)	G_{12} (GPa)	G_{13} (GPa)	G_{23} (GPa)	ν_{12}	ν_{13}	ν_{23}
218 K	120	11.67	11.67	7.3	7.3	2.672	0.33	0.33	0.35
258 K	120	11.01	11.01	7.149	7.149	2.521	0.33	0.33	0.35
300 K	120	11	11	6.478	6.478	2.518	0.33	0.33	0.35
318 K	120	10.28	10.28	6.167	6.167	2.355	0.33	0.33	0.35
355 K	120	9.65	9.65	5.713	5.713	2.21	0.33	0.33	0.35

Table 2. Seven distribution patterns of bio-inspired materials.

Distribution pattern	Fiber angle distribution	Description
UD	$[0^{\circ} / 0^{\circ} / 0^{\circ} / \dots 0^{\circ} / 0^{\circ} / 0^{\circ}]$	

CP	$\left[0^\circ / 90^\circ\right]_{8s}$	
QI	$\left[0^\circ / 45^\circ / 90^\circ / -45^\circ\right]_{4s}$	
LH	$\left[0^\circ / 24^\circ / 48^\circ / \dots / 360^\circ\right]_s$	
HR	$\left[0^\circ / 3^\circ / 9^\circ / \dots / 360^\circ\right]_s - \theta_n^\circ$ $= \theta_{n-1}^\circ + 3(n-1)$	
HS	$\left[0^\circ / 64.6^\circ / 89.8^\circ / \dots / 180^\circ\right]_s$ $\theta_n = \sqrt{\theta_{max}^2 - (12(n-1) - \theta_{max})^2}$, $\theta_{max} = 180^\circ$	

FH	$\begin{bmatrix} 0^\circ / 10^\circ / 10^\circ / 20^\circ / 30^\circ / \\ 50^\circ / 80^\circ / 130^\circ / 210^\circ / \\ 340^\circ / 190^\circ / 170^\circ / 360^\circ / \\ 50^\circ / 170^\circ / 170^\circ / 340^\circ \end{bmatrix}$	
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3. Governing Equations

Reddy's higher order shear deformation plate theory [23] is employed to establish the fundamental governing equations for the stability analysis of the sandwich plate. The strain components at points located at a distance z from the mid-surface are given as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} k_x^1 \\ k_y^1 \\ k_{xy}^1 \end{bmatrix} + z^3 \begin{bmatrix} k_x^3 \\ k_y^3 \\ k_{xy}^3 \end{bmatrix}, \quad \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{bmatrix} + z^2 \begin{bmatrix} k_{xz}^2 \\ k_{yz}^2 \end{bmatrix}, \quad (1)$$

where

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{I}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \frac{\partial v}{\partial y} + \frac{I}{2} \left(\frac{\partial w}{\partial y} \right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{bmatrix}, \quad \begin{bmatrix} \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x} + \phi_x \\ \frac{\partial w}{\partial y} + \phi_y \end{bmatrix}, \quad \begin{bmatrix} k_x^1 \\ k_y^1 \\ k_{xy}^1 \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{bmatrix}, \quad (2)$$

$$\begin{bmatrix} k_x^3 \\ k_y^3 \\ k_{xy}^3 \end{bmatrix} = -c_I \begin{bmatrix} \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}, \quad \begin{bmatrix} k_{xz}^2 \\ k_{yz}^2 \end{bmatrix} = -3c_I \begin{bmatrix} \frac{\partial w}{\partial x} + \phi_x \\ \frac{\partial w}{\partial y} + \phi_y \end{bmatrix},$$

and $\varepsilon_x^0, \varepsilon_y^0$ denote the normal strain components; $\gamma_{xy}^0, \gamma_{xz}^0, \gamma_{yz}^0$ are the shear strain components; u, v, w are the corresponding mid-surface displacement components, ϕ_x, ϕ_y are the rotations of the mid-surface normal about the y and x axes, respectively; moreover, $c_I = 4 / 3h^2$.

For the core layer made of a bio-inspired material, the relationship between the stress and strain components is expressed according to Hooke's law as follows [1, 7, 8]

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix}_c = \begin{pmatrix} \overline{Q_{11}^c} & \overline{Q_{12}^c} & 0 & 0 & \overline{Q_{16}^c} \\ \overline{Q_{21}^c} & \overline{Q_{22}^c} & 0 & 0 & \overline{Q_{26}^c} \\ 0 & 0 & \overline{Q_{44}^c} & \overline{Q_{45}^c} & 0 \\ 0 & 0 & \overline{Q_{45}^c} & \overline{Q_{55}^c} & 0 \\ \overline{Q_{16}^c} & \overline{Q_{26}^c} & 0 & 0 & \overline{Q_{66}^c} \end{pmatrix} \begin{pmatrix} \varepsilon_x - \alpha_{11} \Delta T \\ \varepsilon_y - \alpha_{22} \Delta T \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix}_c, \quad (3)$$

where ΔT is the temperature increment, the components $\overline{Q_{ij}^c}$ of the stiffness matrix are defined by $\overline{Q^c} = S Q^c S^T$ with

$$Q^c = \begin{pmatrix} Q_{11}^c & Q_{12}^c & 0 & 0 & 0 \\ Q_{21}^c & Q_{22}^c & 0 & 0 & 0 \\ 0 & 0 & Q_{44}^c & 0 & 0 \\ 0 & 0 & 0 & Q_{55}^c & 0 \\ 0 & 0 & 0 & 0 & Q_{66}^c \end{pmatrix}, \quad (4)$$

$$S = \begin{pmatrix} \cos^2 \alpha & \sin^2 \alpha & 0 & 0 & -2 \cos \alpha \sin \alpha \\ \sin^2 \alpha & \cos^2 \alpha & 0 & 0 & 2 \cos \alpha \sin \alpha \\ 0 & 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\ \cos \alpha \sin \alpha & -\cos \alpha \sin \alpha & 0 & 0 & \cos^2 \alpha - \sin^2 \alpha \end{pmatrix},$$

in which α is the angle between the fiber direction and the x direction and

$$Q_{11}^c = \frac{E_1^{(c)}}{1 - \nu_{12}^{(c)} \nu_{21}^{(c)}}, Q_{22}^c = \frac{E_2^{(c)}}{1 - \nu_{12}^{(c)} \nu_{21}^{(c)}}, Q_{21}^c = Q_{12}^c = \frac{\nu_{12}^{(c)} E_1^{(c)}}{1 - \nu_{12}^{(c)} \nu_{21}^{(c)}}, Q_{66}^c = G_{12}^{(c)}, Q_{44}^c = G_{23}^{(c)}, Q_{55}^c = G_{13}^{(c)}. \quad (5)$$

The constitutive stress-strain equations of piezoelectric layers are [10]

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{pmatrix}_p = \begin{pmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}_p \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}_p - \begin{pmatrix} \alpha_p \\ \alpha_p \\ 0 \\ 0 \\ 0 \end{pmatrix} \Delta T$$

$$- \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \quad (6)$$

where

$$\begin{aligned}
C_{11} = C_{22} &= \frac{E_p}{1 - \nu_p^2}, \quad C_{12} = \frac{\nu_p E_p}{1 - \nu_p^2}, \quad C_{44} = C_{55} = C_{66} = \frac{E_p}{2(1 + \nu_p)}, \\
e_{31} &= d_{31}C_{11} + d_{32}C_{12}, \quad e_{32} = d_{31}C_{12} + d_{32}C_{22}, \quad e_{24} = d_{24}C_{44}, \quad e_{15} = d_{15}Q_{55}^a, \\
\begin{bmatrix} E_x & E_y & E_z \end{bmatrix}^T &= \begin{bmatrix} 0 & 0 & V_p / h_p \end{bmatrix}^T,
\end{aligned} \tag{7}$$

with V_p is the voltage applied to two piezoelectric layers.

The internal force and moment components of the sandwich plate can be calculated using the following equations

$$\begin{aligned}
(N_j, M_j, P_j) &= \int_{-\frac{h_c}{2}}^{-\frac{h_c}{2} - h_p} \sigma_j^p(l, z, z^3) dz + \sum_{l=1}^N \int_{z_{l-1}}^{z_l} \sigma_j^c(l, z, z^3) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \sigma_j^p(l, z, z^3) dz, \quad j = x, y, xy, \\
(Q_j, R_j) &= \int_{-\frac{h_c}{2}}^{-\frac{h_c}{2} - h_p} \sigma_{jz}^p(l, z^2) dz + \sum_{l=1}^N \int_{z_{l-1}}^{z_l} \sigma_{jz}^c(l, z^2) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \sigma_{jz}^p(l, z^2) dz, \quad j = x, y,
\end{aligned} \tag{8}$$

Substituting Eqs. (1), (2) into Eqs. (3) and (6) then inserting results into Eq. (8), the results are obtained as the system of equations given below

$$\begin{aligned}
(N_x, N_y, N_{xy}) &= (A_{11}, A_{14}, A_{17}) \varepsilon_{0x} + (A_{12}, A_{15}, A_{18}) \varepsilon_{0y} + (A_{13}, A_{16}, A_{19}) \gamma_{0xy} \\
&+ (B_{11}, B_{14}, B_{17}) k_x^l + (B_{12}, B_{15}, B_{18}) k_y^l + (B_{13}, B_{16}, B_{19}) k_{xy}^l + (S_{11}, S_{14}, S_{17}) k_x^3 \\
&+ (S_{12}, S_{15}, S_{18}) k_y^3 + (S_{13}, S_{16}, S_{19}) k_{xy}^3 + (\Gamma_{11}, \Gamma_{12}, \Gamma_{13}) \Delta T + (\Phi_{11}, \Phi_{12}, 0) V_p,
\end{aligned} \tag{9a}$$

$$\begin{aligned}
(M_x, M_y, M_{xy}) &= (A_{20}, A_{23}, A_{26}) \varepsilon_{0x} + (A_{21}, A_{24}, A_{27}) \varepsilon_{0y} + (A_{22}, A_{25}, A_{28}) \gamma_{0xy} \\
&+ (B_{20}, B_{23}, B_{26}) k_x^l + (B_{21}, B_{24}, B_{27}) k_y^l + (B_{22}, B_{25}, B_{28}) k_{xy}^l + (S_{20}, S_{23}, S_{26}) k_x^3 \\
&+ (S_{21}, S_{24}, S_{27}) k_y^3 + (S_{22}, S_{25}, S_{28}) k_{xy}^3 + (\Gamma_{14}, \Gamma_{15}, \Gamma_{16}) \Delta T + (\Phi_{13}, \Phi_{14}, 0) V_p,
\end{aligned} \tag{9b}$$

$$\begin{aligned}
(P_x, P_y, P_{xy}) &= (A_{29}, A_{32}, A_{35}) \varepsilon_{0x} + (A_{30}, A_{33}, A_{36}) \varepsilon_{0y} + (A_{31}, A_{34}, A_{37}) \gamma_{0xy} \\
&+ (B_{29}, B_{32}, B_{35}) k_x^l + (B_{30}, B_{33}, B_{36}) k_y^l + (B_{31}, B_{34}, B_{37}) k_{xy}^l + (S_{29}, S_{32}, S_{35}) k_x^3 \\
&+ (S_{30}, S_{33}, S_{36}) k_y^3 + (S_{31}, S_{34}, S_{37}) k_{xy}^3 + (\Gamma_{17}, \Gamma_{18}, \Gamma_{19}) \Delta T + (\Phi_{15}, \Phi_{16}, 0) V_p,
\end{aligned} \tag{9c}$$

$$\begin{aligned}
(Q_x, Q_y, R_x, R_y) &= (F_{11}, F_{15}, F_{19}, F_{23}) \gamma_{0xz} + (F_{12}, F_{16}, F_{20}, F_{24}) k_{xz}^2 \\
&+ (F_{13}, F_{17}, F_{21}, F_{25}) \gamma_{0yz} + (F_{14}, F_{18}, F_{22}, F_{26}) k_{yz}^2,
\end{aligned} \tag{9d}$$

where the coefficients $A_{ij}, B_{ij}, S_{ij}, \Gamma_{ij}, \Phi_{ij}$ are given in Appendix A.

The equilibrium equations of the sandwich plate made of a bio-inspired material integrated with two piezoelectric layers, based on Reddy's higher order shear deformation theory, are expressed as follows [23]

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad (10a)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \quad (10b)$$

$$\begin{aligned} & \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - 3c_l \left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} \right) + c_l \left(\frac{\partial^2 P_x}{\partial x^2} + 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_y}{\partial y^2} \right) \\ & + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - k_l w + k_2 \nabla^2 w = 0, \end{aligned} \quad (10c)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + 3c_l R_x - c_l \left(\frac{\partial P_x}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) = 0, \quad (10d)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + 3c_l R_y - c_l \left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_y}{\partial y} \right) = 0. \quad (10e)$$

From Eq. (9a), the strain components at the mid-surface can be determined as

$$\begin{aligned} \varepsilon_x^0 &= N_{xy} I_{11} + N_y I_{12} + N_x I_{13} + k_x^l I_{14} + k_y^l I_{15} + k_{xy}^l I_{16} + k_x^3 I_{17} + k_y^3 I_{18} + k_{xy}^3 I_{19} \\ &+ \Delta T I_{20} + V_p I_{21}, \\ \varepsilon_y^0 &= N_{xy} I_{22} + N_y I_{23} + N_x I_{24} + k_x^l I_{25} + k_y^l I_{26} + k_{xy}^l I_{27} + k_x^3 I_{28} + k_y^3 I_{29} + k_{xy}^3 I_{30} \\ &+ \Delta T I_{31} + V_p I_{32}, \\ \gamma_{xy}^0 &= N_{xy} I_{33} + N_y I_{34} + N_x I_{35} + k_x^l I_{36} + k_y^l I_{37} + k_{xy}^l I_{38} + k_x^3 I_{39} + k_y^3 I_{40} + k_{xy}^3 I_{41} \\ &+ \Delta T I_{42} + V_p I_{43}, \end{aligned} \quad (11)$$

where the coefficients I_{ij} are given in Appendix B.

The stress function is chosen to satisfy the following condition as

$$N_x = \frac{\partial^2 f}{\partial y^2}, N_y = \frac{\partial^2 f}{\partial x^2}, N_{xy} = -\frac{\partial^2 f}{\partial x \partial y}. \quad (12)$$

Substituting Eqs. (11) and (12) into Eqs. (9b)–(9d), and then inserting the resulting expressions into Eqs. (10c)–(10e), yields the following equations

$$\begin{aligned} & G_{11} \frac{\partial^2 w}{\partial x^2} + G_{12} \frac{\partial^2 w}{\partial x \partial y} + G_{13} \frac{\partial^2 w}{\partial y^2} + G_{14} \frac{\partial \phi_x}{\partial x} + G_{15} \frac{\partial \phi_y}{\partial x} + G_{16} \frac{\partial \phi_x}{\partial y} + G_{17} \frac{\partial \phi_y}{\partial y} + G_{18} \frac{\partial^4 f}{\partial x^4} + G_{19} \frac{\partial^4 f}{\partial x^3 \partial y} \\ & + G_{20} \frac{\partial^4 f}{\partial x^2 \partial y^2} + G_{21} \frac{\partial^4 f}{\partial x \partial y^3} + G_{22} \frac{\partial^4 f}{\partial y^4} + G_{23} \frac{\partial^3 \phi_x}{\partial x^3} + G_{24} \frac{\partial^3 \phi_x}{\partial x^2 \partial y} + G_{25} \frac{\partial^3 \phi_x}{\partial x \partial y^2} + G_{26} \frac{\partial^3 \phi_x}{\partial y^3} + G_{27} \frac{\partial^3 \phi_y}{\partial x^3} \\ & + G_{28} \frac{\partial^3 \phi_y}{\partial x^2 \partial y} + G_{29} \frac{\partial^3 \phi_y}{\partial x \partial y^2} + G_{30} \frac{\partial^3 \phi_y}{\partial y^3} + G_{31} \frac{\partial^4 w}{\partial x^4} + G_{32} \frac{\partial^4 w}{\partial x^3 \partial y} + G_{33} \frac{\partial^4 w}{\partial x^2 \partial y^2} + G_{34} \frac{\partial^4 w}{\partial x \partial y^3} + G_{35} \frac{\partial^4 w}{\partial y^4} \\ & - k_l w + k_2 \nabla^2 w + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0, \end{aligned}$$

$$\begin{aligned}
& G_{36} \frac{\partial^3 f}{\partial x^3} + G_{37} \frac{\partial^3 f}{\partial x^2 \partial y} + G_{38} \frac{\partial^3 f}{\partial x \partial y^2} + G_{39} \frac{\partial^3 f}{\partial y^3} + G_{40} \frac{\partial^2 \phi_x}{\partial x^2} + G_{41} \frac{\partial^2 \phi_x}{\partial x \partial y} + G_{42} \frac{\partial^2 \phi_x}{\partial y^2} + G_{43} \frac{\partial^2 \phi_y}{\partial x^2} \\
& + G_{44} \frac{\partial^2 \phi_y}{\partial x \partial y} + G_{45} \frac{\partial^2 \phi_y}{\partial y^2} + G_{46} \frac{\partial^3 w}{\partial x^3} + G_{47} \frac{\partial^3 w}{\partial x^2 \partial y} + G_{48} \frac{\partial^3 w}{\partial x \partial y^2} + G_{49} \frac{\partial^3 w}{\partial y^3} + G_{50} \frac{\partial w}{\partial x} \\
& + G_{51} \frac{\partial w}{\partial y} + G_{52} \phi_x + G_{53} \phi_y = 0, \\
& G_{54} \frac{\partial^3 f}{\partial x^3} + G_{55} \frac{\partial^3 f}{\partial x^2 \partial y} + G_{56} \frac{\partial^3 f}{\partial x \partial y^2} + G_{57} \frac{\partial^3 f}{\partial y^3} + G_{58} \frac{\partial^2 \phi_x}{\partial x^2} + G_{59} \frac{\partial^2 \phi_x}{\partial x \partial y} + G_{60} \frac{\partial^2 \phi_x}{\partial y^2} + G_{61} \frac{\partial^2 \phi_y}{\partial x^2} \\
& + G_{62} \frac{\partial^2 \phi_y}{\partial x \partial y} + G_{63} \frac{\partial^2 \phi_y}{\partial y^2} + G_{64} \frac{\partial^3 w}{\partial x^3} + G_{65} \frac{\partial^3 w}{\partial x^2 \partial y} + G_{66} \frac{\partial^3 w}{\partial x \partial y^2} + G_{67} \frac{\partial^3 w}{\partial y^3} + G_{68} \frac{\partial w}{\partial x} + G_{69} \frac{\partial w}{\partial y} \\
& + G_{70} \phi_x + G_{71} \phi_y = 0,
\end{aligned} \tag{13}$$

in which coefficients G_{ij} are expressed in Appendix C.

The geometric compatibility equation of the smart sandwich plate is expressed by the following equation

$$\frac{\partial^2 \varepsilon_{0x}}{\partial y^2} + \frac{\partial^2 \varepsilon_{0y}}{\partial x^2} - \frac{\partial^2 \gamma_{0xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}. \tag{14}$$

Substituting Eqs. (11) and (12) into Eq. (14), the geometric compatibility equation becomes

$$\begin{aligned}
& \frac{\partial^4 f}{\partial x^4} H_{10} + \frac{\partial^4 f}{\partial x^3 \partial y} H_{11} + \frac{\partial^4 f}{\partial x^2 \partial y^2} H_{12} + \frac{\partial^4 f}{\partial x \partial y^3} H_{13} + \frac{\partial^4 f}{\partial y^4} H_{14} + \frac{\partial^3 \phi_x}{\partial x^3} H_{15} + \frac{\partial^3 \phi_x}{\partial x^2 \partial y} H_{16} \\
& + \frac{\partial^3 \phi_x}{\partial x \partial y^2} H_{17} + \frac{\partial^3 \phi_x}{\partial y^3} H_{18} + \frac{\partial^3 \phi_y}{\partial x^3} H_{19} + \frac{\partial^3 \phi_y}{\partial x^2 \partial y} H_{20} + \frac{\partial^3 \phi_y}{\partial x \partial y^2} H_{21} + \frac{\partial^3 \phi_y}{\partial y^3} H_{22} + \frac{\partial^4 w}{\partial x^4} H_{23} \\
& + \frac{\partial^4 w}{\partial x^3 \partial y} H_{24} + \frac{\partial^4 w}{\partial x^2 \partial y^2} H_{25} + \frac{\partial^4 w}{\partial x \partial y^3} H_{26} + \frac{\partial^4 w}{\partial y^4} H_{27} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2},
\end{aligned} \tag{15}$$

in which coefficients H_{ij} are expressed in Appendix D.

Four edges of the smart sandwich plate are assumed to be simply supported. In this case, the boundary conditions are

$$\begin{aligned}
w = u = \phi_y = M_x = N_{xy} = 0, \quad N_x = N_{x0} \quad \text{at } x = 0, a, \\
w = v = \phi_x = M_y = N_{xy} = 0, \quad N_y = N_{y0} \quad \text{at } y = 0, b,
\end{aligned} \tag{16}$$

Based on the above boundary conditions, the solutions are assumed in approximate form as the following double trigonometric functions as

$$\begin{aligned}
w(x, y, t) &= W(t) \sin \lambda_m x \sin \delta_n y, \\
\phi_x(x, y, t) &= \Phi_x(t) \cos \lambda_m x \sin \delta_n y, \\
\phi_y(x, y, t) &= \Phi_y(t) \sin \lambda_m x \cos \delta_n y,
\end{aligned} \tag{17}$$

with $\lambda_m = \frac{m\pi}{a}, \delta_n = \frac{n\pi}{b}$, m, n are the natural numbers and W, Φ_x, Φ_y are unknown functions.

Substituting Eq. (17) into Eq. (15), the form of the stress function is obtained as follows

$$f = L_1 \cos(2\lambda_m x) + L_2 \cos(2\delta_n y) - L_3 \sin(\lambda_m x) \sin(\delta_n y) + \frac{I}{2} N_{x0} y^2 + \frac{I}{2} N_{y0} x^2, \quad (18)$$

where

$$L_1 = Y_{11} W^2, L_2 = Y_{12} W^2, L_3 = Y_{13} W + Y_{14} \Phi_x + Y_{15} \Phi_y, L_4 = Y_{16} W + Y_{17} \Phi_x + Y_{18} \Phi_y, \quad (19)$$

with the coefficients Y_{ij} are given in Appendix E.

Substituting Eqs. (17) and (18) into the equilibrium equations (13) and applying the Bubnov–Galerkin method, three equations of the equilibrium system are obtained as

$$\begin{aligned} U_{10} W^2 + U_{11} W + U_{12} \Phi_x + U_{13} \Phi_y - N_{x0} \frac{m^2 \pi^2 b}{4a} W - N_{y0} \frac{n^2 \pi^2 a}{4b} W &= 0, \\ U_{14} W^2 + U_{15} W + U_{16} \Phi_x + U_{17} \Phi_y &= 0, \\ U_{18} W^2 + U_{19} W + U_{20} \Phi_x + U_{21} \Phi_y &= 0, \end{aligned} \quad (20)$$

where coefficients U_{ij} are given in Appendix F.

The smart sandwich plate is assumed to be subjected to a uniformly distributed axial compressive load along the x -direction with an intensity F_x . In this case, $N_{x0} = -F_x \cdot h$. The condition that the two edges $y=0, b$ of the smart sandwich plate are immovable can be fulfilled in an average sense as

$$\int_0^a \int_0^b \frac{\partial v}{\partial y} dy dx = 0. \quad (21)$$

By inserting Eqs. (17) and (18) into Eqs. (2) and (11) then subsequently substituting the obtained expressions into Eq. (21), we obtain

$$N_{y0} = \left(-\frac{X_1}{I_{23}ab} W^2 - \frac{X_2}{I_{23}ab} W + \frac{I_{24}h}{I_{23}} F_x - \Delta T \frac{I_{31}}{I_{23}} - V_p \frac{I_{32}}{I_{23}} \right), \quad (22)$$

in which coefficients X_1, X_2 are given in Appendix G.

By substituting Eq. (22) into Eq. (20) and eliminating the rotational amplitude unknowns, the axial compressive load–deflection amplitude relationship is obtained as given by the following equation

$$F_x = Z_0 W^2 + Z_1 W + Z_2 + Z_3 \Delta T + Z_4 V_p, \quad (23)$$

where

$$\begin{aligned} Z_0 &= \frac{X_{10}}{[X_{16} - X_{15}]}, Z_1 = \frac{X_{11}}{[X_{16} - X_{15}]}, \\ Z_2 &= \frac{X_{12}}{[X_{16} - X_{15}]}, Z_3 = \frac{X_{13}}{[X_{16} - X_{15}]}, Z_4 = \frac{X_{14}}{[X_{16} - X_{15}]}, \end{aligned} \quad (24)$$

The upper critical axial compressive load for the piezoelectric sandwich plate can be obtained by taking the limit of the function F_x as $W \rightarrow 0$, as follows

$$F_{xcr} = Z_2 + Z_3 \Delta T + Z_4 V_p. \quad (25)$$

4. Results and Discussion

To validate the reliability of the present results, the values of critical axial compressive load F_{xcr} (GPa) of a five layer composite plate with a fiber orientation sequence of $[0^\circ / 90^\circ / 0^\circ / 90^\circ / 0^\circ]$ are compared with the numerical results obtained using analytical methods reported by Lu et al., [8] and Reddy et al., [23], as summarized in Table 3. The material properties considered are $G_{12} = 0.6E_{22}$, $G_{23} = 0.5E_{22}$ and $\nu_{12} = \nu_{21} = 0.25$. In addition, the geometric dimensions are chosen as $a/b = 1$ and $b/h = 10$. The results demonstrate good agreement between the present results and those reported in the literature, with a maximum deviation of 5.99%. This confirms the accuracy of the theoretical formulation and the governing equations employed in this study.

Table 3. Comparison of the critical axial compressive load F_{xcr} (GPa) of five-layer composite plate.

Source	E_{11} / E_{22}			
	10	20	30	40
Present	10.454	16.488	21.629	26.067
Lu et al., [8]	9.960	15.653	20.466	24.593
Reddy et al., [23]	10.150	16.008	20.999	25.308
Maximum error	4.96%	5.33%	5.68%	5.99%

Table 4 presents the effects of the temperature increment ΔT (K), ratio a/b , two elastic foundation parameters and different fiber distribution patterns on the critical axial compressive load F_{xcr} (GPa) of a piezoelectric sandwich plate. From the parametric results, it can be concluded that the critical axial compressive load increases moderately with increasing Winkler and Pasternak foundation parameters due to the enhancement of the overall stiffness of the structure. However, as the temperature increment ΔT increases, the critical axial compressive load decreases significantly because of the reduction in stiffness and elastic moduli. Moreover, an increase in the ratio a/b leads to a decreasing trend in the critical buckling load, since a larger length-to-width ratio makes the plate more prone to bending and deformation, thereby reducing its load carrying capacity. Among the fiber distribution patterns considered, the UD and FH configurations exhibit the best critical buckling load, whereas the QI, LH, and HS configurations show the poorest performance.

The effects of the ratio b/h , applied voltage V_p (V) and different fiber distribution patterns on the critical axial compressive load F_{xcr} (GPa) of a piezoelectric sandwich plate are presented in Table 5. It is observed that the critical load decreases sharply as the ratio b/h increases, since the plate becomes thinner and its stiffness is reduced. On the other hand, the influence of the applied voltage V_p is negligible; specifically, increasing the voltage results in only a very small change in the critical load, although a slight reduction can be observed. This is because the thickness of the piezoelectric layers is small compared to that of the bio-inspired core, and the elastic modulus of the piezoelectric material is also lower. Considering the influence of fiber distribution patterns on the critical load, it is evident that the UD and FH configurations consistently provide the highest critical axial compressive loads, whereas the HS fiber distribution leads to the lowest critical axial compressive load.

Table 4. Effects of the temperature increment ΔT , elastic foundation's coefficients k_1, k_2 , the ratio a/b and fiber distribution patterns on the critical axial compressive load of piezoelectric bio-inspired sandwich plate

ΔT (K)	a/b	Fiber distribution patterns	k_1 (GPa/m), k_2 (GPa.m)			
			0, 0	0.1, 0.01	0.3, 0.02	0.5, 0.04
0	1	UD	13.2784	19.9914	24.6156	28.4657
		CP	10.0664	15.7975	20.6031	24.7487
		QI	8.2623	13.0324	17.1108	20.6399
		LH	8.5991	13.5260	17.6941	21.2950
		HR	11.3598	17.6205	22.6247	26.9078
		HS	8.4623	13.2324	17.3108	20.8399
		FH	12.5985	18.0986	23.3658	26.9871
100 K	1.5	UD	13.2228	19.9358	24.5600	28.4101
		CP	9.8974	15.6285	20.4341	24.5797
		QI	8.1553	12.9254	17.0037	20.5329
		LH	8.4942	13.4211	17.5892	21.1901
		HR	11.3293	17.5900	22.5942	26.8773
		HS	8.3553	13.3254	17.2037	20.7329
		FH	12.5985	18.9896	23.3568	26.9871
218 K	2	UD	13.1036	19.8165	24.4408	28.2099
		CP	9.5353	15.2664	20.0720	24.2176
		QI	7.9260	12.6961	16.7744	20.3036
		LH	8.2694	13.1963	17.3644	20.9653
		HR	11.2637	17.5244	22.5286	26.8117
		HS	8.1260	13.1961	16.9744	20.5036
		FH	12.3933	18.7845	23.1517	26.7819

Table 5. Effect of the ratio b/h , applied voltage V_p (V) and different fiber distribution patterns ratio on the critical axial compressive load of piezoelectric bio-inspired sandwich plate

b/h	V_p (V)	Fiber distribution patterns						
		UD	CP	QI	LH	HR	HS	FH
5	0	28.9046	22.0552	16.9699	18.7132	24.2733	17.1455	27.4210
	200	28.9045	22.0551	16.9699	18.7132	24.2733	17.1454	27.4210
	500	28.9045	22.0551	16.9698	18.7132	24.2732	17.1454	27.4209
10	0	21.9228	16.1746	12.8672	13.8122	20.3485	13.9778	19.7900
	200	21.9227	16.1746	12.8671	13.8121	20.3484	13.9777	19.7900
	500	21.9226	16.1744	12.8671	13.8121	20.3484	13.9776	19.7898
15	0	19.5818	15.3971	12.8039	13.3081	16.9423	12.6881	17.6701
	200	19.5817	15.3969	12.8038	13.3081	16.9422	12.6880	17.6700
	500	19.5816	15.3968	12.8037	13.3079	16.9420	12.6878	17.6698

The effects of the Winkler k_1 and Pasternak k_2 elastic foundation coefficients with the FH fiber distribution pattern on the axial compressive load–deflection amplitude curves of the piezoelectric sandwich plate are illustrated in Figures 2 and 3, respectively. The results indicate that increasing the Winkler and Pasternak foundation coefficients significantly enhances the loading carrying capacity of the sandwich plate. This improvement arises because the elastic foundation provides additional support,

thereby increasing the overall stiffness of the sandwich plate and consequently, its load carrying capability.

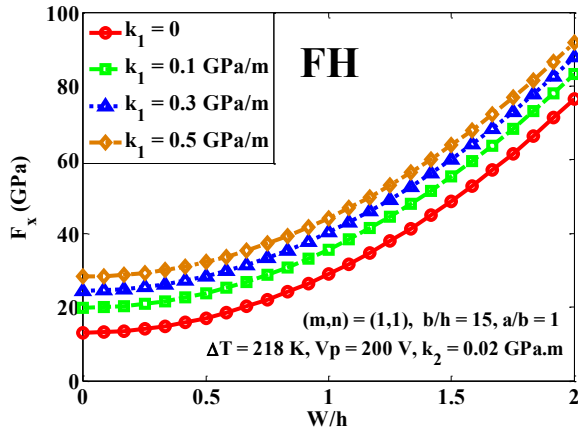


Figure 2. Effect of Winkler foundation coefficient on the axial compressive load–deflection amplitude curves of piezoelectric bio-inspired sandwich plate.

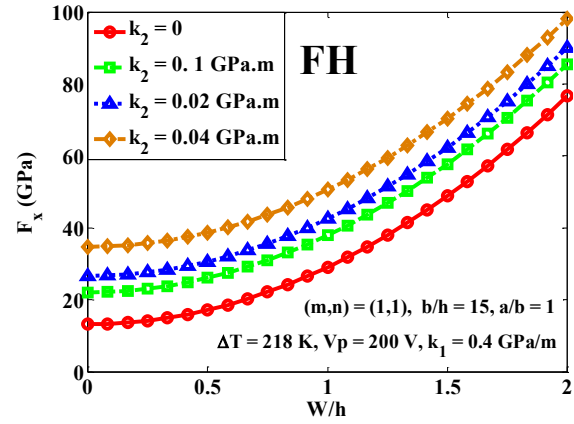


Figure 3. Effect of Pasternak foundation coefficient on the axial compressive load–deflection amplitude curves of piezoelectric bio-inspired sandwich plate.

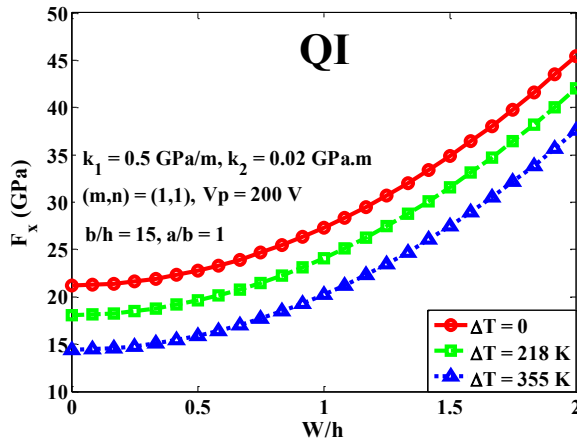


Figure 4. Effect of temperature increment on the axial compressive load–deflection amplitude curves of piezoelectric bio-inspired sandwich plate.

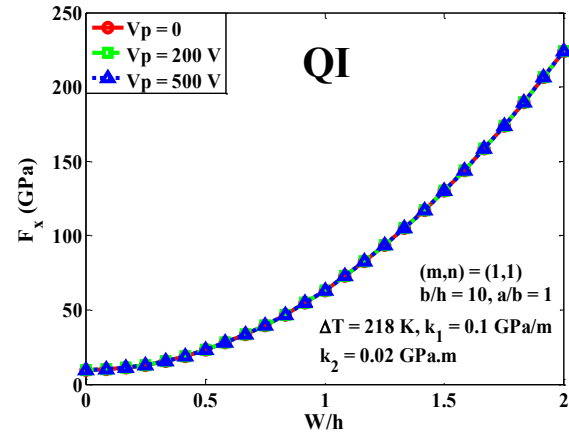


Figure 5. Effect of the applied voltage on the axial compressive load–deflection amplitude curves of piezoelectric bio-inspired sandwich plate.

Figure 4 illustrates the effects of the temperature increment $\Delta T (K)$ and with fiber distribution pattern QI on the axial compressive load–deflection amplitude curves of a bio-inspired piezoelectric sandwich plate. Three different values of the temperature increment, 0 , $100 K$ and $218 K$ are considered. The results clearly indicate that as the temperature increment increases, the load carrying capacity of the plate decreases significantly. This behavior can be explained by the fact that an increase in ΔT reduces the overall stiffness of the system, making the plate more prone to instability.

Figure 5 depicts the effects of the applied voltage $V_p (V)$ with fiber distribution pattern QI on the axial compressive load–deflection amplitude curves of a piezoelectric sandwich plate. It is readily observed that the influence of the applied voltage on the axial compressive load–deflection curves is

negligible; however, as V_p increases, the load-carrying capacity of the sandwich plate slightly decreases. This behavior can be attributed to the fact that the thickness of the two piezoelectric layers is much smaller than that of the core layer. Moreover, with increasing V_p , the electromechanical coupling alters the stress state in a manner that slightly reduces the load-carrying capacity.

Figure 6 illustrates the effects of the plate length-to-width ratio a/b with fiber distribution pattern HR on the axial compressive load–deflection amplitude curves of a piezoelectric sandwich plate. Three different values of the ratio $a/b = 1, 1.5, 2$ are considered. The results show that as the ratio a/b increases, the load-carrying capacity decreases significantly. This can be explained by the fact that an increase in a/b makes the plate longer and more susceptible to bending, thereby reducing its resistance to axial compressive loading.

The influence of seven different fiber distribution patterns on the axial compressive load–deflection amplitude curves of the piezoelectric sandwich plate is illustrated in Figure 7. The results indicate that, under identical material parameter conditions, sandwich plates with a UD fiber distribution pattern exhibit the highest load-carrying capacity. This can be explained by the fact that all fibers are aligned parallel to the loading direction, allowing the applied load to be transferred directly and maximally to the constituent fibers, which possess the highest strength and stiffness. In addition, sandwich plates with FH and HR fiber distribution patterns also demonstrate relatively superior load-carrying capacity, whereas the HS-type plate is the most susceptible to deformation among the seven distribution patterns.

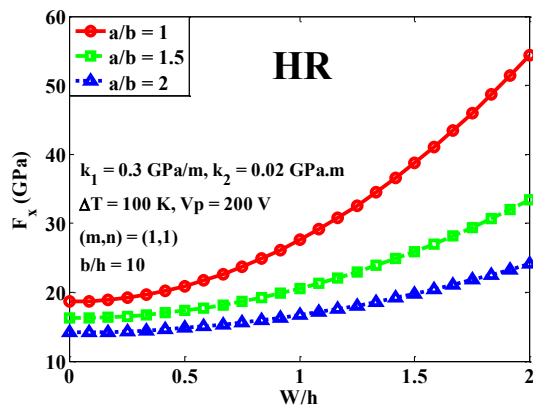


Figure 6. Effect of ratio a/b on the axial compressive load–deflection amplitude curves of piezoelectric bio-inspired sandwich plate.

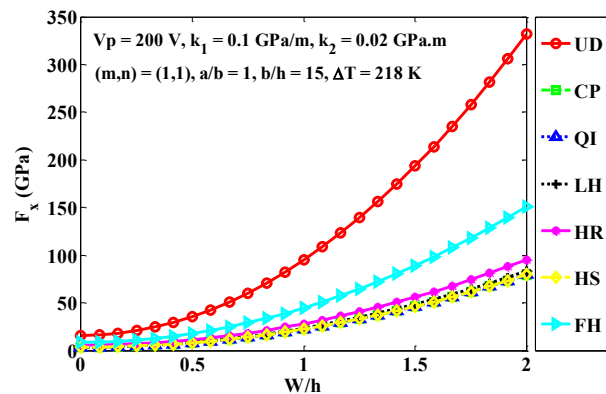


Figure 7. Effect of different fiber distribution patterns on the axial compressive load–deflection amplitude curves of piezoelectric bio-inspired sandwich plate.

5. Conclusions

The study employs Reddy's higher order shear deformation plate theory to establish the governing equations investigating the static stability of a bio-inspired sandwich plate integrated with two symmetrically bonded piezoelectric layers resting on elastic foundations. The critical buckling load and relationship between axial compressive load and deflection amplitude are determined. The main findings of the paper can be summarized as follows:

- The effects of temperature and applied voltage are manifested through a reduction in stiffness, critical load and load-carrying capacity. While the temperature increment has a pronounced influence, the applied voltage exhibits a negligible effect.

- Elastic foundations play beneficial role in enhancing the stiffness of the sandwich plate. Consequently, increasing the two foundation coefficients leads to higher critical buckling load and improved load carrying capacity.

- Geometrical parameters have a significant impact on the static stability of the sandwich plate. Specifically, increasing the ratios a/b and b/h results in a reduction in the critical load and load carrying capacity.

- Among the seven fiber distribution patterns of the bio-inspired material, sandwich plates with UD and FH distributions exhibit the highest critical buckling loads and load carrying capacities, whereas those with QI, LH and HS distributions show the poorest load carrying performance.

In this study, the formulations are developed under idealized assumptions regarding material behavior, geometry and boundary conditions, which may limit its applicability to more complex practical structures. Future studies could complement the present analytical approach by employing numerical techniques, such as the finite element method, to analyze more complicated geometries, loading conditions and coupled electromechanical effects.

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Appendixes

Appendix A

$$\begin{aligned}
 (A_{11}, B_{11}, S_{11}) &= C_{11} \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} (l, z, z^3) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (l, z, z^3) dz \right) + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{11}^f} (l, z, z^3) dz, \\
 (A_{12}, B_{12}, S_{12}) &= C_{12} \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} (l, z, z^3) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (l, z, z^3) dz \right) + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{12}^f} (l, z, z^3) dz, \\
 (A_{13}, B_{13}, S_{13}) &= \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{16}^f} (l, z, z^3) dz, \quad (A_{16}, B_{16}, S_{16}) = \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{26}^f} (l, z, z^3) dz, \\
 (A_{14}, B_{14}, S_{14}) &= C_{12} \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} (l, z, z^3) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (l, z, z^3) dz \right) + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{21}^f} (l, z, z^3) dz, \\
 (A_{15}, B_{15}, S_{15}) &= C_{22} \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} (l, z, z^3) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (l, z, z^3) dz \right) + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{22}^f} (l, z, z^3) dz, \\
 (A_{17}, B_{17}, S_{17}) &= \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{16}^f} (l, z, z^3) dz, \quad (A_{18}, B_{18}, S_{18}) = \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{26}^f} (l, z, z^3) dz, \\
 (A_{19}, B_{19}, S_{19}) &= C_{66} \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} (l, z, z^3) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (l, z, z^3) dz \right) + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{66}^f} (l, z, z^3) dz, \\
 (A_{20}, B_{20}, S_{20}) &= C_{11} \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} (z, z^2, z^4) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (z, z^2, z^4) dz \right) + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{11}^f} (z, z^2, z^4) dz, \\
 (A_{21}, B_{21}, S_{21}) &= C_{12} \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} (z, z^2, z^4) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (z, z^2, z^4) dz \right) + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{12}^f} (z, z^2, z^4) dz, \\
 \Gamma_{11} &= \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} \begin{pmatrix} -C_{11}\alpha_p \\ -C_{12}\alpha_p \end{pmatrix} dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} \begin{pmatrix} -C_{11}\alpha_p \\ -C_{12}\alpha_p \end{pmatrix} dz \right. \\
 &\quad \left. + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \left(\overline{Q_{11}^f} (-\alpha_{11}) + \overline{Q_{12}^f} (-\alpha_{22}) \right) dz \right), \quad \Phi_{11} = \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}} (-1) dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (-1) dz \end{pmatrix} \frac{e_{31}}{h_p},
 \end{aligned}$$

$$\begin{aligned}
\Gamma_{12} &= \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} \begin{pmatrix} -C_{12}\alpha_p \\ -C_{22}\alpha_p \end{pmatrix} dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} \begin{pmatrix} -C_{12}\alpha_p \\ -C_{22}\alpha_p \end{pmatrix} dz \right. \\
&\quad \left. + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \left(\overline{Q_{21}^f}(-\alpha_{11}) + \overline{Q_{22}^f}(-\alpha_{22}) \right) dz \right), \quad \Phi_{12} = \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} (-1) dz \right. \\
&\quad \left. + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (-1) dz \right) \frac{e_{32}}{h_p}, \\
\Gamma_{13} &= \left(\sum_{i=1}^N \int_{z_{i-1}}^{z_i} \left(-\overline{Q_{16}^f}\alpha_{11} - \overline{Q_{26}^f}\alpha_{22} \right) dz \right), \quad \Gamma_{16} = \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \left(-\overline{Q_{16}^f}\alpha_{11} - \overline{Q_{26}^f}\alpha_{22} \right) z dz, \\
\Gamma_{14} &= \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} \begin{pmatrix} -C_{11}\alpha_p \\ -C_{12}\alpha_p \end{pmatrix} z dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} \begin{pmatrix} -C_{11}\alpha_p \\ -C_{12}\alpha_p \end{pmatrix} z dz \right. \\
&\quad \left. + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \left(\overline{Q_{11}^f}(-\alpha_{11}) + \overline{Q_{12}^f}(-\alpha_{22}) \right) z dz \right), \quad \Phi_{13} = \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} (-z) dz \right. \\
&\quad \left. + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (-z) dz \right) \frac{e_{31}}{h_p}, \\
\Gamma_{18} &= \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} \begin{pmatrix} -C_{12}\alpha_p \\ -C_{22}\alpha_p \end{pmatrix} z dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} \begin{pmatrix} -C_{12}\alpha_p \\ -C_{22}\alpha_p \end{pmatrix} z dz \right. \\
&\quad \left. + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \left(\overline{Q_{21}^f}(-\alpha_{11}) + \overline{Q_{22}^f}(-\alpha_{22}) \right) z dz \right), \quad \Phi_{14} = \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} (-z) dz \right. \\
&\quad \left. + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (-z) dz \right) \frac{e_{32}}{h_p}, \\
\Gamma_{17} &= \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} \begin{pmatrix} -C_{11}\alpha_p \\ -C_{12}\alpha_p \end{pmatrix} z^3 dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} \begin{pmatrix} -C_{11}\alpha_p \\ -C_{12}\alpha_p \end{pmatrix} z^3 dz \right. \\
&\quad \left. + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \left(\overline{Q_{11}^f}(-\alpha_{11}) + \overline{Q_{12}^f}(-\alpha_{22}) \right) z^3 dz \right), \quad \Phi_{15} = \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} (-z^3) dz \right. \\
&\quad \left. + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (-z^3) dz \right) \frac{e_{31}}{h_p}, \\
\Gamma_{18} &= \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} \begin{pmatrix} -C_{12}\alpha_p \\ -C_{22}\alpha_p \end{pmatrix} z^3 dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} \begin{pmatrix} -C_{12}\alpha_p \\ -C_{22}\alpha_p \end{pmatrix} z^3 dz \right. \\
&\quad \left. + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \left(\overline{Q_{21}^f}(-\alpha_{11}) + \overline{Q_{22}^f}(-\alpha_{22}) \right) z^3 dz \right), \quad \Phi_{16} = \left(\int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} (-z^3) dz \right. \\
&\quad \left. + \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (-z^3) dz \right) \frac{e_{32}}{h_p},
\end{aligned}$$

$$\begin{aligned}
F_{11} &= \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} C_{55} dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} C_{55} dz + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{55}^f} dz \end{pmatrix}, \quad F_{12} = \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} (C_{55} z^2) dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (C_{55} z^2) dz + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{55}^f} z^2 dz \end{pmatrix}, \\
F_{15} &= \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} C_{44} dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} C_{44} dz + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{44}^f} dz \end{pmatrix}, \quad F_{16} = \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} C_{44} z^2 dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} C_{44} z^2 dz + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{44}^f} z^2 dz \end{pmatrix}, \\
F_{17} &= \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{45}^f} dz, \quad F_{18} = \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{45}^f} z^2 dz, \\
F_{19} &= \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} C_{55} z^2 dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} C_{55} z^2 dz + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{55}^f} z^2 dz \end{pmatrix}, \quad F_{20} = \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} (C_{55} z^4) dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} (C_{55} z^4) dz + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{55}^f} z^4 dz \end{pmatrix}, \\
F_{21} &= \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{45}^f} z^2 dz, \quad F_{22} = \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{45}^f} z^4 dz, \\
F_{23} &= \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} C_{44} z^2 dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} C_{44} z^2 dz + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{44}^f} z^2 dz \end{pmatrix}, \quad F_{24} = \begin{pmatrix} \int_{\frac{-h_c}{2}}^{\frac{-h_c}{2}-h_p} C_{44} z^4 dz \\ \int_{\frac{h_c}{2}}^{\frac{h_c}{2}+h_p} C_{44} z^4 dz + \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{44}^f} z^4 dz \end{pmatrix}, \\
F_{25} &= \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{45}^f} z^2 dz, \quad F_{26} = \sum_{i=1}^N \int_{z_{i-1}}^{z_i} \overline{Q_{45}^f} z^4 dz.
\end{aligned}$$

Appendix B

$$I_{22} = \frac{(-A_{13}A_{14} + A_{11}A_{16})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})},$$

$$\begin{aligned}
I_{23} &= \frac{(A_{13}A_{17} - A_{11}A_{19})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{24} &= \frac{(-A_{16}A_{17} + A_{14}A_{19})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{25} &= \frac{((A_{16}A_{17} - A_{14}A_{19})B_{11} + (A_{11}A_{19} - A_{13}A_{17})B_{14} + (A_{13}A_{14} - A_{11}A_{16})B_{17})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{26} &= \frac{((A_{16}A_{17} - A_{14}A_{19})B_{12} + (A_{11}A_{19} - A_{13}A_{17})B_{15} + (A_{13}A_{14} - A_{11}A_{16})B_{18})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{27} &= \frac{((A_{16}A_{17} - A_{14}A_{19})B_{13} + (A_{11}A_{19} - A_{13}A_{17})B_{16} + (A_{13}A_{14} - A_{11}A_{16})B_{19})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{28} &= \frac{((A_{16}A_{17} - A_{14}A_{19})S_{11} + (A_{11}A_{19} - A_{13}A_{17})S_{14} + (A_{13}A_{14} - A_{11}A_{16})S_{17})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{29} &= \frac{((A_{16}A_{17} - A_{14}A_{19})S_{12} + (A_{11}A_{19} - A_{13}A_{17})S_{15} + (A_{13}A_{14} - A_{11}A_{16})S_{18})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{30} &= \frac{((A_{16}A_{17} - A_{14}A_{19})S_{13} + (A_{11}A_{19} - A_{13}A_{17})S_{16} + (A_{13}A_{14} - A_{11}A_{16})S_{19})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{31} &= \frac{((A_{16}A_{17} - A_{14}A_{19})\Gamma_{11} + (A_{11}A_{19} - A_{13}A_{17})\Gamma_{12} + (A_{13}A_{14} - A_{11}A_{16})\Gamma_{13})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}, \\
I_{32} &= \frac{((A_{16}A_{17} - A_{14}A_{19})\Phi_{11} + (A_{11}A_{19} - A_{13}A_{17})\Phi_{12})}{A_{13}(A_{15}A_{17} - A_{14}A_{18}) + A_{12}(-A_{16}A_{17} + A_{14}A_{19}) + A_{11}(A_{16}A_{18} - A_{15}A_{19})}.
\end{aligned}$$

Appendix C

$$\begin{aligned}
G_{11} &= (D_{77} - 3c_l D_{81}), \quad G_{12} = (D_{78} - 3c_l D_{83}) + (D_{79} - 3c_l D_{85}), \quad G_{13} = (D_{80} - 3c_l D_{87}), \\
G_{14} &= (D_{77} - 3c_l D_{82}), \quad G_{15} = (D_{78} - 3c_l D_{84}), \quad G_{16} = (D_{79} - 3c_l D_{86}), \\
G_{17} &= (D_{80} - 3c_l D_{88}), \quad G_{18} = c_l D_{45}, \quad G_{19} = c_l (2D_{67} - D_{44}), \\
G_{20} &= c_l (D_{46} - 2D_{66} + D_{56}), \quad G_{21} = c_l (2D_{68} - D_{55}), \quad G_{22} = c_l D_{57}, \quad G_{23} = c_l \overline{D_{47}}, \\
G_{24} &= c_l (\overline{D_{49}} + 2\overline{D_{69}}), \quad G_{25} = c_l (\overline{D_{58}} + 2\overline{D_{71}}), \quad G_{26} = c_l \overline{D_{60}}, \quad G_{27} = c_l \overline{D_{50}}, \\
D_{11} &= (A_{20}I_{11} + A_{21}I_{22} + A_{22}I_{33}), \quad D_{12} = (A_{20}I_{12} + A_{21}I_{23} + A_{22}I_{34}), \\
D_{13} &= (A_{20}I_{13} + A_{21}I_{24} + A_{22}I_{35}), \quad D_{14} = (A_{20}I_{14} + A_{21}I_{25} + A_{22}I_{36} + B_{20}), \\
D_{15} &= (A_{20}I_{15} + A_{21}I_{26} + A_{22}I_{37} + B_{21}), \quad D_{16} = (A_{20}I_{16} + A_{21}I_{27} + A_{22}I_{38} + B_{22}), \\
D_{17} &= (A_{20}I_{17} + A_{21}I_{28} + A_{22}I_{39} + S_{20}), \quad D_{18} = (A_{20}I_{18} + A_{21}I_{29} + A_{22}I_{40} + S_{21}), \\
D_{19} &= (A_{20}I_{19} + A_{21}I_{30} + A_{22}I_{41} + S_{22}), \quad D_{20} = (A_{20}I_{20} + A_{21}I_{31} + A_{22}I_{42} + \Gamma_{14}),
\end{aligned}$$

$$\begin{aligned}
\overline{D_{20}} &= -2c_1 D_{19}, \quad D_{22} = (A_{23}I_{11} + A_{24}I_{22} + A_{25}I_{33}), \quad D_{23} = (A_{23}I_{12} + A_{24}I_{23} + A_{25}I_{34}), \\
D_{24} &= (A_{23}I_{13} + A_{24}I_{24} + A_{25}I_{35}), \quad D_{25} = (A_{23}I_{14} + A_{24}I_{25} + A_{25}I_{36} + B_{23}), \\
D_{26} &= (A_{23}I_{15} + A_{24}I_{26} + A_{25}I_{37} + B_{24}), \quad D_{27} = (A_{23}I_{16} + A_{24}I_{27} + A_{25}I_{38} + B_{25}), \\
D_{28} &= (A_{23}I_{17} + A_{24}I_{28} + A_{25}I_{39} + S_{23}), \quad D_{29} = (A_{23}I_{18} + A_{24}I_{29} + A_{25}I_{40} + S_{24}), \\
D_{30} &= (A_{23}I_{19} + A_{24}I_{30} + A_{25}I_{41} + S_{25}), \quad D_{31} = (A_{23}I_{20} + A_{24}I_{31} + A_{25}I_{42} + \Gamma_{15}), \\
D_{32} &= (A_{23}I_{21} + A_{24}I_{32} + A_{25}I_{43} + \Phi_{14}), \quad \overline{D_{25}} = (D_{25} - c_1 D_{28}), \\
\overline{D_{26}} &= (D_{26} - c_1 D_{29}), \quad \overline{D_{27}} = (D_{27} - c_1 D_{30}), \quad \overline{D_{28}} = (D_{27} - c_1 D_{30}), \\
\overline{D_{29}} &= -c_1 D_{28}, \quad \overline{D_{30}} = -c_1 D_{29}, \quad \overline{D_{31}} = -2c_1 D_{30}, \quad D_{33} = (A_{26}I_{11} + A_{27}I_{22} + A_{28}I_{33}), \\
D_{34} &= (A_{26}I_{12} + A_{27}I_{23} + A_{28}I_{34}), \quad D_{35} = (A_{26}I_{13} + A_{27}I_{24} + A_{28}I_{35}), \\
D_{36} &= (A_{26}I_{14} + A_{27}I_{25} + A_{28}I_{36} + B_{26}), \quad D_{37} = (A_{26}I_{15} + A_{27}I_{26} + A_{28}I_{37} + B_{27}), \\
D_{38} &= (A_{26}I_{16} + A_{27}I_{27} + A_{28}I_{38} + B_{28}), \quad D_{39} = (A_{26}I_{17} + A_{27}I_{28} + A_{28}I_{39} + S_{26}), \\
D_{40} &= (A_{26}I_{18} + A_{27}I_{29} + A_{28}I_{40} + S_{27}), \quad D_{41} = (A_{26}I_{19} + A_{27}I_{30} + A_{28}I_{41} + S_{28}), \\
D_{42} &= (A_{26}I_{20} + A_{27}I_{31} + A_{28}I_{42} + \Gamma_{16}), \quad D_{43} = (A_{26}I_{21} + A_{27}I_{32} + A_{28}I_{43}), \\
\overline{D_{36}} &= (D_{36} - c_1 D_{39}), \quad \overline{D_{37}} = (D_{37} - c_1 D_{40}), \quad \overline{D_{38}} = (D_{38} - c_1 D_{41}), \\
\overline{D_{39}} &= (D_{38} - c_1 D_{41}), \quad \overline{D_{40}} = -c_1 D_{39}, \quad \overline{D_{41}} = -c_1 D_{40}, \quad \overline{D_{42}} = -2c_1 D_{41}, \\
D_{44} &= (A_{29}I_{11} + A_{30}I_{22} + A_{31}I_{33}), \quad D_{45} = (A_{29}I_{12} + A_{30}I_{23} + A_{31}I_{34}), \\
D_{46} &= (A_{29}I_{13} + A_{30}I_{24} + A_{31}I_{35}), \quad D_{47} = (A_{29}I_{14} + A_{30}I_{25} + A_{31}I_{36} + B_{29}), \\
D_{48} &= (A_{29}I_{15} + A_{30}I_{26} + A_{31}I_{37} + B_{30}), \quad D_{49} = (A_{29}I_{16} + A_{30}I_{27} + A_{31}I_{38} + B_{31}), \\
D_{50} &= (A_{29}I_{17} + A_{30}I_{28} + A_{31}I_{39} + S_{29}), \quad D_{51} = (A_{29}I_{18} + A_{30}I_{29} + A_{31}I_{40} + S_{30}), \\
D_{52} &= (A_{29}I_{19} + A_{30}I_{30} + A_{31}I_{41} + S_{31}), \quad D_{53} = (A_{29}I_{20} + A_{30}I_{31} + A_{31}I_{42} + \Gamma_{17}), \\
D_{54} &= (A_{29}I_{21} + A_{30}I_{32} + A_{31}I_{43} + \Phi_{15}), \quad \overline{D_{47}} = (D_{47} - c_1 D_{50}), \quad \overline{D_{48}} = (D_{48} - c_1 D_{51}), \\
\overline{D_{49}} &= (D_{49} - c_1 D_{52}), \quad \overline{D_{50}} = (D_{49} - c_1 D_{52}), \quad \overline{D_{51}} = -c_1 D_{50}, \quad \overline{D_{52}} = -c_1 D_{51}, \quad \overline{D_{53}} = -2c_1 D_{52}, \\
D_{55} &= (A_{32}I_{11} + A_{33}I_{22} + A_{34}I_{33}), \quad D_{56} = (A_{32}I_{12} + A_{33}I_{23} + A_{34}I_{34}), \\
D_{57} &= (A_{32}I_{13} + A_{33}I_{24} + A_{34}I_{35}), \quad D_{58} = (A_{32}I_{14} + A_{33}I_{25} + A_{34}I_{36} + B_{32}), \\
D_{59} &= (A_{32}I_{15} + A_{33}I_{26} + A_{34}I_{37} + B_{33}), \quad D_{60} = (A_{32}I_{16} + A_{33}I_{27} + A_{34}I_{38} + B_{34}), \\
D_{61} &= (A_{32}I_{17} + A_{33}I_{28} + A_{34}I_{39} + S_{32}), \quad D_{62} = (A_{32}I_{18} + A_{33}I_{29} + A_{34}I_{40} + S_{33}), \\
D_{63} &= (A_{32}I_{19} + A_{33}I_{30} + A_{34}I_{41} + S_{34}), \quad D_{64} = (A_{32}I_{20} + A_{33}I_{31} + A_{34}I_{42} + \Gamma_{18}), \\
D_{65} &= (A_{32}I_{21} + A_{33}I_{32} + A_{34}I_{43} + \Phi_{16}), \quad \overline{D_{58}} = (D_{58} - c_1 D_{61}), \quad \overline{D_{59}} = (D_{59} - c_1 D_{62}), \\
\overline{D_{60}} &= (D_{60} - c_1 D_{63}), \quad \overline{D_{61}} = (D_{60} - c_1 D_{63}), \quad \overline{D_{62}} = -c_1 D_{61}, \quad \overline{D_{63}} = -c_1 D_{62}, \quad \overline{D_{64}} = -2c_1 D_{63}, \\
D_{66} &= (A_{35}I_{11} + A_{36}I_{22} + A_{37}I_{33}), \quad D_{67} = (A_{35}I_{12} + A_{36}I_{23} + A_{37}I_{34}), \\
D_{68} &= (A_{35}I_{13} + A_{36}I_{24} + A_{37}I_{35}), \quad D_{69} = (A_{35}I_{14} + A_{36}I_{25} + A_{37}I_{36} + B_{35}), \\
D_{70} &= (A_{35}I_{15} + A_{36}I_{26} + A_{37}I_{37} + B_{36}), \quad D_{71} = (A_{35}I_{16} + A_{36}I_{27} + A_{37}I_{38} + B_{37}),
\end{aligned}$$

$$\begin{aligned}
D_{72} &= (A_{35}I_{17} + A_{36}I_{28} + A_{37}I_{39} + S_{35}), \quad D_{73} = (A_{35}I_{18} + A_{36}I_{29} + A_{37}I_{40} + S_{36}), \\
D_{74} &= (A_{35}I_{19} + A_{36}I_{30} + A_{37}I_{41} + S_{37}), \quad D_{75} = (A_{35}I_{20} + A_{36}I_{31} + A_{37}I_{42} + \Gamma_{19}), \\
D_{76} &= (A_{35}I_{21} + A_{36}I_{32} + A_{37}I_{43}), \quad \overline{D_{69}} = (D_{69} - c_l D_{72}), \quad \overline{D_{70}} = (D_{70} - c_l D_{73}), \\
\overline{D_{71}} &= (D_{71} - c_l D_{74}), \quad \overline{D_{72}} = (D_{71} - c_l D_{74}), \quad \overline{D_{73}} = -c_l D_{72}, \quad \overline{D_{74}} = -c_l D_{73}, \quad \overline{D_{75}} = -2c_l D_{74}, \\
D_{77} &= (F_{11} - 3c_l F_{12}), \quad D_{77} = (F_{11} - 3c_l F_{12}), \quad D_{78} = (F_{13} - 3c_l F_{14}), \quad D_{78} = (F_{13} - 3c_l F_{14}), \\
D_{79} &= (F_{15} - 3c_l F_{16}), \quad D_{79} = (F_{15} - 3c_l F_{16}), \quad D_{80} = (F_{17} - 3c_l F_{18}), \quad D_{80} = (F_{17} - 3c_l F_{18}), \\
D_{81} &= (F_{19} - 3c_l F_{20}), \quad D_{82} = (F_{19} - 3c_l F_{20}), \quad D_{83} = (F_{21} - 3c_l F_{22}), \quad D_{84} = (F_{21} - 3c_l F_{22}), \\
D_{85} &= (F_{23} - 3c_l F_{24}), \quad D_{86} = (F_{23} - 3c_l F_{24}), \quad D_{87} = (F_{25} - 3c_l F_{26}), \quad D_{88} = (F_{25} - 3c_l F_{26}).
\end{aligned}$$

Appendix D

$$\begin{aligned}
H_{10} &= I_{23}, \quad H_{11} = (-I_{22} - I_{34}), \quad H_{12} = (I_{33} + I_{24} + I_{12}), \quad H_{13} = (-I_{35} - I_{11}), \\
H_{14} &= I_{13}, \quad H_{15} = (I_{25} - c_l I_{28}), \quad H_{16} = (I_{27} - I_{36} + c_l I_{39} - c_l I_{30}), \\
H_{17} &= (c_l I_{41} + I_{14} - I_{38} - c_l I_{17}), \quad H_{18} = (I_{16} - c_l I_{19}), \quad H_{19} = (I_{27} - c_l I_{30}), \\
H_{20} &= (I_{26} - c_l I_{29} + c_l I_{41} - I_{38}), \quad H_{21} = (I_{16} - I_{37} - c_l I_{19} + c_l I_{40}), \quad H_{22} = (I_{15} - c_l I_{18}), \\
H_{23} &= -c_l I_{28}, \quad H_{24} = (c_l I_{39} - 2c_l I_{30}), \quad H_{25} = (2c_l I_{41} - c_l I_{17} - c_l I_{29}), \\
H_{26} &= (c_l I_{40} - 2c_l I_{19}), \quad H_{27} = -c_l I_{18}.
\end{aligned}$$

Appendix E

$$\begin{aligned}
Y_{11} &= \frac{\delta_n^2}{32H_{10}\lambda_m^2}, \quad Y_{12} = \frac{\lambda_m^2}{32H_{14}\delta_n^2}, \\
\text{mau_}L_3 &= H_{14}^2\delta_n^8 - H_{13}^2\delta_n^6\lambda_m^2 + H_{12}^2\delta_n^4\lambda_m^4 - 2H_{11}H_{13}\delta_n^4\lambda_m^4 - H_{11}^2\delta_n^2\lambda_m^6 + 2H_{10}H_{12}\delta_n^2\lambda_m^6 \\
&\quad + H_{10}^2\lambda_m^8 + 2H_{14}(H_{12}\delta_n^6\lambda_m^2 + H_{10}\delta_n^4\lambda_m^4), \\
Y_{13} &= \frac{H_{14} \begin{pmatrix} H_{27}\delta_n^8 \\ +H_{25}\delta_n^6\lambda_m^2 \\ +H_{23}\delta_n^4\lambda_m^4 \end{pmatrix} + \lambda_m^2 \left[-H_{13} \begin{pmatrix} H_{26}\delta_n^6 \\ +H_{24}\delta_n^4\lambda_m^2 \end{pmatrix} + H_{12} \begin{pmatrix} H_{27}\delta_n^6 \\ +H_{25}\delta_n^4\lambda_m^2 + H_{23}\delta_n^2\lambda_m^4 \end{pmatrix} \right. \\
&\quad \left. + \lambda_m^2 \begin{pmatrix} -H_{11} \begin{pmatrix} H_{26}\delta_n^4 \\ +H_{24}\delta_n^2\lambda_m^2 \end{pmatrix} + H_{10} \begin{pmatrix} H_{27}\delta_n^4 \\ +H_{25}\delta_n^2\lambda_m^2 + H_{23}\lambda_m^4 \end{pmatrix} \end{pmatrix} \right]}{H_{14}^2\delta_n^8 - H_{13}^2\delta_n^6\lambda_m^2 + H_{12}^2\delta_n^4\lambda_m^4 - 2H_{11}H_{13}\delta_n^4\lambda_m^4 - H_{11}^2\delta_n^2\lambda_m^6 + 2H_{10}H_{12}\delta_n^2\lambda_m^6 \\
&\quad + H_{10}^2\lambda_m^8 + 2H_{14}(H_{12}\delta_n^6\lambda_m^2 + H_{10}\delta_n^4\lambda_m^4)}, \\
Y_{14} &= \frac{\lambda_m \left[H_{14} \begin{pmatrix} H_{17}\delta_n^6 + H_{15}\delta_n^4\lambda_m^2 \end{pmatrix} - H_{13} \begin{pmatrix} H_{18}\delta_n^6 + H_{16}\delta_n^4\lambda_m^2 \end{pmatrix} \right. \\
&\quad \left. + \lambda_m^2 \left[H_{10}\lambda_m^2 \begin{pmatrix} H_{17}\delta_n^2 \\ +H_{15}\lambda_m^2 \end{pmatrix} + H_{12} \begin{pmatrix} H_{17}\delta_n^4 \\ +H_{15}\delta_n^2\lambda_m^2 \end{pmatrix} - H_{11} \begin{pmatrix} H_{18}\delta_n^4 \\ +H_{16}\delta_n^2\lambda_m^2 \end{pmatrix} \right] \right]}{\text{mau_}L_3},
\end{aligned}$$

$$\begin{aligned}
Y_{15} &= \frac{\delta_n \left(H_{14} \begin{pmatrix} H_{22} \delta_n^6 \\ + H_{20} \delta_n^4 \lambda_m^2 \end{pmatrix} + \lambda_m^2 \begin{pmatrix} -H_{13} (H_{21} \delta_n^4 + H_{19} \delta_n^2 \lambda_m^2) + H_{12} \begin{pmatrix} H_{22} \delta_n^4 \\ + H_{20} \delta_n^2 \lambda_m^2 \end{pmatrix} \\ + \lambda_m^2 \begin{pmatrix} -H_{11} (H_{21} \delta_n^2 + H_{19} \lambda_m^2) + H_{10} \begin{pmatrix} H_{22} \delta_n^2 \\ + H_{20} \lambda_m^2 \end{pmatrix} \end{pmatrix} \right)}{mau_L_3}, \\
Y_{16} &= \frac{\left((H_{26} \delta_n^3 \lambda_m + H_{24} \delta_n \lambda_m^3) (-H_{14} \delta_n^4 - H_{12} \delta_n^2 \lambda_m^2 - H_{10} \lambda_m^4) \right. \\ &\quad \left. + (H_{13} \delta_n^3 \lambda_m + H_{11} \delta_n \lambda_m^3) (H_{27} \delta_n^4 + H_{25} \delta_n^2 \lambda_m^2 + H_{23} \lambda_m^4) \right)}{\left(-H_{13} \delta_n^3 \lambda_m - H_{11} \delta_n \lambda_m^3 \right)^2 - \left(-H_{14} \delta_n^4 - H_{12} \delta_n^2 \lambda_m^2 - H_{10} \lambda_m^4 \right)^2}, \\
mau_L_4 &= \left(-H_{13} \delta_n^3 \lambda_m - H_{11} \delta_n \lambda_m^3 \right)^2 - \left(-H_{14} \delta_n^4 - H_{12} \delta_n^2 \lambda_m^2 - H_{10} \lambda_m^4 \right)^2, \\
Y_{17} &= \frac{\left((H_{17} \delta_n^2 \lambda_m + H_{15} \lambda_m^3) (H_{13} \delta_n^3 \lambda_m + H_{11} \delta_n \lambda_m^3) \right. \\ &\quad \left. + (H_{18} \delta_n^3 + H_{16} \delta_n \lambda_m^2) (-H_{14} \delta_n^4 - H_{12} \delta_n^2 \lambda_m^2 - H_{10} \lambda_m^4) \right)}{mau_L_4}, \\
Y_{18} &= \frac{\left((H_{22} \delta_n^3 + H_{20} \delta_n \lambda_m^2) (H_{13} \delta_n^3 \lambda_m + H_{11} \delta_n \lambda_m^3) \right. \\ &\quad \left. + (H_{21} \delta_n^2 \lambda_m + H_{19} \lambda_m^3) (-H_{14} \delta_n^4 - H_{12} \delta_n^2 \lambda_m^2 - H_{10} \lambda_m^4) \right)}{mau_L_4},
\end{aligned}$$

Appendix F

$$\begin{aligned}
U_{10} &= - \left(\frac{64ab}{3\pi^2} G_{18} \lambda_m^4 Y_{11} + \frac{64ab}{3\pi^2} G_{22} \delta_n^4 Y_{12} \right), \\
U_{11} &= \frac{ab}{4} \left(G_{31} \lambda_m^4 + G_{33} \delta_n^2 \lambda_m^2 + G_{35} \delta_n^4 - G_{11} \lambda_m^2 - G_{13} \delta_n^2 - G_{18} \lambda_m^4 Y_{13} - G_{19} \lambda_m^3 \delta_n Y_{16} \right. \\ &\quad \left. - G_{20} \lambda_m^2 \delta_n^2 Y_{13} - G_{21} \lambda_m \delta_n^3 Y_{16} - G_{22} \delta_n^4 Y_{13} - k_1 - k_2 (\lambda_m^2 + \delta_n^2) \right), \\
U_{12} &= \frac{ab}{4} \left(G_{23} \lambda_m^3 + G_{25} \lambda_m \delta_n^2 - G_{14} \lambda_m - G_{18} \lambda_m^4 Y_{14} - G_{19} \lambda_m^3 \delta_n Y_{17} - G_{20} \lambda_m^2 \delta_n^2 Y_{14} \right. \\ &\quad \left. - G_{21} \lambda_m \delta_n^3 Y_{17} - G_{22} \delta_n^4 Y_{14} \right), \\
U_{13} &= \frac{ab}{4} \left(G_{28} \lambda_m^2 \delta_n + G_{30} \delta_n^3 - G_{17} \delta_n - G_{18} \lambda_m^4 Y_{15} - G_{19} \lambda_m^3 \delta_n Y_{18} - G_{20} \lambda_m^2 \delta_n^2 Y_{15} \right. \\ &\quad \left. - G_{21} \lambda_m \delta_n^3 Y_{18} - G_{22} \delta_n^4 Y_{15} \right), \\
U_{14} &= G_{36} Y_{11} \lambda_m^3 \frac{64ab}{3\pi^2}, \\
U_{15} &= \frac{ab}{4} \left(G_{36} \lambda_m^3 Y_{13} + G_{37} \lambda_m^2 \delta_n Y_{16} + G_{38} \lambda_m \delta_n^2 Y_{13} + G_{39} \delta_n^3 Y_{16} - G_{46} \lambda_m^3 \right. \\ &\quad \left. - G_{48} \lambda_m \delta_n^2 + G_{50} \lambda_m \right), \\
U_{16} &= \frac{ab}{4} \left(G_{36} \lambda_m^3 Y_{14} + G_{37} \lambda_m^2 \delta_n Y_{17} + G_{38} \lambda_m \delta_n^2 Y_{14} + G_{39} \delta_n^3 Y_{17} - G_{40} \lambda_m^2 \right. \\ &\quad \left. - G_{42} \delta_n^2 + G_{52} \right),
\end{aligned}$$

$$\begin{aligned}
U_{17} &= \frac{ab}{4} (G_{36} \lambda_m^3 Y_{15} + G_{37} \lambda_m^2 \delta_n Y_{18} + G_{38} \lambda_m \delta_n^2 Y_{15} + G_{39} \delta_n^3 Y_{18} - G_{44} \lambda_m \delta_n), \\
U_{18} &= G_{57} Y_{12} \delta_n^3 \frac{64ab}{3\pi^2}, \\
U_{19} &= \frac{ab}{4} \left(G_{54} \lambda_m^3 Y_{16} + G_{55} \lambda_m^2 \delta_n Y_{13} + G_{56} \lambda_m \delta_n^2 Y_{16} + G_{57} \delta_n^3 Y_{13} - G_{65} \lambda_m^2 \delta_n \right), \\
U_{20} &= \frac{ab}{4} (G_{55} \lambda_m^2 \delta_n Y_{14} + G_{54} \lambda_m^3 Y_{17} + G_{56} \lambda_m \delta_n^2 Y_{17} + G_{57} \delta_n^3 Y_{14} - G_{59} \lambda_m \delta_n), \\
U_{21} &= \frac{ab}{4} \left(G_{57} \delta_n^3 Y_{15} + G_{56} \lambda_m \delta_n^2 Y_{18} + G_{54} \lambda_m^3 Y_{18} + G_{55} \lambda_m^2 \delta_n Y_{15} - G_{61} \lambda_m^2 \right), \\
U_a &= \frac{(U_{15} U_{21} - U_{17} U_{19})}{U_{17} U_{20} - U_{16} U_{21}}, \quad U_b = \frac{(U_{14} U_{21} - U_{17} U_{18})}{\text{mau_Phi}}, \\
U_c &= \frac{(U_{16} U_{19} - U_{15} U_{20})}{\text{mau_Phi}}, \quad U_d = \frac{(U_{16} U_{18} - U_{14} U_{20})}{\text{mau_Phi}},
\end{aligned}$$

Appendix G

$$\begin{aligned}
X_1 &= \left(\frac{4ab}{mn\pi^2} (I_{23} \lambda_m^2 Y_{14} - I_{22} \lambda_m \delta_n Y_{17} + I_{24} \delta_n^2 Y_{14} - I_{25} \lambda_m + I_{28} c_l \lambda_m) + U_b \right. \\
&\quad \left. + \frac{4ab}{mn\pi^2} (I_{23} \lambda_m^2 Y_{15} - I_{22} \lambda_m \delta_n Y_{18} + I_{24} \delta_n^2 Y_{15} - I_{26} \delta_n + I_{29} c_l \delta_n) U_d - \frac{ab}{8mn} \delta_n^2 \right), \\
X_2 &= \left(\frac{4ab}{mn\pi^2} (I_{23} \lambda_m^2 Y_{13} - I_{22} \lambda_m \delta_n Y_{16} + I_{24} \delta_n^2 Y_{13} + I_{28} c_l \lambda_m^2 + I_{29} c_l \delta_n^2) \right. \\
&\quad + \frac{4ab}{mn\pi^2} (I_{23} \lambda_m^2 Y_{14} - I_{22} \lambda_m \delta_n Y_{17} + I_{24} \delta_n^2 Y_{14} - I_{25} \lambda_m + I_{28} c_l \lambda_m) U_a \\
&\quad \left. + \frac{4ab}{mn\pi^2} (I_{23} \lambda_m^2 Y_{15} - I_{22} \lambda_m \delta_n Y_{18} + I_{24} \delta_n^2 Y_{15} - I_{26} \delta_n + I_{29} c_l \delta_n) U_c \right).
\end{aligned}$$