VNU Journal of Science: Mathematics – Physics, Vol. 30, No. 2 (2014) 33-46

The Systems for Generalized of Vector Quasiequilibrium Problems and Its Applications

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Received 22 March 2014
Revised 20 May 2014; Accepted 30 June 2014

Abstract: In this paper, we study the systems of generalized quasiequilibrium problems which includes as special cases the generalized vector quasi-equilibrium problems, vector quasiequilibrium problems, and establish the existence results for its solutions by using fixed-point theorem. Moreover, we also discuss the closedness of the solution sets of systems of generalized quasiequilibrium problems. As special cases, we also derive the existence results for vector quasiequilibrium problems and vector quasivariational inequality problems. Our results are new and improve recent existing ones in the literature.

Keywords: Systems of generalized quasiequilibrium problems, quasiequilibrium problems, quasivariational inequality problem, fixed-point theorem, existence, closedness.

1. Introduction and preliminaries

The systems of generalized quasiequilibrium problems includes as special cases the systems of generalized vector equilibrium problems, vector quasi-equilibrium problems, the systems of implicit vector variational inequality problems, etc. In recent years, a lot of results for existence of solutions for systems vector quasiequilibrium problems, vector quasiequilibrium problems and vector variational inequalities have been established by many authors in different ways. For example, the systems equilibrium problems [1-5], equilibrium problems [3,6-8], variational and optimization problems [9-11] and the references therein.

Let X, Y, Z be real locally convex Hausdorff topological vector spaces $A \subseteq X$ and $B \subseteq Y$ are nonempty compact convex subsets and $C \subset Z$ is a nonempty closed compact convex cone. Let $S_i, P_i : A \times A \to 2^A, T_i : A \times A \to 2^B$ and $F_i : A \times B \times A \to 2^Z, i = 1, 2$ be multifunctions.

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We consider the following systems of generalized quasiequilibrium problems (in short, (SGQEP$_1$) and (SGQEP$_2$)), respectively.

(SGQEP$_1$): Find $(\bar{x},\bar{u})\in A\times A$ and $\bar{z}\in T_1(\bar{x},\bar{u}),\bar{v}\in T_2(\bar{x},\bar{u})$ such that $\bar{x}\in S_1(\bar{x},\bar{u}),\bar{u}\in S_2(\bar{x},\bar{u})$ satisfying

$$F_1(\bar{x},\bar{z},y)\cap(Z\ ,\ -\text{intC})\neq\emptyset,\forall y\in P_1(\bar{x},\bar{u}),$$

$$F_2(\bar{u},\bar{v},y)\cap(Z\ ,\ -\text{intC})\neq\emptyset,\forall y\in P_2(\bar{x},\bar{u}).$$

$(SGQEP_2)$ Find $(\bar{x},\bar{u})\in A\times A$ and $\bar{z}\in T_1(\bar{x},\bar{u}),\bar{v}\in T_2(\bar{x},\bar{u})$ such that $\bar{x}\in S_1(\bar{x},\bar{u}),\bar{u}\in S_2(\bar{x},\bar{u})$ satisfying

$$F_1(\bar{x},\bar{z},y)\subset Z\ ,\ -\text{intC},\forall y\in P_1(\bar{x},\bar{u}),$$

$$F_2(\bar{u},\bar{v},y)\subset Z\ ,\ -\text{intC},\forall y\in P_2(\bar{x},\bar{u}).$$

We denote that $\Sigma_1(F)$ and $\Sigma_2(F)$ are the solution sets of (SGQEP$_1$) and (SGQEP$_2$), respectively.

If $P_i(x,u)=S_i(x,u)=S_i(x)$ for each $(x,u)\in A\times A$ and replace "Z , $-$intC" by C then (SGQEP$_2$) becomes systems vector quasiequilibrium problem (in short,(SQVEP)).

This problem has been studied in [4].

Find $(\bar{x},\bar{u})\in A\times A$ and $\bar{z}\in T_1(\bar{x})$, $\bar{v}\in T_2(\bar{u})$ such that $\bar{x}\in S_1(\bar{x})$, $\bar{u}\in S_2(\bar{u})$ and

$$F_1(\bar{x},\bar{z},y)\subset C,\forall y\in S_1(\bar{x})$$

$$F_2(\bar{u},\bar{v},y)\subset C,\forall y\in S_2(\bar{u}).$$

If $S_1(x,u)=S_2(x,u)=P_1(x,u)=P_2(x,u)=S(x),T_1(x,u)=T_2(x,u)=T(x)$ for each $x\in A$ and $S:A\to 2^A,T:A\to 2^B$ be multifunctions and replace "Z , $-$intC" by "C" then (SGQEP$_2$) becomes vector quasiequilibrium problem (in short,(QVEP)). This problem has been studied in [8].

Find $\bar{x}\in A$ and $\bar{z}\in T(\bar{x})$ such that $\bar{x}\in S(\bar{x})$ and $F(\bar{x},\bar{z},y)\subset C,\forall y\in S(\bar{x})$.

If $S_1(x,u)=S_2(x,u)=P_1(x,u)=P_2(x,u)=S(x),T_1(x,u)=T_2(x,u)=\{z\}$ for each $x\in A$ and $S:A\to 2^A$ be multifunction and replace "Z , $-$intC" by "C", then (SGQEP$_2$) becomes quasiequilibrium problem (in short,(QEP)). This problem has been studied in [5].

Find $\bar{x}\in A$ such that $\bar{x}\in S(\bar{x})$ and $F(\bar{x},y)\subset C,\forall y\in S(\bar{x})$.

In this paper we establish some existence theorems by using fixed-point theorem for systems of generalized quasiequilibrium problems with set-valued mappings in real locally convex Hausdorff topological vector spaces. Moreover, we discuss the closedness of the solution sets of these problems. The results presented in the paper are new; however in the special case, then some results in this paper are improve the main results of Plubtieng and Sitthithakerngkietet [4], Long et al [8], Yang and Pu [5].

The structure of our paper is as follows. In the first part of this article, we introduce the models systems of generalized quasiequilibrium problems and some related models and we recall definitions for later uses. In Section 2, we establish some existence and closedness theorems for these problems. In Section 3 and Section 4, applications of the main results in Section 2 for vector quasiequilibrium problems and vector quasivariational inequality problems.

In this section, we recall some basic definitions and their some properties.

Definition 1.1. ([12], Difinition 1.20)

Let X, Y be two topological vector spaces, A be a nonempty subset of X *and* $F: A \to 2^Y$ is a set-valued mapping.

(i) F is said to be *lower semicontinuous (lsc) at* $x_0 \in A$ if $F(x_0) \cap U \neq \varnothing$ for some open set $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that $F(x) \cap U \neq \varnothing, \forall x \in N$. F is said to be lower semicontinuous in A if it is lower semicontinuous at all $x_0 \in A$.

(ii) F is said to be *upper semicontinuous (usc) at* $x_0 \in A$ if for each open set $U \supseteq G(x_0)$,

there is a neighborhood N of x_0 such that $U \supseteq F(x), \forall x \in N$. F is said to be upper semicontinuous in A if it is upper semicontinuous at all $x_0 \in A$.

(iii) F is said to be *continuous at* $x_0 \in A$ if it is both lsc and usc at x_0. F is said to be continuous in A if it is both lsc and usc at each $x_0 \in A$.

(vi) F is said to be *closed at* $x_0 \in A$ if $\mathrm{Graph(F)}=\{(\mathrm{x},\mathrm{y}): \mathrm{x} \in \mathrm{A}, \mathrm{y} \in \mathrm{F(x)}\}$ is a closed subset in $A \times Y$. F is said to be closed in A if it is closed at all $x_0 \in A$.

Lemma 1.2. ([12]) *Let X and Y be two Hausdorff topological spaces and $F: X \to 2^Y$* be a set-valued mapping.

(i) If F is upper semicontinuous with closed values, then F is closed;

(ii) If F is closed and Y is compact, then F is upper semicontinuous.

The following Lemma 1.3 can be found in [13].

Lemma 1.3. *Let X and Y be two Hausdorff topological spaces and $F: X \to 2^Y$ be a set-valued mapping.*

(i) *F is said to be closed at x_0 if and only if $\forall x_n \to x_0, \forall y_n \to y_0$ such that $y_n \in F(x_n)$, we have $y_0 \in F(x_0)$;*

(ii) *If F has compact values, then F is usc at x_0 if and only if for each net $\{x_\alpha\} \subseteq A$ which converges to x_0 and for each net $\{y_\alpha\} \subseteq F(x_\alpha)$, there are $y \in F(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$.*

Definition 1.4. ([5], Definition 2.1)

Let X, Y be two topological vector spaces and A a nonempty subset of X and let $F: A \to 2^Y$ be a set-valued mappings, with $C \subset Y$ is a nonempty closed compact convex cone.

(i) F is called *upper* C*-continuous at* $x_0 \in A$, if for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that, for all $x \in V$,

$$F(x) \subset F(x_0) + U + C, \forall x \in V.$$

(ii) F is called *lower* C*-continuous at* $x_0 \in A$, if for any neighborhood U of the origin in Y, there is a neighborhood V of x_0 such that, for all $x \in V$,

$$F(x_0) \subset F(x) + U - C, \forall x \in V.$$

Definition 1.5. ([5], Definition 2.2)

Let X and Y be two topological vector spaces and A a nonempty convex subset of X. A set-valued mapping $F : A \to 2^Y$ is said to be *properly* C*-quasiconvex* if, for any $x, y \in A$ and $t \in [0,1]$, we have

$$\text{either} \quad F(x) \subset F(tx+(1-t)y)+C$$
$$\text{or} \quad F(y) \subset F(tx+(1-t)y)+C.$$

The following Lemma is obtained from Ky Fan's Section Theorem, see Lemma 4 of [10]. Moreover, we can be found in Lemma 1.3 in [14], and Lemma 2.4 of [6].

Lemma 1.6. *Let* A *be a nonempty convex compact subset of Hausdorff topological vector space* X *and* Ω *be a subset of* $A \times A$ *such that*

(i) for each at $x \in A, (x,x) \notin \Omega$;

(ii) for each at $y \in A$, *the set* $\{x \in A : (x,y) \in \Omega\}$ *is open in* A;

(iii) for each at $x \in A$, *the set* $\{y \in A : (x,y) \in \Omega\}$ *is convex (or empty).*

Then, there exists $x_0 \in A$ *such that* $(x_0, y) \notin \Omega$ *for all* $y \in A$.

Lemma 1.7. *([12], Theorem 1.27).*

Let A be a nonempty compact subset of a locally convex Hausdorff vector topological space Y. If $M : A \to 2^A$ is upper semicontinuous and for any $x \in A, M(x)$ is nonempty, convex and closed, then there exists an $x^* \in A$ such that $x^* \in M(x^*)$.

2. Existence of solutions

In this section, we give some new existence theorems of the solution sets for systems of generalized quasiequilibrium problems (SGQEP$_1$) and (SGQEP$_2$).

Definition 2.1 Let A, X and Z be as above and $C \subset Z$ is a nonempty closed convex cone. Suppose $F : A \to 2^Z$ be a multifunction.

(i) F is said to be *generalized type I* C*-quasiconvex in* A if $\forall x_1, x_2 \in A, \forall \lambda \in [0,1]$, $F(x_1) \cap (Z, -\text{intC}) \neq \emptyset$ and $F(x_2) \cap (Z, -\text{intC}) \neq \emptyset$. Then, it follows that

$$F(\lambda x_1 + (1-\lambda)x_2) \cap (Z\ ,\ \ -\text{intC}) \neq \varnothing.$$

(ii) F is said to be *generalized type II* C*-quasiconvex in* A if $\forall x_1, x_2 \in A, \forall \lambda \in [0,1]$, $F(x_1) \subset Z\ ,\ \ -\text{intC}$ and. Then, it follows that

$$F(\lambda x_1 + (1-\lambda)x_2) \subset Z\ ,\ \ -\text{intC}.$$

Theorem 2.2 *For each* $\{i = 1,2\}$*, assume for the problem (SGQEP* $_1$*) that*

(i) S_i *is upper semicontinuous in* $A \times A$ *with nonempty closed convex values and* P_i *is lower semicontinuous in* $A \times A$ *with nonempty closed values;*

(ii) T_i *is upper semicontinuous in* $A \times A$ *with nonempty convex compact values;*

(iii) for all $(x,z) \in A \times B$*,* $F_i(x,z,x) \cap (Z\ ,\ \ -\text{intC}) \neq \varnothing$*;*

(iv) the set $\{(z,y) \in B \times A : F_i(.,z,y) \cap (Z\ ,\ \ -\text{intC}) = \varnothing\}$ *is convex;*

(v) for all $(z,y) \in B \times A$*,* $F_i(.,z,y)$ *is generalized type I* C*-quasiconvex;*

(vi) the set $\{(x,z,y) \in A \times B \times A : F_i(x,z,y) \cap (Z\ ,\ \ -\text{intC}) \neq \varnothing\}$ *is closed.*

Then, the (SGQEP $_1$*) has a solution. Moreover, the solution set of the (SGQEP* $_1$*) is closed.*

Proof. For all $(x,z,u,v) \in A \times B \times A \times B$, define mappings: $\Psi_1, \Psi_2 : A \times B \times A \to 2^A$ by

$$\Psi_1(x,z,u) = \{\alpha \in S_1(x,u) : F(\alpha,z,y) \cap (Z\ ,\ \ -\text{intC}) \neq \varnothing, \forall \text{y} \in \text{P}_1(\text{x},\text{u})\},$$

and

$$\Psi_2(x,v,u) = \{\beta \in S_2(x,u) : F(\beta,z,y) \cap (Z\ ,\ \ -\text{intC}) \neq \varnothing, \forall \text{y} \in \text{P}_2(\text{x},\text{u})\}.$$

(I) Show that $\Psi_1(x,z,u)$ and $\Psi_2(x,v,u)$ are nonempty.

Indeed, for all $(x,u) \in A \times A$, $S_i(x,u), P_i(x,u)$ are nonempty convex sets.

Set $\Omega = \{(a,y) \in S_1(x,u) \times P_1(x,u) : F(\alpha,z,y) \cap (Z\ ,\ \ -\text{intC}) = \varnothing\}$.

(a) By the condition (iii) we have, for any $a \in S_1(x,u), (a,a) \notin \Omega$.

(b) By the condition (iv) implies that, for any $a \in S_1(x,u), \{y \in P_1(x,u) : (a,y) \in \Omega\}$ is convex in $S_1(x,u)$.

(c) By the condition (vi), we have for any $a \in S_1(x,u), \{y \in P_1(x,u) : (a,y) \in \Omega\}$ is open in $S_1(x,u)$.

By Lemma 1.6 there exists $a \in S_1(x,u)$ such that $(a,y) \notin \Omega$, for all $y \in P_1(x,u)$, i.e., $F(\alpha,z,y) \cap (Z\ ,\ \ -\text{intC}) \neq \varnothing, \forall \text{y} \in \text{P}_1(\text{x},\text{u})$. Thus, $\Psi_1(x,z,u) \neq \varnothing$. Similarly, we also have $\Psi_2(x,z,u) \neq \varnothing$.

(II) Show that $\Psi_1(x,z,u)$ and $\Psi_2(x,v,u)$ are nonempty convex sets.

Let $a_1, a_2 \in \Psi(x,z,u)$ and $\alpha \in [0,1]$ and put $a = \alpha a_1 + (1-\alpha)a_2$. Since $a_1, a_2 \in S_1(x,u)$ and $S_1(x,u)$ is a convex set, we have $a \in S_1(x,u)$. Thus, for $a_1, a_2 \in \Psi(x,z,u)$, it follows that

$$F_1(a_1,z,y) \cap (Z , \quad -\mathrm{intC}) \neq \varnothing, \forall \mathrm{y} \in \mathrm{P}_1(\mathrm{x},\mathrm{u}),$$

$$F_1(a_2,z,y) \cap (Z , \quad -\mathrm{intC}) \neq \varnothing, \forall \mathrm{y} \in \mathrm{P}_1(\mathrm{x},\mathrm{u}).$$

By (v) F_1(., z, y)$ is generalized type I C-quasiconvex.

$$F_1(\alpha a_1 + (1-\alpha)a_2, z, y) \cap (Z , \quad -\mathrm{intC}) \neq \varnothing, \forall \alpha \in [0,1],$$

i.e., $a \in \Psi(x,z)$. Therefore, $\Psi_1(x,z,u)$ is a convex set. Similarly, we have $\Psi_2(x,v,u)$ is a convex set.

(III) We will prove Ψ_1 and Ψ_2 are upper semicontinuous in $A \times B \times A$ with nonempty closed values.

First, we show that Ψ_1 is upper semicontinuous in $A \times B \times A$ with nonempty closed values. Indeed, since A is a compact set, by Lemma 1.2 (ii), we need only show that Ψ_1 is a closed mapping. Indeed, let a net $\{(x_n, z_n, u_n) : n \in I\} \subset A \times B$ such that $(x_n, z_n, u_n) \to (x,z,u) \in A \times B \times A$, and let $\alpha_n \in \Psi_1(x_n, z_n, u_n)$ such that $\alpha_n \to \alpha_0$. Now we need to show that $\alpha_0 \in \Psi_1(x,z,u)$. Since $\alpha_n \in S_1(x_n, u_n)$ and S_1 is upper semicontinuous with nonempty closed values, by Lemma 1.2 (i), hence S_1 is closed, thus, we have $\alpha_0 \in S_1(x,u)$. Suppose to the contrary $\alpha_0 \notin \Psi_1(x,z,u)$. Then, $\exists y_0 \in P_1(x,u)$ such that

$$F_1(\alpha_0, z, y_0) \cap (Z , \quad -\mathrm{intC}) = \varnothing. \tag{2.1}$$

By the lower semicontinuity of P_1, there is a net $\{y_n\}$ such that $y_n \in P_1(x_n, u_n)$, $y_n \to y_0$. Since $\alpha_n \in \Psi_1(x_n, z_n, u_n)$, we have

$$F_1(\alpha_n, z_n, y_n) \cap (Z , \quad -\mathrm{intC}) \neq \varnothing. \tag{2.2}$$

By the condition (v) and (2.2), we have

$$F_1(\alpha_0, z, y_0) \cap (Z , \quad -\mathrm{intC}) \neq \varnothing. \tag{2.3}$$

This is the contradiction between (2.1) and (2.3).

Thus, $\alpha_0 \in \Psi_1(x,z,u)$. Hence, Ψ_1 is upper semicontinuous in $A \times B \times A$ with nonempty closed values. Similarly, we also have $\Psi_2(x,z,u)$ is upper semicontinuous in $A \times B \times A$ with nonempty closed values.

(IV) Now we need to the solutions set $\Sigma_1(F) \neq \varnothing$.

Define the set-valued mappings $\Phi_1, \Phi_2 : A \times B \times A \to 2^{A \times B}$ by

$$\Phi_1(x,z,u) = (\Psi_1(x,z,u), T_1(x,u)), \forall (x,z,u) \in A \times B \times A$$

and

$$\Phi_2(x,v,u) = (\Psi_2(x,v,u), T_2(x,u)), \forall (x,v,u) \in A \times B \times A.$$

Then Φ_1, Φ_2 are upper semicontinuous and $\forall(x,z,u)\in A\times B\times A$, $\forall(x,v,u)\in A\times B\times A$, $\Phi_1(x,z,u)$ and $\Phi_2(x,v,u)$ are nonempty closed convex subsets of $A\times B\times A$.

Define the set-valued mapping $E:(A\times B)\times(A\times B)\to 2^{(A\times B)\times(A\times B)}$ by

$$E((x,z),(u,v))=(\Phi_1(x,z,u),\Phi_2(x,v,u)),\forall((x,z),(u,v))\in(A\times B)\times(A\times B).$$

Then E is also upper semicontinuous and $\forall((x,z),(u,v))\in(A\times B)\times(A\times B)$, $E((x,z),(u,v))$ is a nonempty closed convex subset of $(A\times B)\times(A\times B)$.

By Lemma 1.7, there exists a point $((\hat{x},\hat{z}),(\hat{u},\hat{v}))\in(A\times B)\times(A\times B)$ such that $((\hat{x},\hat{z}),(\hat{u},\hat{v}))\in E((\hat{x},\hat{z}),(\hat{u},\hat{v}))$, that is

$$(\hat{x},\hat{z})\in\Phi_1(\hat{x},\hat{z},\hat{u}),\ (\hat{u},\hat{v})\in\Phi_2(\hat{x},\hat{v},\hat{u}),$$

which implies that $\hat{x}\in\Psi_1(\hat{x},\hat{z},\hat{u}),\hat{z}\in T_1(\hat{x},\hat{u}),\hat{u}\in\Psi_2(\hat{x},\hat{v},\hat{u})$ and $\hat{v}\in T_2(\hat{x},\hat{u})$. Hence, there exist $(\hat{x},\hat{u})\in A\times A,\hat{z}\in T_1(\hat{x},\hat{u})$, $\hat{v}\in T_2(\hat{x},\hat{u})$ such that $\hat{x}\in S_1(\hat{x},\hat{u}),\hat{u}\in S_2(\hat{x},\hat{u})$, satisfying

$$F_1(\hat{x},\hat{z},y)\cap(Z\ ,\ \ -\mathrm{int}C)\neq\emptyset,\forall y\in P_1(\hat{x},\hat{u}),$$

and

$$F_2(\hat{u},\hat{v},y)\cap(Z\ ,\ \ -\mathrm{int}C)\neq\emptyset,\forall y\in P_2(\hat{x},\hat{u}),$$

i.e., ($SGQEP_1$) has a solution.

(V) Now we prove that $\Sigma_1(F)$ is closed.

Indeed, let a net $\{(x_n,u_n),n\in I\}\in\Sigma_1(F)$: $(x_n,u_n)\to(x_0,u_0)$. As $(x_n,u_n)\in\Sigma_1(F)$, there exist $z_n\in T_1(x_n,u_n),v_n\in T_2(x_n,u_n),x_n\in S_1(x_n,u_n),u_n\in S_2(x_n,u_n)$ such that

$$F_1(x_n,z_n,y)\cap(Z\ ,\ \ -\mathrm{int}C)\neq\emptyset,\forall y\in P_1(x_n,u_n).$$

and

$$F_2(u_n,v_n,y)\cap(Z\ ,\ \ -\mathrm{int}C)\neq\emptyset,\forall y\in P_2(x_n,u_n).$$

Since S_1,S_2 are upper semicontinuous with nonempty closed values, by Lemma 1.2 (i), we have S_1,S_2 are closed. Thus, $x_0\in S_1(x_0,u_0),u_0\in S_2(x_0,u_0)$. Since T_1,T_2 are upper semicontinuous and $T_1(x_0,u_0),T_2(x_{0},_0)$ are compact. There exist $z\in T_1(x_0,u_0)$ and $v\in T_2(x_0,u_0)$ such that $z_n\to z,v_n\to v$ (taking subnets if necessary). By the condition (vi) and $(x_n,z_n,u_n,v_n)\to(x_0,z,u_0,v)$, we have

$$F_1(x_0,z,y)\cap(Z\ ,\ \ -\mathrm{int}C)\neq\emptyset,\forall y\in P_1(x_0,u_0),$$

and

$$F_2(u_0,v,y)\cap(Z\ ,\ \ -\mathrm{int}C)\neq\emptyset,\forall y\in P_2(x_0,u_0).$$

This means that

$(x_0,u_0)\in\Sigma_1(F)$. Thus $\Sigma_1(F)$ is a closed set. □

Theorem 2.3. *Assume for the problem (SGQEP$_1$) assumptions (i), (ii), (iii), (iv) and (v), as in Theorem 2.2 and replace (vi) by (vi')*

(vi') for each $i = \{1,2\}$, F_i *is upper semicontinuous in* $A \times B \times A$ *and has compact valued.*

Then, the (SGQEP$_1$) has a solution. Moreover, the solution set of the (SGQEP$_1$) is closed.

Proof. We omit the proof since the technique is similar as that for Theorem 2.2 with suitable modifications. □

The following example shows that all assumptions of Theorem 2.2 are satisfied. However, Theorem 2.3 does not work. The reason is that F_i is not upper semicontinuous

Example 2.4. Let $X = Y = Z = R, A = B = [0,1], C = [0,+\infty)$ and let $S_1(x,u) = S_2(x,u) = T_1(x,u) = T_2(x,u) = [0,1]$

and

$$\mathrm{F}_1(\mathrm{x,z,y}) = \mathrm{F}_2(\mathrm{u,v,y}) = \mathrm{F}(\mathrm{x,z,y}) = \begin{cases} [\frac{1}{9}, \frac{1}{3}] & \text{if } x_0 = z_0 = y_0 = \frac{1}{2}, \\ [\frac{1}{e^3}, \frac{2}{3}] & otherwise. \end{cases}$$

We show that all assumptions of Theorem 2.2 are satisfied. However, F is not upper semicontinuous at $x_0 = \frac{1}{2}$. Also, Theorem 2.3 is not satisfied.

Passing to problem (SGQEP$_2$) we have.

Theorem 2.5 *For each* $\{i = 1,2\}$, *assume for the problem (SGQEP$_2$) that*

(i) S_i *is upper semicontinuous in* $A \times A$ *with nonempty closed convex values and* P_i *is lower semicontinuous in* $A \times A$ *with nonempty closed values;*

(ii) T_i *is upper semicontinuous in* $A \times A$ *with nonempty convex compact values;*

(iii) for all $(x,z) \in A \times B$, $F_i(x,z,x) \subset Z$, $-\mathrm{intC}$;

(iv) the set $\{(z,y) \in B \times A : F_i(.,z,y) \subset Z, \ -\mathrm{intC}\}$ *is convex;*

(v) for all $(z,y) \in B \times A$, $F_i(.,z,y)$ *is generalized type II* C*-quasiconvex;*

(vi) the set $\{(x,z,y) \in A \times B \times A : F_i(x,z,y) \subset Z, \ -\mathrm{intC}\}$ *is closed.*

Then, the (SGQEP$_2$) has a solution. Moreover, the solution set of the (SGQEP$_2$) is closed.

Proof. We can adopt the same lines of proof as in Theorem 2.2 with new multifunctions $\Delta_1(x,z,u)$ and $\Delta_2(x,v,u)$ defined as: $\Delta_1, \Delta_2 : A \times B \times A \to 2^A$ by

$$\Delta_1(x,z,u) = \{a \in S_1(x,u) : F_1(a,z,y) \subset Z, \ -\mathrm{intC}, \forall \mathrm{y} \in \mathrm{P}_1(\mathrm{x,u})\},$$

and

$$\Delta_2(x,v,u) = \{b \in S_2(x,u) : F_2(b,z,y) \subset Z, \ -\mathrm{intC}, \forall \mathrm{y} \in \mathrm{P}_2(\mathrm{x,u})\}. \quad □$$

If $P_i(x,u) = S_i(x,u) = S(x)$ for each $(x,u) \in A \times B$, then (SGQEP$_2$) becomes the system vector quasiequilibrium problem (in short, (SQVEP)), we have the following Corollary.

Corollary 2.6 *For each* $\{i = 1,2\}$, *assume for the problem (SQVEP) that*

(i) S_i *is continuous in* $A \times A$ *with nonempty closed convex;*

(ii) T_i *is upper semicontinuous in* $A \times A$ *with nonempty convex compact values;*

(iii) for all $(x,z) \in A \times B$, $F_i(x,z,x) \subset Z$, $-\text{intC}$;

(iv) the set $\{(z,y) \in B \times A : F_i(.,z,y) \subset Z, -\text{intC}\}$ *is convex;*

(v) for all $(z,y) \in B \times A$, $F_i(.,z,y)$ *is generalized type II* C*-quasiconvex;*

(vi) the set $\{(x,z,y) \in A \times B \times A : F_i(x,z,y) \subset Z, -\text{intC}\}$ *is closed.*

Then, the (SQVEP) has a solution. Moreover, the solution set of the (SQVEP) is closed.

Proof. The result is derived from the technical proof for Theorem 2.5. □

If $S_1(x,u) = S_2(x,u) = P_1(x,u) = P_2(x,u) = S(x), T_1(x,u) = T_2(x,u) = T(x)$, $F_1(x,z,y) = F_2(u,v,y) = F(x,z,y)$ for each $(x,u) \in A \times A$ and $S : A \to 2^A, T : A \to 2^B, F : A \times B \times A \to 2^Z$ be multifunctions, then (SGQEP$_2$) becomes vector quasiequilibrium problem (in short,(QVEP)), we have the following Corollary.

Corollary 2.7. **A***ssume for the problem (QVEP) that*

(i) S *is continuous in* A *with nonempty closed convex;*

(ii) T *is upper semicontinuous in* A *with nonempty convex compact values;*

(iii) for all $(x,z) \in A \times B$, $F(x,z,x) \subset Z$, $-\text{intC}$;

(iv) the set $\{(z,y) \in B \times A : F(.,z,y) \subset Z, -\text{intC}\}$ *is convex;*

(v) for all $(z,y) \in B \times A$, $F(.,z,y)$ *is generalized type II* C*-quasiconvex;*

(vi) the set $\{(x,z,y) \in A \times B \times A : F(x,z,y) \subset Z, -\text{intC}\}$ *is closed.*

Then, the (SQVEP) has a solution. Moreover, the solution set of the (QVEP) is closed.

Proof. The result is derived from the technical proof for Theorem 2.5. □

If $S_1(x,u) = S_2(x,u) = P_1(x,u) = P_2(x,u) = S(x), T_1(x,u) = T_2(x,u) = \{z\}$, $F_1(x,z,y) = F_2(u,v,y) = F(x,y)$ for each $(x,u) \in A \times A$ and $S : A \to 2^A, F : A \times A \to 2^Z$ be two multifunctions, then (SGQEP$_2$) becomes quasiequilibrium problem (in short,(QEP)), we have the following Corollary.

Corollary 2.8 *Assume for the problem (QEP) that*

(i) S *is continuous in* A *with nonempty closed convex;*

(ii) for all $x \in A$, $F(x,x) \subset Z$, $-\text{intC}$;

(iii) the set $\{x \in A : F(., y) \subset Z \setminus -\text{intC}\}$ *is convex;*

(iv) for all $x \in A$*,* $F(., y)$ *is generalized type II* C*-quasiconvex;*

(v) the set $\{(x, y) \in A \times A : F(x, y) \subset Z \setminus -\text{intC}\}$ *is closed.*

Then, the (QEP) has a solution. Moreover, the solution set of the (QEP) is closed.

Remark 2.9 Note that, the models (SQVEP), (QVEP) and (QEP) are different from the models (SGSVQEPs), (GSVQEP) and (SVQEP) in [4], [8] and [5], respectively. However, if we replace "$Z \setminus -\text{intC}$" by "C", then Corollary 2.6, Corollary 2.7 and Corollary 2.8 reduces to Theorem 3.1 in [4], Theorem 3.1 in [8] and Theorem 3.3 in [5], respectively. But, our Corollary 2.6, Corollary 2.7 and Corollary 2.8 are stronger than Theorem 3.1 in [4], Theorem 3.1 in [8] and Theorem 3.3 in [5], respectively.

The following example shows that all the assumptions in Corollary 2.6, Corollary 2.7 and Corollary 2.8 are satisfied. However, Theorem 3.1 in [4], Theorem 3.1 in [8] and Theorem 3.3 in [5] are not satisfied. It gives also cases where Corollary 2.6, Corollary 2.7 and Corollary 2.8 can be applied but Theorem 3.1 in [4], Theorem 3.1 in [8] and Theorem 3.3 in [5] do not work.

Example 2.10. Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = [0,+\infty)$ and let $K, T : [0,1] \to 2^{\mathbb{R}}$, $F : [0,1] \to 2^{\mathbb{R}}$, $S_1(x,u) = S_2(x,u) = P_1(x,u) = P_2(x,u) = K(x) = [0,1], T_1(x,u) = T_2(x,u) = T(x) = [0,1]$ and

$$F_1(x,z,y) = F_2(u,v,y) = F(x) = \begin{cases} [\frac{3}{2}, 2] & \text{if } x_0 = \frac{1}{5}, \\ [\frac{1}{3}, \frac{1}{2}] & \textit{otherwise}. \end{cases}$$

We show that all the assumptions in Corollary 2.6, Corollary 2.7 and Corollary 2.8 are satisfied. However, Theorem 3.1 in [4], Theorem 3.1 in [8] and Theorem 3.3 in [5] are not satisfied. The reason is that F is neither upper C-continuous nor properly C-quasiconvex at $x_0 = \frac{1}{5}$. Thus, it gives cases where Corollary 2.6, Corollary 2.7 and Corollary 2.8 can be applied but Theorem 3.1 in [4], Theorem 3.1 in [8] and Theorem 3.3 in [5] do not work.

Theorem 2.11. *Assume for the problem (SGQEP$_2$) assumptions (i), (ii), (iii), (iv) and (v), as in Theorem 2.2 and replace (vi) by (vi')*

(vi') for each $i = \{1,2\}$, F_i is lower semicontinuous in $A \times B \times A$.

Then, the (SGQEP$_2$) has a solution. Moreover, the solution set of the (SGQEP$_2$) is closed.

Proof. We omit the proof since the technique is similar as that for Theorem 2.5 with suitable modifications. □

The following example shows that all assumptions of Theorem 2.5 are satisfied. However, Theorem 2.11 does not work. The reason is that F_i is not lower semicontinuous

Example 2.12. Let $X = Y = Z = R, A = B = [0,1],\ C = [0,+\infty)$ and let $S_1(x,u) = S_2(x,u) = T_1(x,u) = T_2(x,u) = [0,1]$ and

$$\mathrm{F}_1(\mathrm{x,z,y}) = \mathrm{F}_2(\mathrm{u,v,y}) = \mathrm{F}(\mathrm{x,z,y}) = \begin{cases} [\frac{1}{10}, \frac{2}{3}] & \text{if } x_0 = z_0 = y_0 = \frac{1}{3}, \\ [\frac{1}{2}, 1] & \textit{otherwise.} \end{cases}$$

We show that all assumptions of Theorem 2.5 are satisfied. However, F is not upper semicontinuous at $x_0 = \frac{1}{2}$. Also, Theorem 2.11 is not satisfied.

3. Applications (I): Quasiequilibrium problems

Let X, Z and A, C be as in Section 1. Let $K : A \to 2^A$ be multifunction and $f : A \times A \to Y$ is a vector-valued function. We consider two the following quasiequilibrium problems (in short, (QEP$_1$) and (QEP$_2$)), respectively.

(QEP$_1$): Find $\bar{x} \in A$ such that $\bar{x} \in K(\bar{x})$ satisfying

$$f(\bar{x}, y) \cap (Z\ ,\quad -\mathrm{int}C) \neq \emptyset, \forall y \in K(\bar{x}),$$

and

(QEP$_2$): Find $\bar{x} \in A$ such that $\bar{x} \in K(\bar{x})$ satisfying

$$f(\bar{x}, y) \in Z\ ,\quad -\mathrm{int}C, \forall y \in K(\bar{x}),$$

Corollary 3.1 Assume for problem (QEP$_1$) that

(i) K is continuous in A with nonempty convex closed values;

(ii) for all $x \in A$, $f(x,x) \cap (Z\ ,\quad -\mathrm{int}C) \neq \emptyset$;

(iii) the set $\{y \in A : f(.,y) \cap (Z\ ,\quad -\mathrm{int}C) = \emptyset\}$ is convex;

(iv) for all $y \in A$, $f(.,y)$ is generalized type I C-quasiconvex;

(v) the set $\{(x,y) \in A \times A : f(x,y) \cap (Z\ ,\quad -\mathrm{int}C) \neq \emptyset\}$ is closed.

Then, the (QEP$_1$) has a solution. Moreover, the solution set of the (QEP$_1$) is closed.

Proof. Setting $Y = X, B = A$ and $S_1(x,u) = S_2(x,u) = K(x), T_1(x,u) = T_2(x,u) = \{z\}$, $F_1 = F_2 = f$, problem (QEP$_1$) becomes a particular case of (QEP$_1$) and the Corollary 3.1 is a direct consequence of Theorem 2.2. □

Corollary 3.2. *Assume for the problem (QEP$_1$) assumptions (i), (ii), (iii) and (iv) as in Corollary 3.1 and replace (v) by (v')* (v') f *is continuous in* $A\times B\times A$.

Then, the (QEP_1) *has a solution. Moreover, the solution set of the* (QEP_1) *is closed.*

Proof. We omit the proof since the technique is similar as that for Corollary 3.1 with suitable modifications. □

Corollary 3.3. *Assume for problem* (QEP_2) *that*

(i) K *is continuous in* A *with nonempty convex closed values;*

(ii) for all $x\in A$, $f(x,x)\in Z\setminus -\text{int}C$;

(iii) the set $\{y\in A: f(.,y)\in Z\setminus -\text{int}C\}$ *is convex;*

(iv) for all $y\in A$, $f(.,y)$ *is generalized type II* C*-quasiconvex;*

(v) the set $\{(x,y)\in A\times A: f(x,y)\in Z\setminus -\text{int}C\}$ *is closed.*

Then, the (QEP_2) *has a solution. Moreover, the solution set of the* (QEP_2) *is closed.*

Proof. Setting $Y=X, B=A$ and $S_1(x,u)=S_2(x,u)=K(x), T_1(x,u)=T_2(x,u)=\{z\}$, $F_1=F_2=f$, problem (QEP_2) becomes a particular case of (QEP_2) and the Corollary 3.3 is a direct consequence of Theorem 2.5. □

Corollary 3.4. *Assume for the problem* (QEP_2) *assumptions (i), (ii), (iii) and (iv) as in Corollary 3.3 and replace (v) by (v')*

(v') f *is continuous in* $A\times B\times A$.

Then, the (QEP_2) *has a solution. Moreover, the solution set of the* (QEP_2) *is closed.*

Proof. We omit the proof since the technique is similar as that for Corollary 3.3 with suitable modifications. □

4. Applications (II): Quasivariational inequality problems

Let X, Y, Z and A, B, C be as in Section 1. Let $L(X,Z)$ be the space of all linear continuous operators from X into Z, and $K: A\to 2^A$ and $T: A\to 2^{L(X,Z)}$ are set-valued mappings, $H: L(X,Z)\to L(X,Z), \eta: A\times A\to A$ be continuous single-valued mappings. Denoted $\langle z,x\rangle$ by the value of a linear operator $z\in L(X;Y)$ at $x\in X$, we always assume that $\langle .,.\rangle: L(X;Z)\times X\to Z$ is continuous.

We consider the following vector quasivariational inequality problems (in short, (QVIP)).

(QVIP) Find $\bar{x}\in A$ and $\bar{z}\in T(\bar{x})$ such that $\bar{x}\in K(\bar{x})$ satisfying

$$\langle Q(\bar{z}), \eta(y,\bar{x})\rangle \in Z\setminus -\text{int}C, \forall y\in K(\bar{x}),$$

Corollary 4.1. *Assume for the problem (QVIP) that*

(i) K *is continuous in* A *with nonempty convex closed values;*

(ii) T *is upper semicontinuous in* A *with nonempty convex compact values;*

(iii) for all $(x,z) \in A \times B$, $\langle Q(z), \eta(x,x) \rangle \in Z$, $-\text{int}C$ *;*

(iv) the set $\{(y,z) \in A \times B : \langle Q(z), \eta(y,.) \rangle \notin Z$, $-\text{intC}\}$ *is convex;*

(v) for all $(y,z) \in A \times B$ *, the function* $x \mapsto \langle Q(z), \eta(y,x) \rangle$ *is generalized type II* C *-quasiconvex;*

Then, the (QVIP) has a solution. Moreover, the solution set of the (QVIP) is closed.

Proof. Setting $S_1(x,u) = S_2(x,u) = K(x), T_1(x,u) = T_2(x,u) = T(x), F_1(x,z,y) = F_2(u,v,y) =$ $F(x,z,y) = \langle Q(z), \eta(y,x) \rangle$, the problem (QVIP) becomes a particular case of (SGQEP $_2$) and the Corollary 4.1 is a direct consequence of Theorem 2.5. □

Corollary 3.4. Impose the assumptions of Corollary 4.1 and the following additional condition:

(vi) the set $\{(x,z,y) \in A \times B \times A : \langle Q(z), \eta(y,x) \rangle \in Z$, $-\text{int}C\}$ *is closed.*

Then, the (QVIP) has a solution. Moreover, the solution set of the (QVIP) is closed.

Proof. We omit the proof since the technique is similar as that for Corollary 4.1 with suitable modifications.

Acknowledgment

This research is the output of the project "On existence of solution maps for system multivalued quasi-equilibrium problems and its application" under grant number D2014-06 which belongs to University of Information Technology-Vietnam National University HoChiMinh City.

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