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Predator-prey System with the Effect of Environmental Fluctuation

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Abstract: In this paper we study the trajectory behavior of Lotka - Volterra predator - prey systems with periodic coefficients under telegraph noises. We describe the ω - limit set of the solution, give sufficient conditions for the persistence and prove the existence of a Markov periodic solution.

Keywords: Key words and phrases, Lotka-Volterra Equation, Predator - Prey, Telegraph noise, ω - limit set, Markov periodic solution.

1. Introduction

The Kolmogorov equation

$$\begin{cases} \dot{x}(t) = x\, f\left[t, x(t), y(t)\right] \\ \dot{y}(t) = y\, g\left[t, x(t), y(t)\right] \end{cases}$$

with the functions $f\left[t, x(t), y(t)\right]$; $g\left[t, x(t), y(t)\right]$ periodic in t is a strong tool to describe the evolution of prey-predator communities depending on the changing of seasons. There is a lot of work dealing with the asymptotic behavior of such systems as the existence of periodic solutions, the persistence... [1-4] In particular, the classical model for a system consisting of two species in prey-predator relation

$$\begin{cases} \dot{x}(t) = x(t)\left[a(t) - b(t)x(t) - c(t)y(t)\right] \\ \dot{y}(t) = y(t)\left[-d(t) + e(t)x(t) - f(t)y(t)\right] \end{cases} \quad (1.1)$$

with the periodic coefficients *a; b; c; d; e; f* is well investigated in [5-10], where $x(t)$ (resp. $y(t)$) is the quantity of the prey (resp. of predator) at time t.

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In almost of these works, one supposes that the communities develop under an environment without random perturbation. However, it is clear that it is not the case in reality because in general, annual seasonal living conditions of the communities are not the same. Therefore, it is important to take into account not only in the changing of seasons but also in the fluctuation of stochastic factors, which may have important consequences on the dynamics of the communities.

For the stochastic Lotka - Volterra equation, a systematic review has been given in [11-13]. In our separate paper [14], we analyze the Lotka - Volterra predator-prey system with constant coefficients under the telegraph noises, i.e., environmental variability causes the parameter switching between two systems. Then we have described some parts of ω-set of solutions and show out the existence of a stationary distribution.

In this paper, we want to consider predator-prey models under the influence of stochastic fluctuation of environment and changing periodically of season as well. We describe completely the omega limit set of the positive solutions of Equation (1.1) with the periodic coefficients under the telegraph noises. Also, the existence of a Markov periodic solution that attracts the other solutions of Equation (2.4), starting in $\mathbb{R}_+ \times \mathbb{R}_+$ under certain conditions is proved.

The rest of the paper is divided into three sections. Section 2 details the model. Some properties of the solution and the set of omega limit are shown in section 3. The last section is some simulations and discussions.

2. Preliminary

Let (Ω, F, P) be a complete probability space and $\{\xi(t): t \geq 0\}$ be a continuous-time Markov chain defined on (Ω, F, P), whose state space is a two-element set $M = \{-, +\}$ and whose generator is given by

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

with $\alpha > 0$ and $\beta > 0$. It follows that, $\varpi = (p, q)$, the stationary distribution of $\{\xi(t): t \geq 0\}$ satisfying the system of equations

$$\begin{cases} \varpi Q = 0 \\ p + q = 1 \end{cases}$$

is given by

$$\begin{cases} p = \lim\limits_{t \to +\infty} P\{\xi(t) = 1\} = \dfrac{\beta}{\alpha + \beta} \\ q = \lim\limits_{t \to +\infty} P\{\xi(t) = 2\} = \dfrac{\beta}{\alpha + \beta} \end{cases} \tag{2.1}$$

Such a two-state Markov chain is commonly referred to as telegraph noise because of the nature of its sample paths. The trajectory of $\{\xi_t\}$ is piecewise-constant, cadlag functions. Suppose that

$$0=\tau_0<\tau_1<\tau_2<...<\tau_n<... \qquad (2.2)$$

are its jump times. Put

$$\sigma_1:=\tau_1-\tau_0\ ,\ \sigma_2:=\tau_2-\tau_1\ ,\ ...\ ,\sigma_n:=\tau_n-\tau_{n-1} \qquad (2.3)$$

It is known that the sequence $\{\sigma_k\}_{k=1}^{n}$ is an independent random variables in the condition of given sequence $\{\xi_{\tau_k}\}_{k=1}^{n}$ (see [15, 16]). Note that if ξ_0 is given then ξ_{τ_n} is constant since the process $\{\xi_t\}$ takes only two values. Hence, $(\sigma_k)_{k=1}^{\infty}$ is a sequence of conditionally independent random variables, valued in $[0,+\infty]$. Moreover, if $\xi_0=+$ then σ_{2n+1} has the exponential density $\alpha 1_{[0,+\infty)}e^{-\alpha t}$ and σ_{2n+1} has the density $\beta 1_{[0,+\infty)}e^{-\beta t}$. Conversely, if $\xi_0=-$ then σ_{2n} has the exponential density $\alpha 1_{[0,+\infty)}e^{-\alpha t}$ and σ_{2n+1} has the density $\beta 1_{[0,+\infty)}e^{-\beta t}$ (see [15]). Here

$$1_{[0,+\infty)}=\begin{cases}1\ ,\ t\geq 0\\ 0\ ,\ t<0\end{cases}\ .$$

Denote $\Im_0^n=\sigma(\tau_k\ ,k\leq n)$; $\Im_n^{\infty}=\sigma(\tau_k-\tau_n\ ,k>n)$. We see that $\Im_0^n$ is independent of $\Im_n^{\infty}$ for any $n\in\mathbb{N}$ in the condition that ξ_0 given.

Let ξ_0 have the distribution $\mathrm{P}\{\xi_0=+\}=p$; $\mathrm{P}\{\xi_0=-\}=q$ then $\{\xi_t\}$ is a stationary process. Therefore, there exists a group $\theta^t, t\in\mathbb{R}$ of $P-$ preserving measure transformations $\theta^t:\Omega\to\Omega$ such that $\xi_t(\omega)=\xi_0(\theta^t\omega)$, $\omega\in\Omega$.

We consider the periodic predator-prey equation under a random environment. Suppose that the quantity x of the prey and the quantity y of the predator are described by a Lotka - Volterra equation

$$\begin{cases}\dot{x}=x\left[a(\xi_t,t)-b(\xi_t,t)x-c(\xi_t,t)y\right]\\ \dot{y}=y\left[-d(\xi_t,t)+e(\xi_t,t)x-f(\xi_t,t)y\right]\end{cases} \qquad (2.4)$$

where $g:\mathrm{E}\to\mathbb{R}_+$ for $g=a,b,c,d,e,f$ such that $g(i,.)$ are continuous and periodic functions with period T > 0 for any $i\in\mathrm{E}$. Moreover, $m\leq g(i,t)\leq M$; in which m and M are two positive constants.

In case where the noise $\{\xi_t\}$ intervenes virtually into Equation (2.4), it makes a switching between the deterministic periodic system

$$\begin{cases}\dot{x}_+(t)=x_+(t)\left[a(+,t)-b(+,t)x_+(t)-c(+,t)y_+(t)\right]\\ \dot{y}_+(t)=y_+(t)\left[-d(+,t)+e(+,t)x_+(t)-f(+,t)y_+(t)\right]\end{cases} \qquad (2.5)$$

and another

$$\begin{cases}\dot{x}_-(t)=x_-(t)\left[a(-,t)-b(-,t)x_-(t)-c(-,t)y_-(t)\right]\\ \dot{y}_-(t)=y_-(t)\left[-d(-,t)+e(-,t)x_-(t)-f(-,t)y_-(t)\right]\end{cases} \qquad (2.6)$$

Thus, the relationship of these two systems will determine the trajectory behavior of Equation (2.4).

System (2.4) without the noise $\{\xi_t\}$, i.e., $g(\xi_t, t) = g(t)$ for any $g = a, b, ..., f$ is studied in [9]. They show that

Theorem 2.1. Consider the system

$$\begin{cases} \dot{x}(t) = x(t)\left[a(t) - b(t)x(t) - c(t)y(t)\right] \\ \dot{y}(t) = y(t)\left[-d(t) + e(t)x(t) - f(t)y(t)\right] \end{cases} \quad (2.7)$$

where $a, b, ..., f$ are T-periodic functions.

a) If

$$inf\left(\frac{a}{b}\right) > sup\left(\frac{d}{e}\right) \quad (2.8)$$

$$inf\left(\frac{b}{e}\right) > sup\left(\frac{c}{d}\right) \quad (2.9)$$

then system (2.7) has a positive T-periodic solution $(x^*(t), y^*(t))$ satisfying

$$\left(x(t) - x^*(t),\ y(t) - y^*(t)\right) \xrightarrow{t\to\infty} (0,0). \quad (2.10)$$

b) If

$$inf\left(\frac{d}{e}\right) > sup\left(\frac{a}{b}\right) \quad (2.11)$$

then the (unique) periodic solution $u^*(t)$ of the equation $\dot{u}(t) = u(t)\left[a(t) - b(t)u(t)\right]$ is stable and

$$\left(x(t) - u^*(t), y(t)\right) \xrightarrow{t\to+\infty} (0,0) \quad (2.12)$$

for any positive solution $(x(t),\ y(t))$ to (2.7).

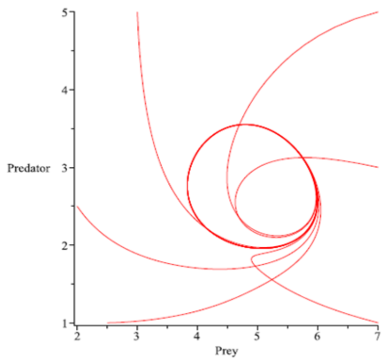

Figure 1. Coexistence of predator and prey.

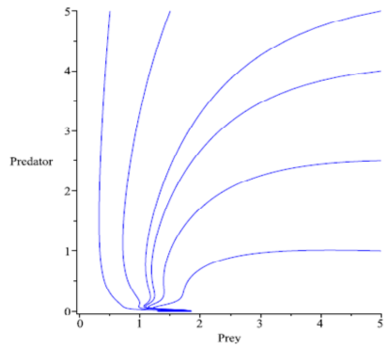

Figure 2. Extinction of predators.

Lemma 2.2. Consider the system

$$\begin{cases} \dot{x}(t) = f(x, y, t) \\ \dot{y}(t) = g(x, y, t) \end{cases}$$

where $f, g : \mathbb{R}^2 \times [0, +\infty) \to \mathbb{R}^2 \times [0, +\infty)$ are T-periodic functions in t.

Suppose that this system has a globally asymptotically stable T- periodic solution

$$\left(x^*(t), y^*(t)\right) := \left(x\left(t, 0, z_0^*\right), y\left(t, 0, z_0^*\right)\right),$$

where $z_0^* := \left(x_0^*, y_0^*\right)$ is the initial point. Then, for every $\varepsilon > 0$ and a compact set K, we can find a $T^* = T^*(\varepsilon, \mathrm{K}) > 0$ such that for all $t \geq T^*$, $s \geq 0$, $(x_0, y_0) \in \mathrm{K}$. we have

$$\left|x(t+s, s, x_0, y_0) - x^*(t+s)\right| + \left|y(t+s, s, x_0, y_0) - y^*(t+s)\right| \leq \varepsilon \tag{2.13}$$

Proof. Since f, g are $T-$periodic, we can suppose that $0 \leq s \leq T$. Moreover, it is easy to show that if $(x_0, y_0) \in \mathrm{K}$ and $0 \leq s \leq T$, there is a compact set K' such that $\left(x(T, s, x_0, y_0), y(T, s, x_0, y_0)\right) \in \mathrm{K}'$. Due to the periodicity of parameters, it is therefore sufficient to verify (2.13) for $s = 0$. Since $\left(x^*(t), y^*(t)\right)$ is stable, we can find a $\delta_\varepsilon > 0$ such that if $\left|x - x_0^*\right| + \left|y - y_0^*\right| \leq \delta_\varepsilon$ then

$$\left|x(t, 0, x, y) - x^*(t)\right| + \left|y(t, 0, x, y) - y^*(t)\right| \leq \varepsilon, \ \forall t \geq 0 \tag{2.14}$$

On the one hand, $\left(x^*(t), y^*(t)\right)$ is globally asymptotic then for every $(x_0, y_0) \in \mathrm{K}$, there exist a $T_{(x_0, y_0)} = k_{(x_0, y_0)} T$, $k_{(x_0, y_0)} \in \mathbb{N}$ such that

$$\left|x\left(T_{(x_0, y_0)}, 0, x, y\right) - x^*\left(T_{(x_0, y_0)}\right)\right| + \left|y\left(T_{(x_0, y_0)}, 0, x, y\right) - y^*\left(T_{(x_0, y_0)}\right)\right| \leq \delta_\varepsilon$$

By the continuous dependence of solutions on the initial data, there is a neighborhood of (x_0, y_0), denoted by V_{x_0, y_0}, such that

$$\left|x\left(T_{(x_0, y_0)}, 0, x, y\right) - x^*\left(T_{(x, y)}\right)\right| + \left|y\left(T_{(x_0, y_0)}, 0, x, y\right) - y^*\left(T_{(x_0, y_0)}\right)\right| \leq \delta_\varepsilon, \ \forall (x, y) \in V_{x_0, y_0} \tag{2.15}$$

As a result of (2.14) and (2.15),

$$\left|x(t, 0, x, y) - x^*(t)\right| + \left|y(t, 0, x, y) - y^*(t)\right| \leq \delta_\varepsilon, \ \forall (x, y) \in V_{x_0, y_0}, \ t \geq T_{(x_0, y_0)} \tag{2.16}$$

The family $\left\{V_{x_0, y_0} : (x_0, y_0) \in \mathrm{K}\right\}$ is an open covering of K. Since K is compact then there is a finite family $\left\{V_{x_0^1, y_0^1}, ..., V_{x_0^n, y_0^n}\right\}$ such that $\mathrm{K} \subset \bigcup_{i=1}^{n} V_{x_0^i, y_0^i}$. By choosing $T^* = \max_{1 \leq i \leq n} T\left(x_0^i, y_0^i\right)$, for any point $(x_0, y_0) \in \mathrm{K}$ and for all $t > T^*$, we have:

$$\left|x(t, 0, x_0, y_0) - x^*(t)\right| + \left|y(t, 0, x_0, y_0) - y^*(t)\right| < \varepsilon.$$

The proof is complete.

3. Dynamic behavior of the solution

Let $(x_0, y_0) \in \mathbb{R}_+^2$. Denote by $(x(t,0,x_0,y_0),\, y(t,0,x_0,y_0))$ the solution of (2.4) satisfying the initial condition $(x(0,0,x_0,y_0),\, y(0,0,x_0,y_0)) = (x_0, y_0)$. For the sake of simplification, we write $(x(t),\, y(t))$ for $(x(t,0,x_0,y_0),\, y(t,0,x_0,y_0))$ if there is no confusion.

Proposition 3.1. The system (2.4) is dissipative and the rectangle $(0, M/m] \times (0, M^2/m^2 - 1]$ is forward invariant.

Proof. By the uniqueness of the solution, it is easy to show that both the nonnegative and positive cones of $\mathbb{R}_+^2$ are positively invariant for (2.4). From the first equation of system (2.4) we see that

$$\dot{x} = x\left[a(\xi_t,t) - b(\xi_t,t)x - c(\xi_t,t)y\right] < x\left[a(\xi_t,t) - b(\xi_t,t)x\right] < x(M - mx).$$

By the comparison theorem, it follows that if $x(0) \geq 0$ then $x(t) \leq M/m,\ \forall t > t_0$ for some $t_0 > 0$. Similarity,

$$\begin{aligned}\dot{y} = y\left[-d(\xi_t,t) + e(\xi_t,t)x - f(\xi_t,t)y\right] &< y\left[-d(\xi_t,t) + e(\xi_t,t)M/m - f(\xi_t,t)y\right] \\ &< y\left(-m + M^2/m - my\right),\end{aligned}$$

which follows that $y(t) \leq M^2/m^2 - 1,\ \forall t > t_1$ for some $t_1 > t_0$.

From these estimates, we also see that the rectangle $(0, M/m] \times (0, M^2/m^2 - 1]$ is forward invariant. The proof is complete.

Proposition 3.2. There exists A $\delta_0 > 0$ such that $\limsup_{t\to+\infty} x(t,0,x_0,y_0) \geq \delta_0$ for any (x_0, y_0) with probability 1.

Proof. By the system (2.4), there exist $\delta_0 > 0,\ \varepsilon_0 > 0$ such that

$$-d(\xi_t,t) + e(\xi_t,t)x - f(\xi_t,t)y < -\varepsilon_0 \ ;\ \forall 0 < x < \delta_0,\ 0 < y \leq M^2/m^2 - 1$$

and

$$a(\xi_t,t) - b(\xi_t,t)x - c(\xi_t,t)y > \varepsilon_0 \ \textit{for all}\ 0 \leq x, y \leq \delta_0. \tag{3.1}$$

Assume that $\limsup_{t\to+\infty} x(t,0,x_0,y_0) < \delta_0$ with a positive probability. Then, there is a $t_3 > 0$ such that $x(t) < \delta_0,\ y(t) \leq M^2/m^2 - 1\ \forall t \geq t_3$, which implies that $\dot{y}(t) < -\varepsilon_0\, y(t)$. Therefore, for some $t_4 > t_3$, $y(t) < \delta_0,\ \forall t \geq t_4$. From (3.1) we see $\dot{x}(t) > \varepsilon_0\, x(t),\ \forall t \geq t_4$, which follows that $\lim_{t\to+\infty} x(t) = \infty$. This contradiction implies the assertion of this proposition.

Proposition 3.3. There exists a positive number $x_{\min}$ satisfying: if $(x_0, y_0) \in \mathbb{R}_+^2$ we can find $\tilde{t} > 0$ such that $x(t,0,x_0,y_0) \geq x_{\min}$ for all $t \geq \tilde{t}$.

Proof. With δ_0 mentioned in 3.2, there exists $\tilde{t}>0$ such that $x(t)>\delta_0$. Let $0<\varepsilon_1\le\delta_0$ such that $-\delta_1:=-m+M\varepsilon_1<0$. If $x(t)\ge\varepsilon_1$ for all $t>\tilde{t}$ then the proposition is proved. Otherwise, $x(t)<\varepsilon_1$ for a $t>\tilde{t}$. Let $h_1=\inf\{s>\tilde{t}: x(s)<\varepsilon_1\}$. We see that if $x(t)\le\varepsilon_1$ for $t>h_1$ then

$$\dot{y}=y\left[-d(\xi_t,t)+e(\xi_t,t)x-f(\xi_t,t)y\right]\le y(-m+M\varepsilon_1)=-\delta_1\, y \;\; \textit{for all} \;\; t\in(h_1,h_2)$$

which implies that

$$y(t)\le y(h_1)\,e^{-\delta_1(t-h_1)}\le y_{\max}\,e^{-\delta_1(t-h_1)}\ ,\ \forall t\in(h_1,h_2)$$

Hence,

$$\dot{x}=x\left[a(\xi_t,t)-b(\xi_t,t)x-c(\xi_t,t)y\right]\ge x\left(m-M\,x-M\,y_{\max}\,e^{-\delta_1(t-h_1)}\right),\ \forall t\in(h_1,h_2)\ .$$

Put

$$n(t)=\int_{h_1}^{t}\left(m-M\,y_{\max}\,e^{-\delta_1(t-h_1)}\right)ds\ ;\ N(t)=\int_{h_1}^{t}e^{n(s)}\,ds$$

By comparison theorem we get

$$x(t)\ge\frac{\varepsilon_1\,e^{n(t)}}{1+\varepsilon M\,N(t)}\ ,\ \forall t\in(h_1,h_2)\,.$$

Let $\alpha=\min\limits_{t>h_1}\dfrac{\varepsilon_1\,e^{n(t)}}{1+\varepsilon M\,N(t)}>0$. It is clear that α does not depend on $(x(0),y(0))$ and h_1. Let $x_{\min}=\min\{\alpha,\varepsilon_1\}$ we see that $x(t)>x_{\min}$, $\forall t>\tilde{t}$. The proof is complete.

As is known, the property of solutions of Lotka -Volterra systems near to boundary is dependent of two marginal equations. In the case where the prey is absent, the quantity $v(t)$ of predator at the time t satisfies the equation $\dot{v}=-d(\xi_t,t)\,v-f(\xi_t,t)v^2$. Thus, $v(t)$ decreases exponentially to 0. Similarly, without the predator, the quantity $u(t)$ of the prey at the time t satisfies the logistic equation

$$\dot{u}=u\left[a(\xi_t,t)-b(\xi_t,t)u\right]\ ,\ 0<u(0)\in\mathbb{R}^+ \tag{3.2}$$

If $u(t)$ is a solution of (3.2) then $\{\xi_t,u(t)\}$ is Markov processes.

A random process $\{\phi_t\}$, valued in a measurable space (S; S), is said to be periodic with period T if for any $t_1,t_2,...,t_n\in\mathbb{R}$, the simultaneous distribution of $\left(\phi_{t_1+kT},\phi_{t_2+kT},...,\phi_{t_n+kT}\right)$ does not depend on $k\in\mathbb{N}$.

We show that Equation (3.2) has a unique solution $u^*(t)$ such that $(\xi_t,u^*(t))$ is a periodic process. Indeed, put

$$u^*(t)=\frac{e^{A(t)}}{\int_{-\infty}^{t}b(\xi_s,s)\,e^{A(s)}\,ds}$$

where $A(t)=\int_0^t a(\xi_s,s)\,ds$. Firstly, we see that

$$u^*(t+T,\omega)=\frac{e^{\int_0^{t+T} a[\xi_s(\omega),\,s]\,ds}}{\int_{-\infty}^{t+T} b\left[\xi_s(\omega),\,s\right] e^{\int_0^s a[\xi_\tau(\omega),\,\tau]\,d\tau}\,ds}$$

$$=\frac{e^{\int_0^{t+T} a\left[\xi_{s-T}(\theta^T\omega),\,s-T\right] ds}}{\int_{-\infty}^{t+T} b\left[\xi_{s-T}(\theta^T\omega),\,s-T\right] e^{\int_0^s a\left[\xi_{\tau-T}(\theta^T\omega),\,\tau-T\right] d\tau}\,ds}$$

$$=\frac{e^{\int_{-T}^{t} a\left[\xi_s(\theta^T\omega),\,s\right] ds}}{e^{\int_{-T}^{0} a\left[\xi_s(\theta^T\omega),\,s\right] ds}\int_{-\infty}^{t} b\left[\xi_s(\theta^T\omega),\,s\right] e^{\int_0^s a\left[\xi_\tau(\theta^T\omega),\,\tau\right] d\tau}\,ds}$$

$$=\frac{e^{\int_0^{t} a\left[\xi_s(\theta^T\omega),\,s\right] ds}}{\int_{-\infty}^{t} e^{\int_0^s a\left[\xi_\tau(\theta^T\omega),\,\tau\right] d\tau}\, b\left[\xi_s(\theta^T\omega),\,s\right]ds}=u^*\left(t,\,\theta^T,\,\omega\right).$$

Hence, by virtue of P - preserving measure property of θ, for any continuous function h, for any $t_1<t_2<\ldots<t_n$; $k\in\mathbb{N}$ we have

$$\mathrm{E}\left\{h\left[\xi_{t_1+kT},\,u^*(t_1+kT),\,\xi_{t_2+kT},\,u^*(t_2+kT),\ldots,\xi_{t_n+kT},\,u^*(t_n+kT)\right]\right\}$$
$$=\mathrm{E}\left\{h\left[\xi_{t_1}\left(\theta^{kT}\right),\,u^*\left(t_1,\,\theta^{kT}\right),\,\xi_{t_2}\left(\theta^{kT}\right),\,u^*\left(t_2,\,\theta^{kT}\right),\ldots,\,\xi_{t_n}\left(\theta^{kT}\right),\,u^*\left(t_n,\,\theta^{kT}\right)\right]\right\}$$
$$=\mathrm{E}\left\{h\left[\xi_{t_1}(.),\,u^*(t_1,.),\,\xi_{t_2}(.),\,u^*(t_2,.),\ldots,\,\xi_{t_n}(.),\,u^*(t_n,.)\right]\right\}.$$

This means that $\left(\xi_t,\,u^*(t)\right)$ is a periodic process with period T . The uniqueness follows from the following lemma:

Lemma 3.4. For any $u_0>0$, $\lim_{t\to+\infty}\left[u(t)-u^*(t)\right]=0$ a.s., where $u(t)$ is the solution of the equation (3.2) satisfying $u(0)=u_0$.

Proof. Put $z=\frac{1}{u}-\frac{1}{u^*}$ we have $\dot z=-a\,z$. Thus, by virtue of the bounded below property by positive constant of z we follow the result.

Lemma 3.5. [Law of large numbers for periodic processes] For any continuous, bounded function $h(t,i,u)$, periodic in t with period T we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t h\left[s,\xi_s,u^*(s)\right]ds = \mathrm{E}\left\{\frac{1}{T}\int_0^T h\left[s,\xi_s,u^*(s)\right]ds\right\} \tag{3.3}$$

Proof. Put

$$X_n = \int_{nT}^{(n+1)T} h\left[s,\xi_s,u^*(s)\right]ds$$

Since $\{\xi_t, u^*(t)\}$ is periodic then $\{X_n\}$ is a stationary process. By the law of large numbers we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} X_k = \mathrm{E}\left[X_0 / J\right] \ a.s.,$$

where J is the $\sigma-$ algebra of the invariant sets. However, (ξ_t) is ergodic and $u^*(t)$ has no non-trivial invariant set then we follow that $J=\{\Phi,\Omega\}$. This implies that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t h\left[s,\xi_s,u^*(s)\right]ds = \lim_{|t|\to+\infty}\frac{1}{T}\frac{1}{[t/T]}\sum_{k=0}^{[t/T]} X_k = \frac{1}{T}\mathrm{E}\left[X_0\right]$$

$$= \frac{1}{T}\mathrm{E}\left[\int_0^T h\left[s,\xi_s,u^*(s)\right]ds\right]$$

Where, $[x]$ denotes the integer number such that $[x] \le x < [x]+1$. Lemma is proved.

We study conditions that ensure the persistence of $y(t)$ of the Equation (2.4) with $x(0)>0$ and $y(0)>0$.

Proposition 3.6. *Put*

$$\lambda := \frac{1}{T}\mathrm{E}\left[\int_0^T\left[-d(\xi_t,t)+e(\xi_t,t)u^*(t)\right]dt\right] \tag{3.4}$$

a) If $\lambda>0$ *then* $\limsup_{t\to+\infty} y(t)>\delta>0$ *with probability 1.*

b) In case $\lambda<0$, $\lim_{t\to+\infty} y(t)=0$ *with probability 1.*

Proof. By comparison theorem, if $x(0)=u(0)$ we have $u(t)\ge x(t)$, $\forall t$. Therefor, by virtue of Lemma 3.4 we have $\liminf_{t\to+\infty}\frac{\ln u^*(t)-\ln x(t)}{t} \ge 0$.

a) From Equations (3.2) and (2.4) we have

$$\frac{\ln u^*(t)-\ln u^*(0)}{t} = \frac{1}{t}\int_0^t a(\xi_s,s)ds - \frac{1}{t}\int_0^t b(\xi_s,s)u^*(s)ds \tag{3.5}$$

$$\begin{aligned}\frac{\ln x(t)-\ln x(0)}{t} &= \frac{1}{t}\int_0^t a(\xi_s,s)ds - \frac{1}{t}\int_0^t b(\xi_s,s)\left[x(s)-u^*(s)\right]ds - \\ &-\frac{1}{t}\int_0^t b(\xi_s,s)\,u^*(s)ds - \frac{1}{t}\int_0^t c(\xi_s,s)\,y(s)d\end{aligned} \tag{3.6}$$

On subtracting (3.6) from (3.5) we obtain

$$0 \le \liminf_{t\to+\infty} \left\{ \frac{1}{t}\int_0^t c(\xi_s,s)\,y(s)\,ds - \frac{1}{t}\int_0^t b(\xi_s,s)\left[u^*(s)-x(s)\right]ds \right\}$$

$$\le \liminf_{t\to+\infty} \left\{ \frac{1}{t}\int_0^t M\; y(s)\,ds - \frac{1}{t}\int_0^t m\left[u^*(s)-x(s)\right]ds \right\}$$

Hence,

$$\liminf_{t\to+\infty} \left\{ \frac{1}{t}\int_0^t \frac{M}{m}\,y(s)\,ds - \frac{1}{t}\int_0^t \left[u^*(s)-x(s)\right]ds \right\} \ge 0 \qquad (3.7)$$

Otherwise, $\frac{\dot{y}(t)}{y(t)} = -d(\xi_t,t)+e(\xi_t,t)x(t)-f(\xi_t,t)\,y(t)$ follows

$$\frac{\ln y(t)-\ln\, y(0)}{t} = \frac{1}{t}\int_0^t \left[-d(\xi_s,s)+e(\xi_s,s)u^*(s)\right]ds -$$

$$-\frac{1}{t}\int_0^t e(\xi_s,s)\left[u^*(s)-x(s)\right]ds - \frac{1}{t}\int_0^t f(\xi_s,s)\,y(s)\,ds$$

and

$$\frac{1}{t}\int_0^t e(\xi_s,s)\left[u^*(s)-x(s)\right]ds + \frac{1}{t}\int_0^t f(\xi_s,s)\,y(s)\,ds =$$

$$= \frac{1}{t}\int_0^t \left[-d(\xi_s,s)+e(\xi_s,s)u^*(s)\right]ds \;-\; \frac{\ln y(t)-\ln\, y(0)}{t}$$

Moreover, $y(t)$ is bounded above then $\liminf_{t\to+\infty} \left\{ -\frac{\ln y(t)-\ln y(0)}{t} \right\} \ge 0$ and we apply the law of large numbers (Lemma 3.5), $\lim_{t\to+\infty} \frac{1}{t}\int_0^t \left[-d(\xi_s,s)+e(\xi_s,s)u^*(s)\right]ds = \lambda$, consequently,

$$\liminf_{t\to+\infty} \left\{ \frac{1}{t}\int_0^t e(\xi_s,s)\left[u^*(s)-x(s)\right]ds + \frac{1}{t}\int_0^t f(\xi_s,s)\,y(s)\,ds \right\} =$$

$$= \liminf_{t\to+\infty} \left\{ \frac{1}{t}\int_0^t \left[-d(\xi_s,s)+e(\xi_s,s)u^*(s)\right]ds - \frac{\ln y(t)-\ln\, y(0)}{t} \right\}$$

$$\ge \liminf_{t\to+\infty} \left\{ \frac{1}{t}\int_0^t \left[-d(\xi_s,s)+e(\xi_s,s)u^*(s)\right]ds \right\} + \liminf_{t\to+\infty} \left\{ -\frac{\ln y(t)-\ln\, y(0)}{t} \right\} \ge \lambda$$

Hence,

$$\liminf_{t\to+\infty} \left\{ \frac{1}{t}\int_0^t \left[u^*(s)-x(s)\right]ds + \frac{1}{t}\int_0^t y(s)\,ds \right\} \ge$$

$$\liminf_{t\to+\infty}\left\{\frac{1}{t}\int_0^t \frac{e(\xi_s,s)}{M}\left[u^*(s)-x(s)\right]ds+\frac{1}{t}\int_0^t \frac{f(\xi_s,s)}{M}y(s)ds\right\}\geq\frac{\lambda}{M}$$

By (3.7) plus (3.8), we obtain

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t\left(\frac{M}{m}+1\right)y(s)ds\geq\liminf_{t\to+\infty}\left\{\frac{1}{t}\int_0^t\frac{M}{m}y(s)\,ds-\frac{1}{t}\int_0^t\left[u^*(s)-x(s)\right]ds\right\}+$$

$$+\liminf_{t\to+\infty}\left\{\frac{1}{t}\int_0^t\left[u^*(s)-x(s)\right]ds+\frac{1}{t}\int_0^t y(s)\,ds\right\}\geq\frac{\lambda}{M}$$

then $\liminf\limits_{t\to+\infty}\frac{1}{t}\int_0^t y(s)ds\geq\frac{m}{M(M+m)}\lambda\geq 0$ and $\limsup\limits_{t\to+\infty} y(t)>\delta>0$.

b) From the second equality of systems (2.4) and $\lambda>0$ we have

$$\limsup_{t\to+\infty}\frac{\ln y(t)-\ln y(0)}{t}=\limsup_{t\to+\infty}\left\{\frac{1}{t}\int_0^t\left[-d(\xi_s,s)+e(\xi_s,s)u^*(s)\right]ds-\right.$$

$$\left.-\frac{1}{t}\int_0^t e(\xi_s,s)\left[u^*(s)-x(s)\right]ds-\frac{1}{t}\int_0^t f(\xi_s,s)y(s)ds\right\}$$

$$\leq\lambda-\limsup_{t\to+\infty}\frac{1}{t}\int_0^t e(\xi_s,s)\left[u^*(s)-x(s)\right]ds-\limsup_{t\to+\infty}\frac{1}{t}\int_0^t f(\xi_s,s)y(s)ds<0$$

which implies that $\lim\limits_{t\to+\infty}y(t)=0$. The proof is complete.

Remark 3.7. The conditions (3.4) is easily to be checked by simulation method based on the law of large numbers. Moreover, by $(\xi_t,u^*(t))$ is solution of equation (3.2), we have

$$\lambda:=\frac{1}{T}\,\mathrm{E}\left\{\int_0^T\left[-d(\xi_t,t)+e(\xi_t,t)u^*(t)\right]dt\right\}$$

$$=\frac{1}{T}\,\mathrm{E}\left\{\int_0^T\left[-d(\xi_t,t)+\frac{e(\xi_t,t)}{b(\xi_t,t)}\,b(\xi_t,t)\,u^*(t)\right]dt\right\}$$

$$\geq\frac{1}{T}\,\mathrm{E}\left\{\int_0^T\left[-d(\xi_t,t)+\inf_{t\to+\infty}\left\{\frac{e(\pm,t)}{b(\xi_t,t)}\right\}b(\xi_t,t)\,u^*(t)\right]dt\right\}$$

$$=\frac{1}{T}\,\mathrm{E}\left\{\int_0^T\left[-d(\xi_t,t)+\inf_{t\to+\infty}\left\{\frac{e(\pm,t)}{b(\xi_t,t)}\right\}a(\xi_t,t)\right]dt\right\}+$$

$$+\inf_{t\to+\infty}\left\{\frac{e(\pm,t)}{b(\xi_t,t)}\right\}\frac{1}{T}\,\mathrm{E}\left\{\int_0^T\left[a(\xi_t,t)-b(\xi_t,t)\,u^*(t)\right]dt\right\}$$

Note that

$$\frac{1}{T}\,\mathrm{E}\left\{\int_0^T \left[\,a(\xi_t,t)-b(\xi_t,t)\,u^*(t)\right]dt\right\}=0$$

and that $-d(\pm,t)+\inf\limits_{t\to+\infty}\left\{\dfrac{e(\pm,t)}{b(\pm,t)}\right\}a(\pm,t)\ge 0$, $\forall t>0$

provided that

$$\inf_{t\to+\infty}\left\{\frac{a(\pm,t)}{b(\pm,t)}\right\}>\sup_{t\to+\infty}\left\{\frac{d(\pm,t)}{e(\pm,t)}\right\} \qquad (3.9)$$

Then, $\lambda>0$ under the condition 3.9, which is similar to (2.8).

From now on, we suppose that $\lambda>0$.

Lemma 3.8. With probability 1, there are infinitely many $s_n=s_n(\omega)>0$ such that $s_n>s_{n-1}$, $\lim\limits_{n\to+\infty}s_n=\infty$ and $x(s_n)\ge\delta$, $y(s_n)\ge\delta$, $\forall n\in\mathbb{N}$.

Proof. By Proposition 3.3 we can find $\bar{t}>0$ such that $x(t)\ge x_{\min}$, for all $t>\bar{t}$. On the other hand, there exists $\delta<x_{\min}$ and a random sequences $\{s_n\}\uparrow^{\infty}$, $s_n>\bar{t}$ such that $y(s_n)>\delta$, $\forall n\in\mathbb{N}$. The proof is complete.

For the sake of simplicity, we suppose $\xi_0=+$ a.s and set $x_n:=x(\tau_n,x,y)$, $y_n:=y(\tau_n,x,y)$ $\Im_0^n=\sigma(\tau_k\,,k\le n)$; $\Im_n^\infty=\sigma(\tau_k-\tau_n\,,k>n)$. It is clear that (x_n,y_n) is $\Im_0^n$ measurable $\Im_0^n$ is independent $\Im_0^\infty$ if ξ_0 is given.

Hypotheses 3.9. On the quadrant $\operatorname{int}\mathbb{R}_+^2$, the system (2.5) has a stable positive $T-$periodic solution (x_+^*,y_+^*) such that

$$\left(x_+(t)-x_+^*(t)\,,\,y_+(t)-y_+^*(t)\right)\xrightarrow{t\to+\infty}(0,0).$$

Lemma 3.10. Suppose that Hypothesis 3.9 holds and $\lambda>0$, we can find an $\Delta>0$ such that with probability 1, there are infinitely many $n\in\mathbb{N}$ such that $\Delta\le x_n\,,y_n\le M^*$. Moreover, we can find $\overline{\Delta}>0$ such that the events $\left\{x_{2k+1}>\overline{\Delta},\ y_{2k+1}>\overline{\Delta}\right\}$ as well as $\left\{x_{2k}>\overline{\Delta},\ y_{2k}>\overline{\Delta}\right\}$ occur infinitely many often.

Proof. Let $\{\Im_t\}$ be the filtration generated by $\{\xi(t)\}$. It is obvious that $\{\xi(t),x(t),y(t)\}$ is a strong Feller-Markov process with respect to the filtration $\{\Im_t\}$. For a stopping time ς, the $\sigma-$algebra at ς is $\Im_\varsigma=\left\{A\in\Im_\infty : A\cap\{\varsigma\le t\}\in\Im_t\,,\ \forall t\in\mathbb{R}_+\right\}$. Fix a $T_1>0$, by Lemma 3.8, we can define almost surely finite stopping times

$$\eta_1=\inf\{t>0:x(t)\ge\delta,y(t)\ge\delta\}$$

$$\eta_2=\inf\{t>\eta_1+T_1:x(t)\ge\delta,y(t)\ge\delta\}$$

....

$$\eta_n=\inf\{t>\eta_{n-1}+T_1:x(t)\ge\delta,y(t)\ge\delta\}$$

For a stopping time ς, we write $\tau(\varsigma)$ for the first jump of $\xi(t)$ after ς, i.e., $\tau(\varsigma)=\inf\{t>\varsigma: \xi(t)\neq\xi(\varsigma)\}$. Let $\overline{\sigma}(\varsigma)=\tau(\varsigma)-\varsigma$ and $A_k=\{\overline{\sigma}(\eta_k)<T_1\}$, $k\in\mathbb{N}$. Obviously, A_k is in the $\sigma-$ algebra generated by $\{\xi(\eta_n+s): s\geq 0\}$ and $A_k\in\Im_{\eta_{k+1}}$ also. Therefore, in view of the strong Markov property of $(\xi(t),x(t),y(t))$ and [see 15, Theorem 5, p. 59] we have

$$\mathrm{P}\left[\overline{A_k}\,\middle|\,\xi(\eta_k)=\pm\right]=\mathrm{P}\left[\overline{\sigma}(0)>T_1\middle|\xi(0)=\pm\right].$$

Hence,

$$\mathrm{P}\left(\overline{A_k}\right)=\mathrm{P}\left[\overline{\sigma}(\eta_k)>T_1\middle|\xi(\eta_k)=+\right]\mathrm{P}\left[\xi(\eta_k)=+\right]+\mathrm{P}\left[\overline{\sigma}(\eta_k)>T_1\middle|\xi(\eta_k)=-\right]\mathrm{P}\left[\xi(\eta_k)=-\right]$$
$$=\mathrm{P}\left[\overline{\sigma}(0)>T_1\middle|\xi(0)=+\right]\mathrm{P}\left[\xi(\eta_k)=+\right]+\mathrm{P}\left[\overline{\sigma}(0)>T_1\middle|\xi(0)=-\right]\mathrm{P}\left[\xi(\eta_k)=-\right]\leq\overline{p}$$

where $\overline{p}=\max\left\{\mathrm{P}\left(\overline{\sigma}(0)>T_1\mid\xi_0=\pm\right)\right\}<1.$ Moreover,

$$\mathrm{E}\left\{1_{\overline{A}_{k+1}}1_{\overline{A}_k}\middle|\left[\xi(\eta_{k+1}),\,x(\eta_{k+1}),y(\eta_{k+1})\right]\right\}=$$
$$=\mathrm{E}\left\{\mathrm{E}\left[1_{\overline{A}_{k+1}}1_{\overline{A}_k}\middle|\Im_{\eta_{k+1}}\right]\middle|\left[\xi(\eta_{k+1}),\,x(\eta_{k+1}),y(\eta_{k+1})\right]\right\}$$
$$=\mathrm{E}\left\{1_{\overline{A}_k}\,\mathrm{E}\left[1_{\overline{A}_{k+1}}\middle|\Im_{\eta_{k+1}}\right]\middle|\left[\xi(\eta_{k+1}),\,x(\eta_{k+1}),y(\eta_{k+1})\right]\right\}$$
$$=\mathrm{E}\left\{1_{\overline{A}_k}\,\mathrm{E}\left[1_{\overline{A}_{k+1}}\middle|\left(\xi(\eta_{k+1}),\,x(\eta_{k+1}),y(\eta_{k+1})\right)\right]\middle|\left[\xi(\eta_{k+1}),\,x(\eta_{k+1}),y(\eta_{k+1})\right]\right\}$$
$$=\mathrm{E}\left\{1_{\overline{A}_k}\middle|\left[\xi(\eta_{k+1}),\,x(\eta_{k+1}),y(\eta_{k+1})\right]\mathrm{E}\left[1_{\overline{A}_{k+1}}\middle|\left(\xi(\eta_{k+1}),\,x(\eta_{k+1}),y(\eta_{k+1})\right)\right]\right\}$$

which implies that

$$\mathrm{P}\left\{\overline{A}_{k+1}\cap\overline{A}_k\middle|\xi(\eta_{k+1})=\pm\right\}=\mathrm{P}\left\{\overline{A}_{k+1}\middle|\xi(\eta_{k+1})=\pm\right\}\mathrm{P}\left\{\overline{A}_k\middle|\xi(\eta_{k+1})=\pm\right\}. \tag{3.10}$$

Therefore, from (3.10) and the equation

$$\mathrm{P}\left(\overline{A}_{k+1}\cap\overline{A}_k\right)=\mathrm{P}\left\{\overline{A}_{k+1}\cap\overline{A}_k\middle|\xi(\eta_{k+1})=+\right\}\mathrm{P}\left\{\xi(\eta_{k+1})=+\right\}+\mathrm{P}\left\{\overline{A}_{k+1}\cap\overline{A}_k\middle|\xi(\eta_{k+1})=-\right\}\mathrm{P}\left\{\xi(\eta_{k+1})=-\right\}$$

it follows

$$\mathrm{P}\left(\overline{A}_{k+1}\cap\overline{A}_k\right)\leq\overline{p}\,\mathrm{P}\left\{\overline{A}_k\middle|\xi(\eta_{k+1})=+\right\}\mathrm{P}\left\{\xi(\eta_{k+1})=+\right\}+\overline{p}\,\mathrm{P}\left\{\overline{A}_k\middle|\xi(\eta_{k+1})=-\right\}\mathrm{P}\left\{\xi(\eta_{k+1})=-\right\}\leq\overline{p}^2$$

Continuing this way, we conclude that

$$\mathrm{P}\left(\bigcup_{i=k}^{n}A_i\right)=1-\mathrm{P}\left(\bigcap_{i=k}^{n}\overline{A}_i\right)\leq 1-\left(\overline{p}\right)^{n-k+1}$$

Consequently,

$$\mathrm{P}\left(\bigcap_{k=1}^{+\infty}\bigcup_{i=k}^{+\infty}A_i\right)=1.$$

Let $\Delta = \min\{x_{\pm}(t+s,t,x_0,y_0), y_{\pm}(t+s,t,x_0,y_0) : t\in[0,T_1], s\in[0,T], x_0,y_0\in[\delta,M]\} > 0$, if A_k occurs, $x_{\tau(\eta_k)} > \Delta$, $y_{\tau(\eta_k)} > \Delta$, which directly implies the first assertion. As a result, we are able to define finite stopping times

$$\bar{\eta}_1 = \inf\{n>0:\ \Delta \le x_n, y_n \le M^*\}$$
$$\bar{\eta}_2 = \inf\{t>\eta_1:\ \Delta \le x_n, y_n \le M^*\}$$
....
$$\bar{\eta}_k = \inf\{t>\eta_{k-1}:\ \Delta \le x_n, y_n \le M^*\}.$$

Put $\bar{\Delta} = \min\{x_{\pm}(t+s,t,x_0,y_0), y_{\pm}(t+s,t,x_0,y_0) : t\in[0,T_1], s\in[0,T], x_0,y_0\in[\Delta,M^*]\} > 0.$

Note that if the event $B_k = \{\sigma_{\bar{\eta}_k+1} < T_1\}$ occurs then $\bar{\Delta} \le x_{\eta_k}, y_{\eta_k}, x_{\eta_k+1}, y_{\eta_k+1} \le M^*$. Using arguments similar to the previous part of this proof, we can show that B_k occurs infinitely often. Consequently, we obtain the second assertion of this lemma due to the fact that η_k is odd then η_k+1 is even and conversely.

Next, we will describe the $\omega-$ limit sets of the system (2.4). Denoted by $\Omega(x,y,\omega)$ the $\omega-$ limit set of the solution $(x(t,0,x,y), y(t,0,x,y))(\omega)$ starting in (x,y). To simplify the notations, for $t\ge s\ge 0$, we denote

$\pi^{+}_{t,s}(x,y) := (x_+(t,s,x,y), y_+(t,s,x,y))$; resp $\pi^{-}_{t,s}(x,y) := (x_-(t,s,x,y), y_-(t,s,x,y))$) is the solution to the system (2.5) (resp. (2.6)) starting at $(x,y)\in\mathbb{R}^2_+$ at time s.

Suppose that the solution starting at $\gamma^*_+(0) = (x^*_+(0), y^*_+(0))$ at time 0 is a periodic solution to the system (2.5), we now describe the pathwise dynamic behavior of the solutions of system (2.4). Put

$$\Gamma = \left\{(x,y) = \pi^{(-1)^n}_{t_n,t_{n-1}} \ldots \pi^{-}_{t_3,t_2}\pi^{+}_{t_3,t_2}\left(\gamma^*_+(0)\right) : 0 = t_1 \le t_2 \le \ldots \le t_n\ ;\ n\in\mathbb{N}\right\} \tag{3.11}$$

where $\gamma^*_+(0)$ is mentioned above. Let us $(x_0,y_0)\in\mathbb{R}^2_+$.

Theorem 3.11. Suppose that on the quadrant $\operatorname{int}\mathbb{R}^2_+$, the system (2.5) has unique stable $T-$periodic solution $(x^*_+(t), y^*_+(t))$ and with λ mentioned in Proposition 3.6, let $\lambda>0$. Then,

a) With probability 1, the closure $\bar{\Gamma}$ of Γ is a subset of the $\omega-$limit set $\Omega(x_0,y_0,\omega)$.

b) If there exists a $\hat{z} = (\hat{x},\hat{y})$ such that the point $\hat{z} = \pi^{+}_{\hat{t},0}[\gamma^*_+(0)]$ satisfies the following condition

$$\det\begin{pmatrix} h_1(+,\hat{t},\hat{z}) & h_1(-,\hat{t},\hat{z}) \\ h_2(+,\hat{t},\hat{z}) & h_2(-,\hat{t},\hat{z}) \end{pmatrix} \ne 0 \tag{3.12}$$

where $\begin{cases} h_1\left(\xi_t,\hat{t},\hat{z}\right)= a\left(\xi_t,\hat{t}\right)-b\left(\xi_t,\hat{t}\right)\hat{x}-c\left(\xi_t,\hat{t}\right)\hat{y} \\ h_2\left(\xi_t,\hat{t},\hat{z}\right)=-d\left(\xi_t,\hat{t}\right)+e\left(\xi_t,\hat{t}\right)\hat{x}-f\left(\xi_t,\hat{t}\right)\hat{y} \end{cases}$

Then, with probability 1, the closure $\overline{\Gamma}$ of Γ is the ω - limit set $\Omega(x_0,y_0,\omega)$. Moreover, Γ absorbs all positive solutions in the sense that for any initial value $(x_0,y_0)\in \operatorname{int}\mathbb{R}^2_+$, the value

$$\gamma(\omega)=\inf\left\{t>0:\left(x(s,0,x_0,y_0,\omega),y(s,0,x_0,y_0,\omega)\right)\in\Gamma,\forall s>t\right\}$$

is finite outside a P-null set.

Proof. Denote $\mathrm{H}:=\mathrm{H}_{\Delta,M^*}$. We construct a sequence of stopping times

$$\eta_1=\inf\left\{2k:(x_{2k},y_{2k})\in\mathrm{H}\right\}$$

$$\eta_2=\inf\left\{2k>\eta_1:(x_{2k},y_{2k})\in\mathrm{H}\right\}$$

....

$$\eta_n=\inf\left\{2k>\eta_{n-1}:(x_{2k},y_{2k})\in\mathrm{H}\right\}.$$

It is easy to see that the events $\{\eta_k=n\}\in\mathfrak{I}_0^n$ for any *k; n*. Thus the event $\{\eta_k=n\}$ is independent of $\mathfrak{I}_0^\infty$

if ξ_0 is given. By the Proposition 3.1 and Lemma 3.10, $\eta_n<\infty$ a.s for all *n*. For simplicity, we put $\operatorname{mod}(t)=t-k_tT$ where k_t is such a integer that $k_tT\le t\le(k_t+1)T$. As a convention, the notation $\operatorname{mod}(t)\in(-\delta,\delta)$ means $\operatorname{mod}(t)\in[0,\delta)\cup(T-\delta,T)$. By $U_\varepsilon(x,y)$, we denote neighborhood of point (x,y) with radius $\varepsilon>0$ and $\phi(t,s,x,y)=\left(x(t,s,x,y),\,y(t,s,x,y)\right)$.

Firstly, we prove that for any $\varepsilon_1>0$, $\delta_1>0$, there are infinitely many odd stopping times such that $(x_{2n+1},y_{2n+1})\in U_{\varepsilon_1}\left[\gamma_+^*(0)\right]$ and $\operatorname{mod}(\tau_{2n+1})\in(-\delta_1,\delta_1)$. We have

$$\pi^+_{t+\tau_{\eta_k+1},\,\tau_{\eta_k+1}}\left(x_{\eta_k+1},y_{\eta_k+1}\right)=\pi^+_{t+\operatorname{mod}(\tau_{\eta_k+1}),\,\operatorname{mod}(\tau_{\eta_k+1})}\left(x_{\eta_k+1},y_{\eta_k+1}\right)=\pi^+_{t-T+\operatorname{mod}(\tau_{\eta_k+1}),\,0}\left(\overline{x},\overline{y}\right)$$

where $\left(\overline{x},\overline{y}\right)=\pi^+_{T,\,\operatorname{mod}(\tau_{\eta_k+1})}\left(x_{\eta_k+1},y_{\eta_k+1}\right)$. Therefore, applying the Lemma 2.2 obtains, for any neighborhood $U_{\varepsilon_1}\left[\gamma_+^*(0)\right]$, there exists $T^*>0$ and δ_2 so that

$$\pi^+_{t+\tau_{\eta_k},\,\tau_{\eta_k}}\left(x_{\eta_k},y_{\eta_k}\right)\in U_{\varepsilon_1}\left[\gamma_+^*(0)\right];\ \forall t>T^*,\ \operatorname{mod}\left(t+\tau_{\eta_k}\right)\in(-\delta_2,\delta_2).$$

This is equivalent to $t\in\left(-\operatorname{mod}\left(\tau_{\eta_k}\right)+KT-\delta_2,\,-\operatorname{mod}\left(\tau_{\eta_k}\right)+KT+\delta_2\right)K\ge\overline{K}$, in which $\overline{K}\in\mathbb{N}$ is the smallest natural number satisfying $-\operatorname{mod}\left(\tau_{\eta_k+1}\right)+KT-\delta_2>T^*$.

Note that, $-\operatorname{mod}\left(\tau_{\eta_k+1}\right)+\overline{K}T<T^*+T:=\overline{T}$.

Now, let $\delta_3=\min\{\delta_1,\delta_2\}$; for any $u>0,\delta_3>0,k\in\mathbb{N}$, put

$$A_k = \left\{\omega :\ \sigma_{\eta_k+1} \in \left(-\operatorname{mod}\left(\tau_{\eta_k+1}\right) + \overline{K}T - \delta_3\ ,\ -\operatorname{mod}\left(\tau_{\eta_k+1}\right) + \overline{K}T + \delta_3\ \right)\right\}.$$

Note that if X has the exponential distribution then $\mathrm{P}\{t < X < t+a\} \geq \mathrm{P}\{s < X < s+a\}$ whenever $t \leq s$. Using the strong Markov property of $\{\xi(t)\ ,\ x(t)\ ,\ y(t)\}$ and noting that we have already known the value of $\xi_{\tau_{\eta_k}}$, we have the estimation

$$\mathrm{P}\left\{\overline{A_k}\right\} = \mathrm{P}\left\{\sigma_{\eta_k+1} \notin \left(-\operatorname{mod}\left(\tau_{\eta_k+1}\right) + \overline{K}T - \delta_3\ ,\ -\operatorname{mod}\left(\tau_{\eta_k+1}\right) + \overline{K}T + \delta_3\right)\right\}$$

$$= \int_0^{+\infty} \mathrm{P}\left\{\sigma_{\eta_k+1} \notin \left(-\operatorname{mod}(t) + \overline{K}T - \delta_3\ ,\ -\operatorname{mod}(t) + \overline{K}T + \delta_3\right)\right\}\ \Big|_{\tau_{\eta_k}=t} \times \mathrm{P}\left\{\tau_{\eta_k} \in d\,t\right\}$$

$$= \int_0^{+\infty} \mathrm{P}\left\{\sigma_{\eta_k+1} \notin \left(-\operatorname{mod}(t) + \overline{K}T - \delta_3\ ,\ -\operatorname{mod}(t) + \overline{K}T + \delta_3\right)\right\}\ \Big|_{\tau_{\eta_k}=t,\ \xi_{\tau_{\eta_k}}=+} \times \mathrm{P}\left\{\tau_{\eta_k} \in d\,t\right\}$$

$$= \int_0^{+\infty} \mathrm{P}\left\{\sigma_{\eta_k+1} \notin \left(-\operatorname{mod}(t) + \overline{K}T - \delta_3\ ,\ -\operatorname{mod}(t) + \overline{K}T + \delta_3\right)\right\}\ \Big|_{\xi_{\tau_{\eta_k}}=+} \times \mathrm{P}\left\{\tau_{\eta_k} \in d\,t\right\}$$

$$\leq \int_0^{+\infty} \mathrm{P}\left\{\sigma_{\eta_k+1} \notin \left(\overline{T} - \delta_3\ ,\ \overline{T} + \delta_3\right)\right\}\ \Big|_{\xi_{\tau_{\eta_k}}=+} \times\ \mathrm{P}\left\{\tau_{\eta_k} \in d\,t\right\}$$

$$= \mathrm{P}\left\{\sigma_3 \notin \left(\overline{T} - \delta_3\ ,\ \overline{T} + \delta_3\right)\right\} \int_0^{+\infty} \mathrm{P}\left\{\tau_{\eta_k} \in dt\right\} = \mathrm{P}\left\{\sigma_3 \notin \left(\overline{T} - \delta_3\ ,\ \overline{T} + \delta_3\right)\right\} := \varphi < 1$$

We now estimate $\mathrm{P}\left\{\overline{A_k} \cap \overline{A_{k+2}}\right\}$. Since $A_k \in \Im_{\eta_{k+2}}$, applying the strong Markov property of $\left(\xi(t)\ ,\ x(t)\ ,\ y(t)\right)$ we have

$$\mathrm{P}\left\{\overline{A_k} \cap \overline{A_{k+1}}\right\} = \mathrm{E}\left[\mathrm{E}\left\{1_{\overline{A_k}} 1_{\overline{A_{k+1}}}\ \left|\Im_{\eta_{k+1}}\right.\right\}\right] = \mathrm{E}\left[1_{\overline{A_k}} \mathrm{E}\left\{1_{\overline{A_{k+1}}}\ \left|\Im_{\eta_{k+1}}\right.\right\}\right]$$

$$= \mathrm{E}\left[1_{\overline{A_k}} \mathrm{E}\left\{1_{\overline{A_{k+1}}}\ \Im_{\eta_{k+1}}\right\}\right] = \mathrm{E}\left[1_{\overline{A_k}} \mathrm{E}\left\{1_{\overline{A_{k+1}}}\ \left|\xi_{\tau_{\eta_{k+1}}} = -\right.\right\}\right] \leq\ \varphi\ \mathrm{E}\left(1_{\overline{A_k}}\right) = \varphi^2.$$

Continuing this way, we have,

$$\mathrm{P}\left\{\cap_{k=1}^{+\infty} \cup_{i=k}^{+\infty} A_i\right\} = \mathrm{P}\left\{\omega :\ \sigma_{\eta_k+1} \in \left(-\operatorname{mod}\left(\tau_{\eta_k+1}\right) + \overline{K}T - \delta_3\ ,\ -\operatorname{mod}\left(\tau_{\eta_k+1}\right) + \overline{K}T + \delta_3\ \right) i.o.\ of\ n\right\} = 1\ .$$

The even A_k occurs infinitely means that, with probability 1, for any $\delta_1 > 0$, for any $U_{\varepsilon_1}\left[\gamma_+^*(0)\right]$, there are infinitely many $n = n(\omega) \in \mathbb{N}$ such that $\left(x_{2n+1}\ ,\ y_{2n+1}\right) \in U_{\varepsilon_1}\left[\gamma_+^*(0)\right]$ and $\operatorname{mod}\left(\tau_{2n+1}\right) \in \left(-\delta_3\ ,\ \delta_3\right) \subset \left(-\delta_1\ ,\ \delta_1\right)$. Thus $\gamma_+^*(0) \in\ \Omega\left(x_0\ ,\ \mathrm{y}_0\ ,\ \omega\right)$.

Secondly, we prove $\left\{\pi_{t,0}^{-}\left[\gamma_+^*(0)\right] :\ t \geq 0\right\} \in\ \Omega\left(x_0\ ,\ \mathrm{y}_0\ ,\ \omega\right)$ a.s. To do this, we show that for $\overline{\gamma} := \pi_{t_1,0}^{-}\left[\gamma_+^*(0)\right], \forall U_{\varepsilon_2}\left(\overline{\gamma}\right), \forall \delta_4 > 0$, there are infinitely many even stopping times such that $\left(x_{2n}\ ,\ y_{2n}\right) \in U_{\varepsilon_2}\left(\overline{\gamma}\right)$ and $\operatorname{mod}\left(\tau_{2n}\right) \in \left(\operatorname{mod}\left(t_1 - \delta_4\right)\ ,\ \operatorname{mod}\left(t_1 + \delta_4\right)\right)$. By continuity of solutions with respect to initial conditions, there are $\varepsilon_3 > 0$, $\delta_5 > 0$, $\delta_6 > 0$ small enough so that if $\forall (x, y) \in U_{\varepsilon_3}\left[\gamma_+^*(0)\right]$, $\forall t \in (t_1 - \delta_5, t_1 + \delta_5)$ and $\forall \operatorname{mod}(s) \in (-\delta_6, \delta_6)$ then

$$\left|\pi_{t,0}^-\left[\gamma_+^*(0)\right] - \pi_{t_1,0}^-\left[\gamma_+^*(0)\right]\right| < \frac{\varepsilon_2}{3}$$

$$\left|\pi_{t,0}^-(x, y) - \pi_{t,0}^-\left[\gamma_+^*(0)\right]\right| < \frac{\varepsilon_2}{3}$$

$$\left|\pi_{t+s,s}^-(x, y) - \pi_{t,0}^-(x, y)\right| < \frac{\varepsilon_2}{3}.$$

Therefore,

$$\left|\pi_{t+s,s}^-(x, y) - \pi_{t_1,0}^-\left[\gamma_+^*(0)\right]\right| \le \left|\pi_{t+s,s}^-(x, y) - \pi_{t,0}^-(x, y)\right|$$

$$+\left|\pi_{t,0}^-(x, y) - \pi_{t,0}^-\left[\gamma_+^*(0)\right]\right| + \left|\pi_{t,0}^-\left[\gamma_+^*(0)\right] - \pi_{t_1,0}^-\left[\gamma_+^*(0)\right]\right| < \varepsilon_2,$$

$$\forall (x, y) \in U_{\varepsilon_3}\left[\gamma_+^*(0)\right], \ \forall t \in (t_1 - \delta_5, t_1 + \delta_5), \ \forall \operatorname{mod}(s) \in (-\delta_6, \delta_6).$$

Put

$$\varsigma_1 = \inf\left\{2k+1 : (x_{2k+1}, y_{2k+1}) \in U_{\varepsilon_3}\left[\gamma_+^*(0)\right], \operatorname{mod}(\tau_{2k+1}) \in (-\delta_6, \delta_6)\right\}$$

$$\varsigma_2 = \inf\left\{2k+1 > \varsigma_1 : (x_{2k+1}, y_{2k+1}) \in U_{\varepsilon_3}\left[\gamma_+^*(0)\right], \operatorname{mod}(\tau_{2k+1}) \in (-\delta_6, \delta_6)\right\}$$

....

$$\varsigma_n = \inf\left\{2k+1 > \varsigma_{n-1} : (x_{2k+1}, y_{2k+1}) \in U_{\varepsilon_3}\left[\gamma_+^*(0)\right], \operatorname{mod}(\tau_{2k+1}) \in (-\delta_6, \delta_6)\right\}.$$

From the previous part of this proof, it follows that $\varsigma_k < +\infty$ and $\lim_{k\to+\infty} \varsigma_k = +\infty$ a.s.. Since $\{\varsigma_k = n\} \in \Im_0^n$, $\{\varsigma_k\}$ is independent of $\Im_n^\infty$. Put $\bar{t} = \min\{\delta_4, \delta_5\}$. By the same argument as above we obtain $P\left\{\omega : \sigma_{\varsigma_n+1} \in \left(t_1 - \bar{t},\ t_1 + \bar{t}\right) i.o.\ of\ n\right\} = 1$. This relation says that $\left(x_{\varsigma_k}, y_{\varsigma_k}\right) \in U_{\varepsilon_3}\left[\gamma_+^*(0)\right]$ and $\sigma_{\varsigma_k+1} \in \left(t_1 - \bar{t},\ t_1 + \bar{t}\right)$ which implies $\left(x_{\varsigma_k+1}, y_{\varsigma_k+1}\right) \in U_{\varepsilon_2}\left(\bar{\gamma}\right)$ for many infinite $k \in \mathbb{N}$ and

$$\operatorname{mod}\left(\tau_{\varsigma_k+1}\right) \in \left(\operatorname{mod}\left(t_1 - \bar{t}\right), \operatorname{mod}\left(t_1 + \bar{t}\right)\right) \subset \left(\operatorname{mod}(t_1 - \delta_4), \operatorname{mod}(t_1 + \delta_4)\right).$$

This means $\bar{\gamma} \in \Omega(x_0, y_0, \omega)$ a.s..

Lastly, by similar way and induction, we conclude that Γ is a subset of $\Omega(x_0, y_0, \omega)$. Because $\Omega(x_0, y_0, \omega)$ is a close set, we have $\bar{\Gamma} \subset \Omega(x_0, y_0, \omega)$ a.s..

b) We now prove the second assertion of this theorem. Let $\hat{z} = \left(\hat{x}, \hat{y}\right)$ satisfying the condition (3.12). By the existence and continuous dependence on the initial values of the solutions, there exist two numbers $a > 0$ and $b > 0$ such that the function $\varphi(s, t) = \pi_{t,s}^+ \pi_{t,s}^-\left(\hat{z}\right)$ is defined and continuously differentiable in $(-a, a) \times (-b, b)$.

We note that

$$det\left(\frac{\partial\varphi}{\partial s},\frac{\partial\varphi}{\partial t}\right)\Bigg|_{(\hat{t},\hat{t})} = det\begin{pmatrix} \hat{x}h_1\left(+,\hat{x},\hat{y}\right) & \hat{x}h_1\left(-,\hat{x},\hat{y}\right) \\ \hat{y}h_2\left(+,\hat{x},\hat{y}\right) & \hat{y}h_2\left(-,\hat{x},\hat{y}\right) \end{pmatrix}$$

$$= \hat{x}\hat{y}\, det\begin{pmatrix} h_1\left(+,\hat{z}\right) & h_1\left(-,\hat{z}\right) \\ h_2\left(+,\hat{z}\right) & h_2\left(-,\hat{z}\right) \end{pmatrix} \neq 0.$$

Therefore, by Theorem of Inverse Function, there exist $0<a_1<a$, $0<b_1<b$ such that $\varphi(s,t)$ is a diffeomorphism between $V=(0,a_1)\times(0,b_1)$ and $U=\varphi(V)$. As a consequence, U is an open set. Moreover, for every $(x,y)\in U$, there exists a $\left(s^*,t^*\right)\in V=(0,a_1)\times(0,b_1)$ such that $(x,y)=\pi^+_{t^*,s^*}\,\pi^-_{t,s^*}\left(\hat{z}\right)\in\Gamma$. Hence, $U\subset\Gamma\subset\Omega(x_0\ ,\ y_0\ ,\ \omega)$. Thus, there is a stopping time $\gamma<+\infty$ a.s. such that $\left(x(\gamma),y(\gamma)\right)\in U$. Since Γ is a forward invariant set and $U\subset\Gamma$, it follows that $\left(x(t),y(t)\right)\in\Gamma$, $\forall t>\gamma$ with probability 1. The fact $\left(x(t),y(t)\right)\in\Gamma$ for all $t>\gamma$ implies that $\Omega(x_0\ ,\ y_0\ ,\ \omega)\subset\overline{\Gamma}$. By combining with the part a) we get $\Omega(x_0\ ,\ y_0\ ,\ \omega)=\overline{\Gamma}$ a.s..The proof is complete.

4. Simulation and discussion

Noting that λ can be estimated by using the law of large number and formula (3.4) for an initial concrete set. We will illustrate the above model by following numerical examples in three cases.

Example I. $\lambda>0$ and the coexistence case presents in both states (see figure 3). It corresponds to

$\alpha=0.6$; $\beta=0.4$; $a(+)=10+sin\,t$; $b(+)=2+\frac{1}{2}cos\,t$; $c(+)=1$;

$d(+)=1-\frac{1}{5}cos\left(t-\frac{\pi}{3}\right)$; $e(+)=1.8$; $f(+)=3.1+\frac{1}{2}sin\left(t+\frac{\pi}{6}\right)$;

$a(-)=11.7-sin(t+\pi)$; $b(-)=1.5+\frac{1}{4}cos\,t$;

$c(-)=1.4-\frac{1}{2}sin\left(t+\frac{\pi}{2}\right)$; $d(-)=2.1+\frac{1}{6}sin(t+\pi)$;

$e(-)=1.2+\frac{1}{4}cos\,t$; $f(-)=2.7-\frac{1}{2}cos\left(t+\frac{\pi}{5}\right)$

the initial condition $(x(0), y(0)) = (2.5\,;\,2.8)$ and number of switching n = 300. In this example, the periodic $T = 2\pi$, the solution of (2.4) switches between two positive periodic orbit of the systems (2.5) and (2.6).

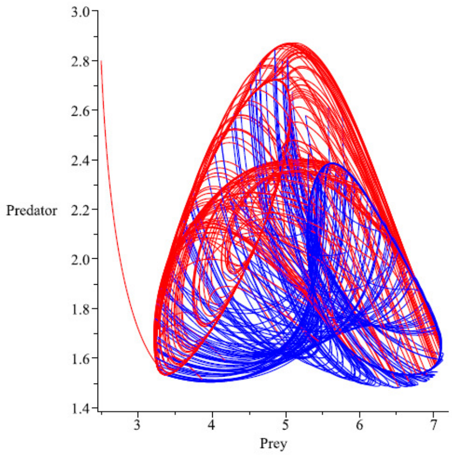

Figure 3. Orbit of the system (2.4) in example I.

Example II. $\lambda > 0$ and one state is coexistence, the other is extinction of predator. The system (2.5) with coefficients

$$a(+) = 12 + \sin \pi t\,;\; b(+) = 2.8 + \frac{1}{2}\cos \pi t\,;$$

$$c(+) = 2.4 + \frac{1}{4}\sin(\pi t + \pi);\;\; d(+) = 1.2 - \frac{1}{3}\cos\left(\pi t - \frac{\pi}{12}\right);$$

$$e(+) = 2.4 - \frac{1}{2}\sin \pi t;\;\; f(+) = 2.4 + \frac{1}{3}\sin\left(\pi t + \frac{\pi}{6}\right)\,;$$

has a stable positive periodic solution and the system (2.6) with coefficients

$$a(-) = 6.1 - \sin(\pi t + \pi)\,;\; b(-) = 1.6 + \frac{1}{3}\cos \pi t\,;$$

$$c(-) = 2.4 - \frac{1}{2}\sin\left(\pi t + \frac{\pi}{2}\right);\; d(-) = 6 + \frac{1}{6}\sin\left(\pi t + \frac{\pi}{2}\right);$$

$$e(-) = 0.5 + \frac{1}{4}\cos \pi t\;\;;\;\; f(-) = 1.9 - \frac{1}{2}\cos\left(\pi t + \frac{\pi}{5}\right)$$

has predator tending to 0. The number of switching $n = 300$, transition intensities $\alpha = 0.3$, $\beta = 0.7$ and initial condition $(x(0)\,,\,y(0)) = (1.2, 3.4)$. Since $\lambda > 0$, the system (2.4) is persistent (see figure 4).

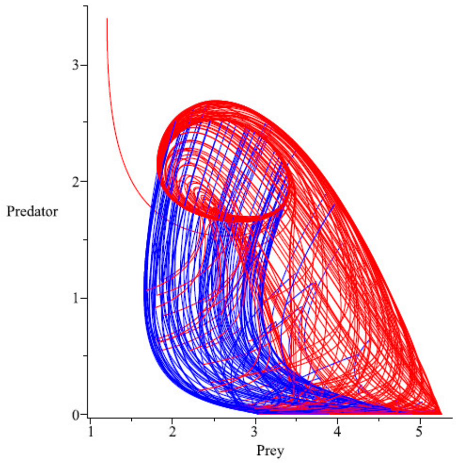

Figure 3. Orbit of the system (2.4) in example II.

This work provides some results about the asymptotic behavior of a system of two coupled deterministic predator-prey models switching at random. The formula for the value λ can not be explicitly computed. However, it is easy to approximate it by simulation. When $\lambda > 0$ the dynamics of the predator-prey system leads to the existence of a periodic Markov process, which plays an important role in the study of the development of communities.

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