



Original Article

Deformation-dependent Inner Pockets in Repulsive-core Double-folding Potentials for α Decay of Even-even Heavy Nuclei

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Abstract: We develop a repulsive-core double-folding (DF+rep) model for the α decays of ^{196}Rn , ^{202}Ra , ^{224}Th , and ^{228}U . Axial deformation of the daughter nucleus is included in both the radius and diffuseness of the Woods–Saxon density. Different deformation configurations are compared to separate the effects of quadrupole and hexadecapole deformation on the inner pocket, Coulomb barrier, and Wentzel–Kramers–Brillouin (WKB) tunneling integral. The repulsive-core term removes the unphysically deep attraction of the pure double-folding (DF) potential and produces a finite pocket in the density-overlap region. For oblate daughter nuclei with small β_4 , diffuseness deformation gives the main change in the potential and half-life, whereas the hexadecapole term becomes more important for prolate daughters. The full 4–4 configuration gives the closest agreement with experiment in all four cases, showing that deformation should be included consistently in both the radius and diffuseness. Since a_{rep} is fitted separately for each nucleus, however, this agreement reflects the ability of the repulsive-core parameter to account for short-range effects rather than an independent test of predictive power. These results show the combined role of surface deformation and short-range repulsion in α -barrier tunneling.

Keywords: α -nucleus potential, repulsive-core effects, heavy nuclei, nuclear deformation.

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1. Introduction

Alpha decay is a major decay mode of heavy and superheavy nuclei. It also provides a key tool for identifying new isotopes, linking decay chains, and probing nuclear structure far from stability [1, 2]. It is commonly described as the tunneling of a preformed α cluster through an effective barrier formed by the nuclear, Coulomb, and centrifugal interactions. Because the penetrability depends exponentially on the WKB integral, the half-life $T_{1/2}^\alpha$ is highly sensitive to the decay energy Q_α and to the shape of the α -daughter potential [3-5].

Phenomenological and semiempirical relations provide compact descriptions of systematic trends in α -decay half-lives, mainly through Geiger–Nuttall-type dependences on charge, mass, and Q_α . Denisov and Khudenko included odd–even and angular-momentum corrections [6], while the Viola–Seaborg–Sobiczewski (VSS) relation has been widely used for heavy and superheavy nuclei [7]. Royer derived separate expressions for the four odd–even classes [8]; Budaca et al., extended the Brown-type systematics and tested their extrapolation to superheavy nuclei [9]; and Akrawy and Poenaru added an explicit dependence on isospin asymmetry [10]. These formulas are useful for broad surveys, but they mainly reflect the data used in their fits and do not describe the α -daughter barrier explicitly. Good agreement within the fitted region therefore does not ensure reliable extrapolation to neutron-rich nuclei, shape-transition regions, or unobserved superheavy systems. Moreover, these relations cannot isolate the roles of the pocket depth, turning points, Coulomb-barrier curvature, and nuclear surface structure in determining $T_{1/2}^\alpha$.

A different class of models builds the tunneling barrier directly from the α -daughter interaction. In proximity models, the nuclear attraction is expressed in terms of the surface separation, surface energy, and an effective curvature radius. The Myers proximity potential gives a compact description of the interaction near contact and has been widely applied to heavy and superheavy systems [11]. Ghodsi et al., systematically compared several proximity potentials and showed that the choice of the universal function, surface-energy coefficient, and geometric parameters can produce sizable changes in calculated half-lives [12]. These results show that the radial shape of the barrier, not only its height, is important for tunneling.

A more microscopic description is provided by the DF method, in which the nuclear potential is built from the α density, the daughter density, and an effective nucleon–nucleon interaction. This approach links the interaction directly to nuclear structure. Xu and Ren developed a deformed DF model that includes the orientation dependence of the α -daughter potential in half-life calculations [13]. For spherical systems, Mohr applied a DF potential to the α decay of the superheavy nucleus $^{296}118$ and found a strong sensitivity to both Q_α and the potential parametrization [3]. Ghodsi and Amiri later combined HFB densities obtained with Skyrme interactions with the DF method to study superheavy nuclei [4].

A purely DF calculation, however, often produces an unphysically deep inner pocket when the two densities strongly overlap. This problem can be reduced by adding a short-range repulsive term that effectively accounts for Pauli blocking, density overlap, and the incompressibility of nuclear matter. Zheng *et al.* recently developed a deformed dynamical DF pocket potential to study the effect of deformation on α decay, with emphasis on the preformation probability P_α [14]. From the pocket geometry and the extracted preformation factors, they found an approximately linear relation between P_α and β_2^2 , where β_2 is the quadrupole deformation parameter, and proposed an empirical expression that includes both shell and deformation effects. Combined with the deformed dynamical DF potential, this relation was used to predict the half-lives of neutron-deficient Pu, Cm, and Cf isotopes near the proton drip line.

In our previous work, we studied a DF+rep model for the α decay of heavy and superheavy nuclei [5]. The repulsive term removes the overly deep pocket of the DF potential, reshapes the strong-overlap region, and produces a more realistic inner wall for the quasibound α -daughter state. The diffuseness a_{rep} , extracted from measured half-lives, acts as an effective measure of short-range corrections. A normalized sensitivity analysis for even–even nuclei with $94 \leq Z \leq 118$ showed that, although Q_α gives the dominant contribution, the calculated half-lives also exhibit sizable sensitivities to isospin asymmetry and a_{rep} . In particular, the sensitivity to a_{rep} highlights the role of short-range repulsion in setting the inner slope and effective width of the tunneling barrier [15]. These studies, however, did not treat nuclear deformation and the repulsive core within the same DF framework. In even–even heavy nuclei, quadrupole and hexadecapole deformations can change the contact radius, surface diffuseness, inner-pocket depth, and Coulomb-barrier shape. Because the WKB penetrability depends exponentially on the tunneling integral, even small changes in the pocket structure may strongly affect $T_{1/2}^\alpha$.

In this paper, we study the α decays of ^{196}Rn , ^{202}Ra , ^{224}Th , and ^{228}U , whose daughter nuclei span representative oblate and prolate shapes. For these systems, we extend the DF+rep model to axially deformed daughter nuclei. The quadrupole and hexadecapole deformations, β_2 and β_4 , are included in both the radius and diffuseness of the Woods–Saxon density. Unlike our previous DF+rep studies, the present work explicitly incorporates β_2 and β_4 deformations into both the radius and diffuseness of the daughter density, enabling their distinct effects on the inner pocket and WKB tunneling to be resolved. This focus also differs from our recent (α)-induced reaction study, where deformation was considered together with nonlocality in the entrance channel. By comparing different deformation configurations, we separate the effects of radius deformation, diffuseness deformation, and the hexadecapole term on the orientation-averaged potential. We examine the pocket depth V_{min} , barrier height V_{bar} , classically forbidden region, and $T_{1/2}^\alpha$. Beyond reproducing the measured half-lives, we aim to clarify how surface deformation reshapes the inner part of the DF+rep potential and affects the extracted a_{rep} . The results are compared with the Denisov and Myers proximity potentials and with the VSS, Royer, Brown, and Akrawy–Poenu relations. This comparison tests whether a consistent treatment of β_2 and β_4 in both the radius and diffuseness is needed to obtain realistic inner pockets and to describe the half-lives of both oblate and prolate daughter systems. test of the predictive capability of the DF+rep model. It should be noted that, since is fitted independently for each decay, the present results primarily assess deformation sensitivity rather than provide parameter-free predictions. Agreement with the measured half-lives constrains the effective short-range repulsion but does not constitute an independent validation of the predictive power of the DF+rep model.

2. Theoretical Framework

2.1. Deformed DF Potential with a Repulsive Core

The α -daughter interaction is described within a local semimicroscopic framework. Here, r is the distance between the two centers, and ϕ is the angle between the symmetry axis of the daughter nucleus and the internuclear axis. The total interaction is written as [16, 17]

$$U(r, \phi) = U_N(r, \phi) + U_C(r, \phi) + U_{\text{rep}}(r, \phi) + U_\ell(r). \quad (1)$$

The centrifugal term is

$$U_\ell(r) = \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2}, \quad (2)$$

where μ is the reduced mass of the system. For favored ground-state-to-ground-state transitions, we set $\ell = 0$.

The α particle is taken to be spherical and is described by the Gaussian matter density

$$\rho_\alpha(r_1) = 0.4229 \exp(-0.7024 r_1^2). \quad (3)$$

The daughter density is represented by an axially deformed Woods–Saxon distribution,

$$\rho_d(r_2, \theta_2) = \frac{\rho_{0d}}{1 + \exp\{[r_2 - R_d(\theta_2)]/a_d(\theta_2)\}}, \quad (4)$$

where θ_2 is the polar angle in the intrinsic frame of the daughter nucleus. The angle-dependent radius and diffuseness are written as

$$R_d^{(L_R)}(\theta_2) = R_{0d} \left[1 + \sum_{\lambda \in \mathcal{S}_{L_R}} \beta_\lambda Y_{\lambda 0}(\theta_2) \right], \quad (5)$$

$$a_d^{(L_a)}(\theta_2) = a_{0d} \left[1 + \sum_{\lambda \in \mathcal{S}_{L_a}} \beta_\lambda Y_{\lambda 0}(\theta_2) \right], \quad (6)$$

with

$$\mathcal{S}_0 = \emptyset, \quad \mathcal{S}_2 = \{2\}, \quad \mathcal{S}_4 = \{2, 4\}. \quad (7)$$

The parameters β_2 and β_4 describe quadrupole and hexadecapole deformation, respectively. We use $R_{0d} = 1.07 A_d^{1/3}$ fm and $a_{0d} = 0.54$ fm, following the density configuration of Ref. [16]. The pair (L_R, L_a) gives the highest multipole retained in R_d and a_d , respectively. Thus, the 0–0, 2–0, 4–0, 2–2, and 4–4 configurations denote, in order, a spherical density; β_2 in the radius only; β_2 and β_4 in the radius only; β_2 in both functions; and both β_2 and β_4 in R_d and a_d .

The nuclear potential is obtained by folding the two densities with a density-dependent effective nucleon–nucleon interaction,

$$U_N(r, \phi) = \int d^3 r_1 d^3 r_2 \rho_\alpha(r_1) F[\rho] v_{NN}(s) \rho_d(\mathbf{r}_2; \phi), \quad (8)$$

where

$$s = |\mathbf{r} + \mathbf{r}_2 - \mathbf{r}_1| \quad (9)$$

is the distance between the interacting nucleons. The factor $F[\rho]$ accounts for the density dependence of the effective M3Y interaction, whereas $\rho_d(\mathbf{r}_2; \phi)$ is the intrinsic daughter density rotated by an angle ϕ relative to the internuclear axis. The orientation dependence of U_N therefore follows directly from the deformed density. The latter may be expanded as

$$\rho_d(r_2, \theta_2) = \sum_{\lambda=0,2,4} \rho_d^{(\lambda)}(r_2) Y_{\lambda 0}(\theta_2), \quad (10)$$

which separates the monopole, quadrupole, and hexadecapole contributions to the DF integral.

The Coulomb potential is obtained consistently by folding the charge densities,

$$U_C(r, \phi) = \int d^3 r_1 d^3 r_2 \rho_\alpha^{\text{ch}}(\mathbf{r}_1) \frac{e^2}{4\pi\epsilon_0 s} \rho_d^{\text{ch}}(\mathbf{r}_2; \phi). \quad (11)$$

This treatment allows the position and height of the Coulomb barrier to change directly with the deformation of the daughter nucleus.

A purely DF potential may produce an overly deep inner pocket when the two densities strongly overlap. We correct this behavior by adding a zero-range repulsive term,

$$U_{\text{rep}}(r) = v_{\text{rep}} \int d^3 r_1 d^3 r_2 \rho_\alpha^{\text{rep}}(\mathbf{r}_1) \delta(\mathbf{r} + \mathbf{r}_2 - \mathbf{r}_1) \rho_d^{\text{rep}}(\mathbf{r}_2). \quad (12)$$

The strength v_{rep} is constrained by the compression energy at full overlap, while a_{rep} sets the radial range of the repulsive core. This term effectively accounts for Pauli blocking, density overlap, and the incompressibility of nuclear matter [5, 18].

For an axially symmetric daughter nucleus, the local potential used in the tunneling calculation is averaged over all orientations,

$$\bar{U}(r) = \frac{1}{2} \int_0^\pi U(r, \phi) \sin \phi d\phi. \quad (13)$$

The resulting potential retains the corrections from the $\lambda = 2$ and 4 components while reducing the tunneling problem to one dimension. The WKB penetrability is

$$\mathcal{P} = \exp \left[-2 \int_{r_{\text{in}}}^{r_{\text{out}}} \sqrt{\frac{2\mu}{\hbar^2} (\bar{U}(r) - Q_\alpha)} dr \right], \quad (14)$$

where r_{in} and r_{out} are the classical turning points defined by

$$\bar{U}(r_i) = Q_\alpha, \quad i = \text{in, out}, \quad (15)$$

for $\ell = 0$. The decay constant and half-life are then

$$\lambda_\alpha = \nu_0 \mathcal{P}, \quad T_{1/2}^\alpha = \frac{\ln 2}{\nu_0 \mathcal{P}}, \quad (16)$$

where ν_0 is the assault frequency. Deformation changes the position, curvature, and width of the barrier, whereas the repulsive core controls the strong-overlap region and removes the unphysical attraction at short distances. Their combined effect enters $T_{1/2}^\alpha$ exponentially through the WKB tunneling integral.

2.2. Reference Models and Semiempirical Formulas

For comparison, we also calculate the α -decay half-lives using two reference models and four semiempirical formulas. The Denisov relations [6] describe ground-state α transitions and were fitted to data for 344 α emitters. Separate parameter sets were obtained for the full data set and for light and heavy nuclei. For odd-mass and odd-odd systems, the model also includes corrections for orbital angular momentum, parity, and electron screening. The Myers proximity model [11] describes the nucleus–nucleus interaction with a universal potential that depends on the separation between the nuclear surfaces, their radii, and the surface tension. Applied to reactions that form superheavy nuclei with $Z = 102$ –118, the model suggests that the Coulomb barrier may fall below the incident energy, thereby increasing the reaction cross section.

The VSS formula keeps the Geiger–Nuttall dependence on $Z/\sqrt{Q_\alpha}$ and adds separate hindrance terms for the four odd–even classes of the parent nucleus. The Royer formula includes an explicit $A^{1/6}Z^{1/2}$ term and uses different coefficients for even–even, even–odd, odd–even, and odd–odd nuclei [8]. The Brown formula gives a more compact charge–energy relation through $(Z - 2)^{0.6}/\sqrt{Q_\alpha}$ [9]. The Akrawy–Poenu formula extends the Royer form by adding linear and quadratic terms in the isospin asymmetry $I = (N - Z)/A$ [10].

In the present DF+rep model, the repulsive diffuseness a_{rep} is fitted separately for each nucleus to reproduce the measured half-life. The agreement with $T_{1/2}^\alpha$ therefore does not provide an independent test of the model’s predictive power. Instead, a_{rep} is treated as an effective quantity extracted from data, which measures the short-range correction required in the strong-overlap region of the α -daughter potential.

3. Numerical Results and Discussion

In this section, we study how quadrupole and hexadecapole deformations of the daughter nuclei affect the corresponding α -nucleus potentials and the α -decay half-lives of ^{196}Rn , ^{202}Ra , ^{224}Th , and ^{228}U . The 2–0, 4–0, 2–2, and 4–4 configurations are used to separate the effects of radius deformation, diffuseness deformation, and the hexadecapole term on the inner pocket, Coulomb barrier, and WKB penetrability. The calculated half-lives are compared with experimental data, the Denisov and Myers

proximity models, and the VSS, Royer, Brown, and Akrawy–Poenu formulas. The four decays are used here as controlled test cases containing two oblate and two prolate daughter nuclei. The deformation trends identified below therefore apply to these four systems and should not be interpreted as universal systematics; a broader survey over isotopic chains and deformation ranges is left for future work.

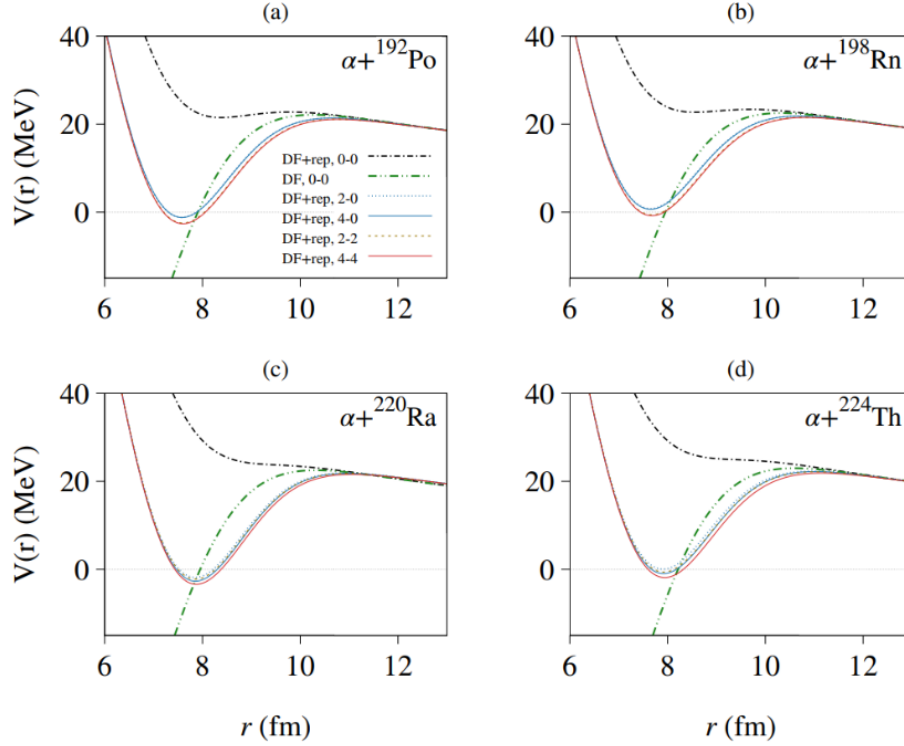


Figure 1. (Color online) Orientation-averaged α -daughter potentials for $\alpha+^{192}\text{Po}$, $\alpha+^{198}\text{Rn}$, $\alpha+^{220}\text{Ra}$, and $\alpha+^{224}\text{Th}$, corresponding to the α decays of ^{196}Rn , ^{202}Ra , ^{224}Th , and ^{228}U , respectively. The spherical DF and DF+rep potentials are compared with the deformed DF+rep results obtained with the 2–0, 4–0, 2–2, and 4–4 density configurations.

Figure 1 shows the orientation-averaged potentials $\bar{U}(r)$ for $\alpha+^{192}\text{Po}$, $\alpha+^{198}\text{Rn}$, $\alpha+^{220}\text{Ra}$, and $\alpha+^{224}\text{Th}$, corresponding to the α decays of ^{196}Rn , ^{202}Ra , ^{224}Th , and ^{228}U , respectively. The pure DF potential falls rapidly to large negative values in the density-overlap region. By contrast, the repulsive core makes the DF+rep potential rise sharply at short distances and removes the unphysically deep attraction. The repulsive term therefore controls the inner wall and produces a finite pocket between the short-range repulsion and the outer Coulomb barrier. At large distances, the potentials merge because the Coulomb interaction dominates and deformation effects become weak. Relative to the spherical 0–0 case, deformation of the daughter density mainly reshapes the inner pocket and the inner side of the barrier. The 2–0 and 4–0 configurations include deformation only in the radius, whereas the 2–2 and 4–4 cases also include deformation of the surface diffuseness. Although orientation averaging reduces the anisotropy, the β_2 and β_4 terms still produce clear changes in the tunneling region. In general, diffuseness deformation deepens the inner pocket and lowers the barrier relative to radius deformation alone.

For the two decays with oblate daughter nuclei, the effect of β_4 is small when it is included only in the radius. For ^{196}Rn , changing from 2–0 to 4–0 shifts $V_{\min}$ only from -1.18 to -1.19 MeV and V_{bar} from 21.37 to 21.36 MeV (Table 1). For ^{202}Ra , the corresponding changes are from 0.80 to 0.66 MeV and from 21.82 to 21.81 MeV. These small differences explain the near overlap of the 2–0 and 4–0 curves in Figs. 1(a) and 1(b). Diffuseness deformation produces a much larger effect. Changing from the 2–0 to the 2–2 configuration lowers $V_{\min}$ for ^{196}Rn from -1.18 to -2.48 MeV and V_{bar} from 21.37 to 21.11 MeV. For ^{202}Ra , the pocket minimum decreases from 0.80 to -0.62 MeV, while the barrier height falls from 21.82 to 21.53 MeV. A similar trend is found when 4–0 is compared with 4–4: $V_{\min}$ decreases by a further 1.43 MeV for ^{196}Rn and 1.45 MeV for ^{202}Ra . Thus, for the two oblate systems, diffuseness deformation gives the main change in the inner pocket, whereas the β_4 term in the radius plays a smaller role.

For the two decays with prolate daughter nuclei, both the hexadecapole term and diffuseness deformation become important. For ^{224}Th , changing from the 2–0 to the 4–0 configuration lowers $V_{\min}$ from -1.84 to -2.67 MeV, whereas V_{bar} decreases only slightly, from 21.88 to 21.80 MeV. Including β_2 in the diffuseness through the 2–2 configuration gives $V_{\min} = -2.23$ MeV and $V_{\text{bar}} = 21.78$ MeV. When both β_2 and β_4 are included in the radius and diffuseness, the full 4–4 configuration produces the deepest pocket, $V_{\min} = -3.37$ MeV, and the lowest barrier, $V_{\text{bar}} = 21.58$ MeV. Relative to the 2–0 case, these values are lower by 1.53 and 0.30 MeV, respectively. The largest deformation effect is found for ^{228}U . The 2–0 configuration gives a shallow pocket with $V_{\min} = 0.04$ MeV. This value decreases to -0.59 MeV for 2–2, -0.95 MeV for 4–0, and -1.86 MeV for the full 4–4 configuration. At the same time, V_{bar} decreases from 22.31 to 21.88 MeV. Thus, from 2–0 to 4–4, the pocket deepens by 1.90 MeV and the barrier is lowered by 0.43 MeV. The clear separation of the curves in Fig. 1(d) shows that the β_4 term and diffuseness deformation act together on both the inner pocket and the inner part of the tunneling barrier.

These results show that deformation cannot be described only through the change in the Coulomb-barrier height. It also modifies the pocket depth, the position of the inner wall, and the curvature and effective width of the classically forbidden region. Diffuseness deformation gives the main effect for the oblate systems, whereas the combined effects of β_4 and diffuseness deformation are stronger for the prolate systems, especially for ^{228}U . Among the four configurations, the full 4–4 treatment therefore gives the most complete description of the surface geometry. Because the barrier penetrability depends exponentially on the WKB tunneling integral, the deeper pockets and lower barriers obtained with the 2–2 and 4–4 configurations can significantly reduce $T_{1/2}^\alpha$. The change in the half-life, however, cannot be inferred from V_{bar} alone. It depends on the full radial shape of the potential, the turning points, and the Q_α value of each decay.

Table 1. Inner-pocket depth $V_{\min}$, barrier height V_{bar} (MeV), and values of a_{rep} (fm) for different daughter-nucleus deformation configurations in the α decays of ^{196}Rn , ^{202}Ra , ^{224}Th , and ^{228}U . The corresponding daughter-nucleus deformation parameters, β_2 and β_4 , are also listed

Configuration	^{196}Rn $a_{\text{rep}} = 0.460$ fm $\beta_2 = -0.217$ $\beta_4 = 0.017$		^{202}Ra $a_{\text{rep}} = 0.498$ fm $\beta_2 = -0.227$ $\beta_4 = 0.019$		^{224}Th $a_{\text{rep}} = 0.396$ fm $\beta_2 = 0.111$ $\beta_4 = 0.081$		^{228}U $a_{\text{rep}} = 0.441$ fm $\beta_2 = 0.144$ $\beta_4 = 0.084$	
	$V_{\min}$	V_{bar}	$V_{\min}$	V_{bar}	$V_{\min}$	V_{bar}	$V_{\min}$	V_{bar}
2–0	-1.18	21.37	0.80	21.82	-1.84	21.88	0.04	22.31
2–2	-2.48	21.11	-0.62	21.53	-2.23	21.78	-0.59	22.14
4–0	-1.19	21.36	0.66	21.81	-2.67	21.80	-0.95	22.20
4–4	-2.62	21.09	-0.79	21.49	-3.37	21.58	-1.86	21.88

Table 2 shows how $\log_{10}(T_{1/2}^\alpha/s)$ depends on the treatment of daughter-nucleus deformation in the radius and surface diffuseness. For each decay, the same a_{rep} value listed in Table 1 is used for all four configurations. The changes in the calculated half-lives therefore arise from the β_2 and β_4 terms rather than from a refit of the repulsive core.

For the α decays of ^{196}Rn and ^{202}Ra , the daughter nuclei are oblate, with $(\beta_2, \beta_4) = (-0.217, 0.017)$ and $(-0.227, 0.019)$, respectively. Because β_4 is small, changing from the 2–0 to the 4–0 configuration changes $\log_{10}(T_{1/2}^\alpha/s)$ only from -2.132 to -2.141 for ^{196}Rn and from -2.159 to -2.174 for ^{202}Ra . This weak effect is consistent with the small changes in V_{min} and V_{bar} between the two configurations in Table 1. A larger effect appears when deformation is included in the diffuseness. The 2–2 configuration gives -2.308 and -2.363 , much closer to the experimental values -2.328 and -2.387 . Thus, for the two oblate daughters, diffuseness deformation affects the half-life more strongly than the β_4 term included only in the radius.

Table 2. Comparison of $\log_{10}(T_{1/2}^\alpha/s)$ obtained with different daughter-nucleus deformation configurations and with experimental data

Configuration	$\log_{10}(T_{1/2}^\alpha/s)$			
	^{196}Rn	^{202}Ra	^{224}Th	^{228}U
2–0	-2.132	-2.159	0.217	3.079
2–2	-2.308	-2.363	0.158	2.971
4–0	-2.141	-2.174	0.141	2.971
4–4	-2.327	-2.389	0.016	2.785
Experiment	-2.328	-2.387	0.017	2.748

For the α decays of ^{224}Th and ^{228}U , the daughter nuclei are prolate, with $(\beta_2, \beta_4) = (0.111, 0.081)$ and $(0.144, 0.084)$, respectively. Their larger β_4 values make the hexadecapole contribution more important. For ^{224}Th , the 4–0 configuration lowers the result from 0.217 to 0.141, while the 2–2 configuration gives 0.158. The similar values indicate that hexadecapole deformation of the radius and quadrupole deformation of the diffuseness produce corrections of comparable size.

The strongest dependence on deformation is found for ^{228}U . Both the 2–2 and 4–0 configurations give $\log_{10}(T_{1/2}^\alpha/s) = 2.971$, below the 2–0 value of 3.079. Despite giving the same half-life, the two configurations produce different pockets, $V_{\text{min}} = -0.59$ MeV for 2–2 and -0.95 MeV for 4–0. This shows that $T_{1/2}^\alpha$ is not fixed by the pocket depth or the barrier height alone, but by the full radial potential and the WKB tunneling integral. Overall, diffuseness deformation gives the main effect for the oblate daughters with small β_4 , whereas the hexadecapole term becomes more important for the prolate daughters with larger β_4 .

When both β_2 and β_4 are included in the radius and diffuseness, the full 4–4 configuration gives the closest agreement with experiment for all four decays. The incomplete configurations give larger values of $\log_{10}(T_{1/2}^\alpha/s)$ and therefore smaller penetrabilities and longer half-lives. The full deformation treatment lowers and reshapes the barrier, reduces the WKB tunneling integral, and increases the barrier penetrability.

Table 3 compares the $\log_{10}(T_{1/2}^\alpha/s)$ values obtained with the DF+rep model using the full 4–4 deformation configuration, the reference models, and experimental data. The DF+rep results are close to experiment for all four decays, whereas the reference models show clearer systematic shifts. The Denisov and Myers proximity models predict longer half-lives than experiment for all four nuclei. In contrast, the VSS, Royer, Brown, and Akrawy–Poenu formulae predict shorter half-lives. The largest

deviations are obtained with VSS, especially for ^{196}Rn and ^{202}Ra , for which the differences are -1.089 and -1.040 logarithmic units, respectively. Among the models without a separate fit for each nucleus, the Akrawy–Poenaru formula gives the best agreement for this set, followed by the Myers proximity model, while VSS gives the largest deviation. This ordering, however, applies only to the four nuclei considered here and does not measure the overall accuracy of these models.

Table 3. Comparison of $\log_{10}(T_{1/2}^\alpha/s)$ for the α decays of ^{196}Rn , ^{202}Ra , ^{224}Th , and ^{228}U . Results from the Denisov model, the Myers proximity potential, the VSS, Royer, Brown, and Akrawy–Poenaru formulas, and the DF+rep model with the full 4–4 deformation configuration are compared with experimental data

Nucleus	Denisov	Proximity	VSS	Royer	Brown	Poenaru	DF+rep (4–4)	Experiment
^{196}Rn	-1.766	-1.992	-3.417	-2.635	-2.745	-2.460	-2.330	-2.328
^{202}Ra	-1.877	-2.093	-3.427	-2.744	-2.858	-2.519	-2.371	-2.387
^{224}Th	0.464	0.393	-0.701	-0.408	-0.216	-0.331	0.017	0.017
^{228}U	3.216	3.121	1.975	2.254	2.331	2.313	2.750	2.748

The values $a_{\text{rep}} = 0.460, 0.498, 0.396$, and 0.441 fm were fitted separately to the measured half-life of each nucleus. The agreement of the 4–4 results with experiment therefore does not provide an independent test of the predictive power of the model. Instead, a_{rep} acts as an effective parameter that accounts for short-range effects associated with density overlap, Pauli blocking, nuclear incompressibility, and deformation-driven surface rearrangement. Within the deformed DF+rep model, the flexibility of a_{rep} allows the measured $T_{1/2}^\alpha$ values to be reproduced after fitting. A predictive test requires a systematic parametrization of a_{rep} in terms of mass, charge, isospin asymmetry, and deformation, followed by calculations for nuclei outside the fit set.

4. Conclusion

We extend the DF model with a repulsive term to the α decays of ^{196}Rn , ^{202}Ra , ^{224}Th , and ^{228}U , including axial deformation of the corresponding daughter nuclei. The repulsive term removes the unphysically deep attraction of the pure folding potential and produces a finite inner pocket. Quadrupole and hexadecapole deformations, β_2 and β_4 , modify the pocket, the inner side of the barrier, and its effective width. Diffuseness deformation gives the main effect for the oblate daughter nuclei with small β_4 , whereas the hexadecapole contribution becomes more important for the prolate daughters with larger β_4 . Including both β_2 and β_4 in the radius and diffuseness through the full 4–4 configuration gives the largest change in the WKB tunneling integral and the half-lives closest to experiment among the four configurations studied.

The agreement should be viewed within the limits of the present fitting procedure because a_{rep} is determined separately from the measured half-life of each nucleus. Thus, a_{rep} is best treated as an effective parameter for short-range corrections due to density overlap, Pauli blocking, nuclear incompressibility, and surface rearrangement. The present results show the joint role of surface geometry and short-range repulsion in α -barrier tunneling. The next step is to develop a common parametrization of a_{rep} based on mass, charge, isospin asymmetry, and deformation, and to test it for nuclei not used in the fit.

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