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Response's Probabilistic Characteristics of a Duffing Oscillator under Harmonic and Random Excitations

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Abstract: Response's probabilistic characteristics of a Duffing oscillator subjected to combined harmonic and random excitations are investigated by a technique combining the stochastic averaging method and the equivalent linearization method. The harmonic excitation frequency is taken to be in the neighborhood of the system's natural frequency. The original equation is averaged by the stochastic averaging method in Cartesian coordinates. Then the equivalent linearization method is applied to the nonlinear averaged equations so that the equations obtained can be solved exactly by the technique of auxiliary function. The theoretical analyses of Duffing oscillator are validated by numerical simulation.

Keywords: Duffing, averaging method, equivalent linearization, harmonic excitation, random excitation.

1. Introduction

Duffing oscillator, a classical system for illustrating the jump phenomenon and other nonlinear behaviors, has been applied to model many mechanical systems. When the system is under only harmonic excitation, one of the popular tools used to study it is the averaging method. This method was originally given by Krylov and Bogoliubov [1] and then it was developed by Bogoliubov and Mitropolskii [2,3] and was extended to systems under a random excitation as in works of Stratonovich [4], Khasminskii [5], Robert and Spanos [6]. Another popular method to find the approximate response of a stochastic nonlinear system is the stochastic equivalent linearization method. The original version of this method was proposed by Caughey [7] and then this method has been developed up to recent years by many authors [8-15].

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It is more complicated when a nonlinear system is under a combination of harmonic and random excitations. One would see that it is only possible to obtain an approximate response of the system within the individual frameworks of methods of stochastic averaging and equivalent linearization just in only a few special cases as shown in works of Dimentberg [16], Mitropolski et al. [17]. Therefore, combinations of various methods need developing to investigate responses of such systems. Some methods have been used for the analyses such as the averaging method and the path integration (see e.g [18]), the combination of multiple scales and second-order closure method [19], the method of harmonic balance and the method of stochastic averaging [20], stochastic averaging and equivalent nonlinearization [21]. In [21], Manohar and Iyenga overcame a difficulty in solving a Fokker Planck (FP) by employing the equivalent nonlinearization method in investigating the Van der Pol oscillator under harmonic and white noise excitations. However, a limitation of this approach is which equivalent nonlinear function can be chosen to the original one. This technique cannot be applied to the Duffing oscillator. In [22], Anh and Hieu investigated Duffing oscillator under periodic and random excitations by the averaging and linearization methods. The information of the response, however, may not be full when coefficients depending on time in a random equation are replaced by their averaged values over one period.

In the present paper, response's probabilistic characteristics of a Duffing oscillator under harmonic and random excitations are analyzed by a new technique using the stochastic averaging and equivalent linearization methods and the technique of auxiliary for FP equation [23]. By using the averaging method in Cartesian coordinates, the averaged Duffing equation is simplified in polynomial forms which can be replaced by linear ones whose solution can be found exactly. Finally, the theoretical analyses of the Duffing system obtained by the proposed technique are validated by numerical simulation results, obtained by Monte-Carlo method.

2. Approximate technique

Let us consider the Duffing oscillator under combined harmonic and random excitations of the form

$$\ddot{x} + \varepsilon h\dot{x} + \varepsilon\gamma x^3 + \omega^2 x = \varepsilon P\cos\nu t + \sqrt{\varepsilon}\,\sigma\xi(t), \qquad (1)$$

where $\omega, h, \gamma, P, \nu, \sigma$ are positive parameters, ε is a small positive parameter, and function $\xi(t)$ is a Gaussian white noise process of unit intensity with the correlation function $R_\xi(\tau) = E\left(\xi(t)\xi(t+\tau)\right) = \delta(\tau)$, where $\delta(\tau)$ is the Dirac delta function, and notation $E(.)$ denotes the mathematical expectation operator. We consider Eq. (1) in primary resonant frequency region, i.e. parameters ω and ν have the relation

$$\omega^2 - \nu^2 = \varepsilon\Delta\,, \qquad (2)$$

where Δ is a detuning parameter. Substituting (2) into Eq. (1) yields

$$\ddot{x} + \nu^2 x = \varepsilon\left(-\Delta x - h\dot{x} - \gamma x^3 + P\cos\nu t\right) + \sqrt{\varepsilon}\sigma\xi(t)\,, \qquad (3)$$

We seek the solution of Eq. (3) in the form of

$$x = b\cos\nu t + d\sin\nu t,\ \dot{x} = -b\nu\sin\nu t + d\nu\cos\nu t, \tag{4}$$

where b and d are slowly varying random processes satisfying an additional condition

$$\dot{b}\cos\nu t + \dot{d}\sin\nu t = 0\,. \tag{5}$$

Substituting (4) into Eq. (3) and then solving the resulting equation and Eq. (5) with respect to the derivatives $\dot{b}$ and $\dot{d}$ yield

$$\begin{aligned}\dot{b} &= -\frac{1}{\nu}\Big(\varepsilon\Big(-\Delta(b\cos\nu t + d\sin\nu t) - h(-b\nu\sin\nu t + d\nu\cos\nu t) - \gamma(b\cos\nu t + d\sin\nu t)^3 + P\cos\nu t\Big) + \sqrt{\varepsilon}\sigma\xi(t)\Big)\sin\nu t,\\ \dot{d} &= \frac{1}{\nu}\Big(\varepsilon\Big(-\Delta(b\cos\nu t + d\sin\nu t) - h(-b\nu\sin\nu t + d\nu\cos\nu t) - \gamma(b\cos\nu t + d\sin\nu t)^3 + P\cos\nu t\Big) + \sqrt{\varepsilon}\sigma\xi(t)\Big)\cos\nu t,\end{aligned} \tag{6}$$

This pair of stochastic differential equations, the system (6), can be simplified by using the stochastic averaging method

$$\begin{aligned}\dot{b} &= \varepsilon H_1(b,d) + \frac{\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}}\xi_1(t),\\ \dot{d} &= \varepsilon H_2(b,d) + \frac{\sqrt{\varepsilon}\sigma}{\nu\sqrt{2}}\xi_2(t).\end{aligned} \tag{7}$$

Here $\xi_1(t)$ and $\xi_2(t)$ are independent white noises with unit intensity, and the drift coefficients $H_1(b,d)$ and $H_2(b,d)$ are determined as follows

$$\begin{aligned}H_1(b,d) &= -\frac{1}{\nu}\Big\langle\Big(-\Delta(b\cos\nu t + d\sin\nu t) - h(-b\nu\sin\nu t + d\nu\cos\nu t) - \gamma(b\cos\nu t + d\sin\nu t)^3 + P\cos\nu t\Big)\sin\nu t\Big\rangle_t,\\ H_2(b,d) &= \frac{1}{\nu}\Big\langle\Big(-\Delta(b\cos\nu t + d\sin\nu t) - h(-b\nu\sin\nu t + d\nu\cos\nu t) - \gamma(b\cos\nu t + d\sin\nu t)^3 + P\cos\nu t\Big)\cos\nu t\Big\rangle_t,\end{aligned} \tag{8}$$

where $\langle . \rangle_t$ is a time-averaging operator over one period defined by

$$\langle \cdot \rangle_t = \frac{1}{T}\int_0^T (.)\,dt\,. \tag{9}$$

From (8), one obtains the drift coefficients of the system (7)

$$\begin{aligned}H_1(b,d) &= -\frac{h}{2}b + \frac{\Delta}{2\nu}d + \frac{3\gamma}{8\nu}\left(b^2 d + d^3\right),\\ H_2(b,d) &= -\frac{\Delta}{2\nu}b - \frac{h}{2}d - \frac{3\gamma}{8\nu}\left(b^3 + bd^2\right) + \frac{P}{2\nu}.\end{aligned} \tag{10}$$

The FP equation written for the stationary PDF $W(b,d)$ associated with the system (7) has the form

$$\frac{\partial}{\partial b}\left(H_1(b,d)W\right) + \frac{\partial}{\partial d}\left(H_2(b,d)W\right) = \frac{\sigma^2}{4\nu^2}\left[\frac{\partial^2}{\partial b^2}(W) + \frac{\partial^2}{\partial d^2}(W)\right]. \tag{11}$$

Solution of (11) is still a difficult problem because functions $H_1(b,d)$ and $H_2(b,d)$ are nonlinear functions in b,d. To overcome this, the equivalent linearization method is employed. Following this method, the nonlinear functions H_1,H_2 are replaced by linear ones. Denote

$$g_1(b,d)=\frac{3\gamma}{8\nu}\left(b^2d+d^3\right),\qquad g_2(b,d)=-\frac{3\gamma}{8\nu}\left(b^3+bd^2\right). \tag{12}$$

According to the stochastic equivalent linearization method, the nonlinear terms (12) are replaced by

$$\overline{g}_1(b,d)=\eta_{11}b+\eta_{12}d+\eta_{13},\qquad \overline{g}_2(b,d)=\eta_{21}b+\eta_{22}d+\eta_{23}, \tag{13}$$

where equivalent coefficients $\eta_{ij}, i=1,2; j=1,2,3$ are to be determined by an optimization criterion. Thus, the functions $H_i, i=1,2$ in (10) are replaced by linear functions

$$\begin{aligned} H_1(b,d)&=\left(-\frac{h}{2}+\eta_{11}\right)b+\left(\frac{\Delta}{2\nu}+\eta_{12}\right)d+\eta_{13},\\ H_2(b,d)&=\left(-\frac{\Delta}{2\nu}+\eta_{21}\right)b+\left(-\frac{h}{2}+\eta_{22}\right)d+\frac{P}{2\nu}+\eta_{23}. \end{aligned} \tag{14}$$

According to the technique of auxiliary function with the auxiliary function taking the form (see [23] for details)

$$u_0=\frac{\sigma^2}{4\nu^2}\frac{-\dfrac{\Delta}{\nu}+\eta_{21}-\eta_{12}}{-h+\eta_{11}+\eta_{22}}, \tag{15}$$

the corresponding FP equation to Eq. (11), where drift coefficients are linear functions (14), has the following exact solution

$$W(b,d)=C\exp\left\{-\tau_1b^2-\tau_2d^2+\tau_3bd+\tau_4b+\tau_5d\right\}, \tag{16}$$

where C is a normalization constant and coefficients $\tau_i, i=\overline{1,5}$ are determined as follows

$$\tau_1=-\Psi\left(\left(-\frac{h}{2}+\eta_{11}\right)\left(-h+\eta_{11}+\eta_{22}\right)+\left(-\frac{\Delta}{2\nu}+\eta_{21}\right)\left(-\frac{\Delta}{\nu}+\eta_{21}-\eta_{12}\right)\right),$$

$$\tau_2=-\Psi\left(\left(-h+\eta_{11}+\eta_{22}\right)\left(-\frac{h}{2}+\eta_{22}\right)-\left(-\frac{\Delta}{\nu}+\eta_{21}-\eta_{12}\right)\left(\frac{\Delta}{2\nu}+\eta_{12}\right)\right),$$

$$\tau_3=2\Psi\left(\left(-\frac{\Delta}{2\nu}+\eta_{21}\right)\left(-\frac{h}{2}+\eta_{22}\right)+\left(\frac{\Delta}{2\nu}+\eta_{12}\right)\left(-\frac{h}{2}+\eta_{11}\right)\right),$$

$$\tau_4=2\Psi\left(\eta_{13}\left(-h+\eta_{11}+\eta_{22}\right)+\left(\frac{P}{2\nu}+\eta_{23}\right)\left(-\frac{\Delta}{\nu}+\eta_{21}-\eta_{12}\right)\right),$$

$$\tau_5 = 2\Psi\left(\left(-\frac{\Delta}{\nu}+\eta_{21}-\eta_{21}\right)\eta_{13}+\left(-h+\eta_{11}+\eta_{22}\right)\left(\frac{P}{2\nu}+\eta_{23}\right)\right), \tag{17}$$

where

$$\Psi = \frac{2\nu^2\left(-h+\eta_{11}+\eta_{22}\right)}{\sigma^2\left[\left(-\frac{\Delta}{\nu}+\eta_{21}-\eta_{12}\right)^2+\left(-h+\eta_{11}+\eta_{22}\right)^2\right]}. \tag{18}$$

It is noted that the joint PDF $W(b,d)$ determined by (16) has finite integral if coefficients τ_1 and τ_2 are positive. Therefore, the approximate stationary PDF of Eq. (11) is determined by (16) whose coefficients are given in (17). It is seen from (16) that random variables b and d are jointly Gaussian. Thus, from (16), one obtains

$$E(b)=\frac{2\tau_2\tau_4+\tau_3\tau_5}{4\tau_1\tau_2-\tau_3^2},\ E(d)=\frac{2\tau_1\tau_5+\tau_3\tau_4}{4\tau_1\tau_2-\tau_3^2},\ \sigma_b^2=\frac{2\tau_2}{4\tau_1\tau_2-\tau_3^2},\ \sigma_d^2=\frac{2\tau_1}{4\tau_1\tau_2-\tau_3^2},\ k_{bd}=\frac{\tau_3}{4\tau_1\tau_2-\tau_3^2}, \tag{19}$$

where σ_b^2 and σ_d^2 are variance of b and d, respectively, and k_{bd} is covariance of b and d. It is seen from (19) that necessary statistics of processes b and d can be computed algebraically based on coefficients of joint PDF $W(b,d)$. Thus, the approximate solution (16) of Eq. (1) is completely determined when the linearization coefficients $\eta_{ij}, i=1,2; j=1,2,3$ are found. There are some criteria for determining the coefficients $\eta_{ij}, i=1,2; j=1,2,3$. The most extensively used criterion is the mean square error criterion which requires that the mean square of the following errors be minimum [7]. From (10), (12), (13), and (14), the errors in this problem will be

$$e_i = g_i(b,d)-\left(\eta_{i1}b+\eta_{i2}d+\eta_{i3}\right), i=1,2. \tag{20}$$

So, the mean square error criterion leads to

$$E\left(e_i^2\right)=E\left\{\left[g_i(b,d)-\left(\eta_{i1}b+\eta_{i2}d+\eta_{i3}\right)\right]^2\right\}\to\min_{\eta_{ij}}, i=1,2;\ j=1,2,3. \tag{21}$$

From

$$\frac{\partial}{\partial\eta_{ij}}E\left(e_i^2\right)=0,\ i=1,2; j=1,2,3, \tag{22}$$

it follows that

$$\begin{aligned}
&E\left(b\,g_1(b,d)\right)-E\left(b^2\right)\eta_{11}-E(bd)\eta_{12}-E(b)\eta_{13}=0,\\
&E\left(d\,g_1(b,d)\right)-E(bd)\eta_{11}-E\left(d^2\right)\eta_{12}-E(d)\eta_{13}=0,\\
&E\left(g_1(b,d)\right)-E(b)\eta_{11}-E(d)\eta_{12}-\eta_{13}=0,\\
&E\left(b\,g_2(b,d)\right)-E\left(b^2\right)\eta_{21}-E(bd)\eta_{22}-E(b)\eta_{23}=0,\\
&E\left(d\,g_2(b,d)\right)-E(bd)\eta_{21}-E\left(d^2\right)\eta_{22}-E(d)\eta_{23}=0,\\
&E\left(g_2(b,d)\right)-E(b)\eta_{21}-E(d)\eta_{22}-\eta_{23}=0,
\end{aligned} \tag{23}$$

where $g_1(b,d)$, $g_2(b,d)$ are given by (12). Solving system (23) in $\eta_{ij}, i=1,2; j=1,2,3$, with noting that higher moments of b and d can be expressed in the first and second moments because b and d are jointly Gaussian (see [24] for details), gives

$$\begin{aligned}\eta_{11}&=-\frac{3\gamma}{4\nu}E(bd), \eta_{12}=-\frac{3\gamma}{8\nu}\left(E(b)+\sigma_b^2+3E(d)+3\sigma_d^2\right), \eta_{13}=\frac{3\gamma}{4\nu}E(d)\left(E^2(b)+E^2(d)\right),\\ \eta_{21}&=\frac{3\gamma}{8\nu}\left(3E(b)+3\sigma_b^2+E(d)+\sigma_d^2\right), \eta_{22}=\frac{3\gamma}{4\nu}\left(k_{bd}+E(b)E(d)\right), \eta_{23}=-\frac{3\gamma}{4\nu}E(b)\left(E^2(b)+E^2(d)\right).\end{aligned} \quad (24)$$

Thus, $\eta_{ij}, i=1,2; j=1,2,3$ are determined from the system of nonlinear equations obtained by combining equations (17), (19), and (24). After being found by solving system (24), the values of coefficients η_{ij}, $i=1,2$; $j=1,2,3$ are substituted into (16) to obtain the approximate stationary PDF in b and d of Duffing equation (1).

From the transformation (4), the mean response of the oscillator can be rewritten in the form

$$E(x)=\sqrt{E^2(b)+E^2(d)}\left[\frac{E(b)}{\sqrt{E^2(b)+E^2(d)}}\cos\nu t+\frac{E(d)}{\sqrt{E^2(b)+E^2(d)}}\sin\nu t\right]=\sqrt{E^2(b)+E^2(d)}\cos(\nu t+\theta), \quad (25)$$

where $\tan\theta=-\dfrac{E(d)}{E(b)}$. Thus it is periodic with amplitude A where

$$A^2=E^2(b)+E^2(d). \quad (26)$$

The mean square response of Eq. (1) can be determined as follows

$$E\left(x^2(t)\right)=E\left(b^2\right)\cos^2\nu t+E\left(d^2\right)\sin^2\nu t+E(bd)\sin 2\nu t\,. \quad (27)$$

Taking averaging with respect to time Eq. (27) gives

$$\left\langle E\left(x^2\right)\right\rangle_t=\frac{1}{2\pi}\int_0^{2\pi}E\left[x^2(t)\right]d(\nu t)=\frac{1}{2}\left[E\left(b^2\right)+E\left(d^2\right)\right]=\frac{1}{2}\left[E^2(b)+\sigma_b^2+E^2(d)+\sigma_d^2\right]. \quad (28)$$

Substituting (19) into (28) and reducing the obtained result yield the time-averaging of mean square response to be

$$\left\langle E\left(x^2\right)\right\rangle_t=\frac{\left(2\tau_2\tau_4+\tau_3\tau_5\right)^2+\left(2\tau_1\tau_5+\tau_3\tau_4\right)^2}{2\left(4\tau_1\tau_2-\tau_3^2\right)^2}+\frac{\tau_1+\tau_2}{4\tau_1\tau_2-\tau_3^2}, \quad (29)$$

where $\tau_i, i=\overline{1,5}$ are given by (17). It is noted from (29) that the approximate time-averaging value of mean square response of Duffing oscillator is calculated algebraically.

3. Numerical results

The various values of response of Duffing equation (1) are compared to the numerical simulation results versus the particular parameter. The numerical simulation of the mean square response $\left\langle x^2\right\rangle_{sim}$

is obtained by 10,000-realization Monte Carlo simulation. The time-averaged mean square response of the Duffing oscillator obtained by the formula (29) is compared to a numerical result in tables below. The responses are evaluated versus the parameter γ and the parameter σ^2 of the random excitation in Table 1 with the system parameters chosen to be $\omega=1$, $P=5$, $h=2$, $\sigma^2=1$, $\varepsilon=0.2$, $\nu=1.01$, and in Table 2 with the input parameters $\omega=1$, $P=5$, $h=2$, $\gamma=1$, $\nu=1.01$, respectively. Table 1 shows that the proposed technique gives a good prediction. Meanwhile, Table 2 shows that the error of the present technique, in general, increases when random intensity σ^2 increases, and that the error decreases when ε decreases. For small values of σ^2, however, the proposed technique gives a good prediction. The error in the tables is defined as

$$\text{Err } = \frac{\left|\left\langle x^2\right\rangle_{sim} - \left\langle E\left(x^2\right)\right\rangle_t\right|}{\left\langle x^2\right\rangle_{sim}} \times 100\%, \tag{30}$$

where $\left\langle E\left(x^2\right)\right\rangle_t$ denote the time-averaging values of mean square response by the present technique. Moreover, the mean response $E\left(x(t)\right)$ and mean square response $E\left(x^2(t)\right)$ obtained by the present technique are compared to ones obtained by Monte-Carlo simulation in Fig. 1. It may be seen that the theoretical predictions and the simulations compare very well.

Table 1. The error between the simulation result and approximate values of the time-averaging of mean square response $E\left(x^2(t)\right)$ versus the parameter γ $\left(\omega=1, P=5, h=2, \sigma^2=1, \varepsilon=0.2, \nu=1.01\right)$.

γ	$\left\langle x^2\right\rangle_{sim}$	$\left\langle E\left(x^2\right)\right\rangle_t$	Err(%)
0.5	2.0307	2.1001	3.42
1	1.4542	1.5005	3.18
2	0.9872	1.0171	3.03
5	0.5679	0.5865	3.27

Table 2. The error between the simulation result and approximate values of the time-averaging of $E\left(x^2(t)\right)$ versus the parameter σ^2 $\left(\omega=1, P=5, h=2, \gamma=1, \nu=1.01\right)$ with various values of the parameter ε.

	$\varepsilon=0.1$			$\varepsilon=0.2$			$\varepsilon=0.3$			$\varepsilon=0.5$		
σ^2	$\left\langle x^2\right\rangle_{sim}$	$\left\langle E\left(x^2\right)\right\rangle_t$	Err(%)	$\left\langle x^2\right\rangle_{sim}$	$\left\langle E\left(x^2\right)\right\rangle_t$	Err(%)	$\left\langle x^2\right\rangle_{sim}$	$\left\langle E\left(x^2\right)\right\rangle_t$	Err(%)	$\left\langle x^2\right\rangle_{sim}$	$\left\langle E\left(x^2\right)\right\rangle_t$	Err(%)
0.1	1.4949	1.5069	0.80	1.4385	1.4720	2.33	1.4302	1.4605	2.12	1.3991	1.4513	3.73
1	1.5134	1.5329	1.29	1.4539	1.5005	3.20	1.4434	1.4898	3.21	1.4109	1.4813	4.99
2	1.5472	1.5745	1.77	1.4802	1.5451	4.38	1.4633	1.5354	4.93	1.4277	1.5277	7.00
3	1.5948	1.6326	2.37	1.5140	1.6066	6.12	1.4919	1.5981	7.12	1.4496	1.5914	9.78
4	1.6509	1.7119	3.70	1.5558	1.6899	8.62	1.5242	1.6827	10.40	1.4750	1.6770	13.69
5	1.7150	1.8162	5.90	1.6044	1.7982	12.08	1.5622	1.7923	14.73	1.5019	1.7877	19.03

In Fig. 2, 3, the square amplitude of the mean response is computed from (24) and (26) with initial values $\eta_{11} = -1$, $\eta_{12} = -1$, $\eta_{13} = 2$, $\eta_{21} = -0.1$, $\eta_{22} = -2$, $\eta_{23} = 10$, and the input parameters $h = 2$, $\omega = 1$, $\gamma = 1$. It can be observed from Fig 2 that the mean response amplitude increases when harmonic excitation increases. For a given value of harmonic excitation, the white noise is seen to reduce the mean response amplitude (Fig. 3).

In Fig. 4, the time-averaging values of mean square response are computed from the Eqs. (24) and (29) with the same initial values η_{ij}. Fig. 4 portrays effects of the noise intensity σ^2 and the external force's amplitude P on the time-averaging values of mean square response. It is seen that the time-averaging values of mean square response increases with increasing of the external force's amplitude P for a given the noise intensity σ^2.

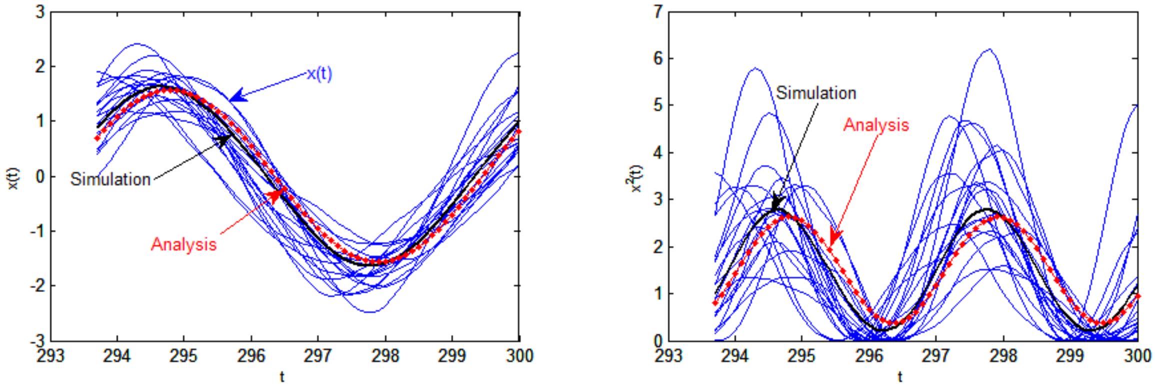

Fig. 1. Analytical results are compared with numerical ones, $h = 2$, $\omega = 1$, $\nu = 1.01$, $P = 5$, $\sigma^2 = 1$.

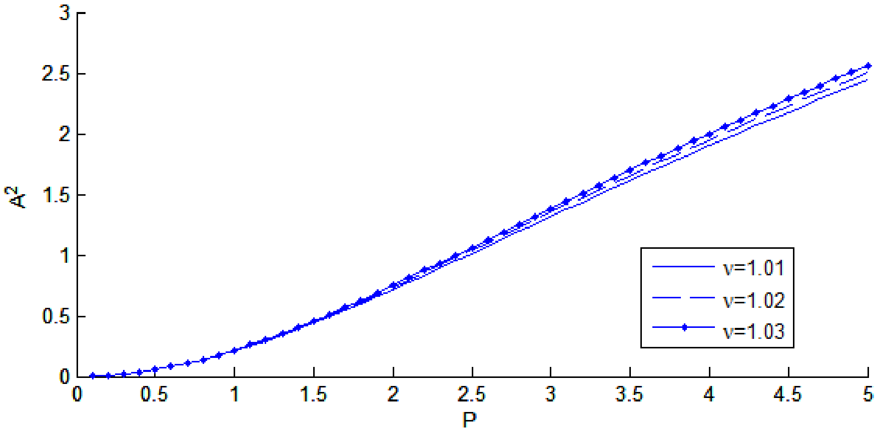

Fig. 2. Mean response amplitude versus P with input parameters $h = 2$, $\omega = 1$, $\sigma^2 = 1$, $\gamma = 1$, and $\nu = 1.01$, $\nu = 1.02$, and $\nu = 1.03$, respectively.

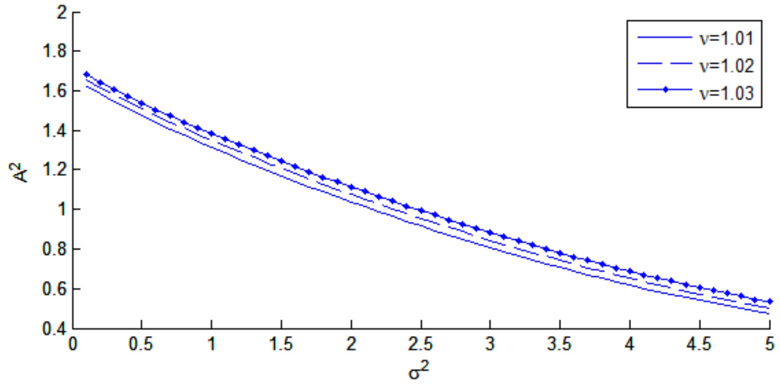

Fig. 3. Mean response amplitude versus σ^2 with input parameters $h=2$, $\omega=1$, $P=3$, $\gamma=1$, and $\nu=1.01$, $\nu=1.02$, and $\nu=1.03$, respectively.

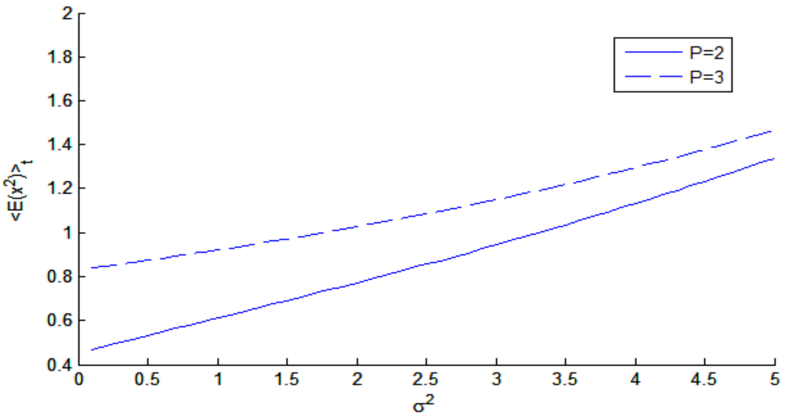

Fig. 4. Time-averaging of mean square response $\langle E(x^2)\rangle_t$ versus the parameter σ^2, $h=2$, $\omega=1$, $\nu=1.01$, $\gamma=1$, and $P=2$, $P=3$, respectively.

4. Summary and conclusions

It is difficult to find an exact solution of a nonlinear system subjected to a combination of harmonic and random excitations but only a few special cases. Thus, the application and development of different methods for such nonlinear systems are very important. It is shown in this paper that the response of the Duffing oscillator is investigated by a new approximate technique using a combination of two typical methods, namely the stochastic averaging and equivalent linearization method. The technique can be summarized as follows. The state coordinates $(x,\dot{x})$ are transformed to Cartesian coordinates (b,d) at first. In this coordinates, the averaged equations are nonlinear ones whose solution is still a difficult problem. The stochastic equivalent linearization method and the technique of auxiliary function are employed to overcome this obstacle. The linearization coefficients of the equivalent linear system are determined from a closed nonlinear algebraic equation system as presented. The exact stationary PDF of the linearized system from which probabilistic characteristics of the response are investigated are obtained. Numerical simulation shows that the analytical results are valid. The significant contribution of the present paper is that the technique gained through it can be helpful for other nonlinear systems.

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