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Ruin Probabilities in Generalized Risk Process with a Marko Chain Interest Model

Nguyen Thi Hien*

Viet Nam University of Comerce, Ho Tung Mau, Mai Dich, Ha Noi, Vietnam

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Abstract: This paper deals with ruin probabilities in generalized discrete time risk process with Markov chain interest model. After, recursive and integral equations for the ruin probabilities are given. When interest rates can be negative and loss distribution have regularly varying tails, the paper built an asymptotic formula for the finite time ruin probability by an inductive approach on the recursive equations.

Keywords: Discrete time risk model; Time of ruin; Ruin probability; Recursive eqution; Rate of interest.

1. Introduction

Recently, estimates for the probability of ruin within finite time or infinite time for a discrete time risk model as the initial capital tends to infinity, with emphasis on heavy-tailed insurance risk and financial risk, have drawn a lot attention. In [1], authors consider the discrete time risk process

$$U_k = U_{k-1}(1+I_k) - Y_k, \qquad k = 1,2,... \tag{1.1}$$

where $U_0 = u > 0$ is the initial capital, $\{Y_k,\ k=1,2,...\}$ (the net loss in period k) is a sequence of independent and identically distributed (i.i.d.) random variables and the interest rate $\{I_k,\ k=0,1,\ldots\}$in period k is a sequence of random variables and independent of $\{Y_k,\ k=1,2,...\}$. Thus U_k given by (1.1) is the surplus of an insurer at the end of period k. The main results dealt with in [1] are to first derive a recursive equation of the finite time ruin probabilities and then an integral equation for infinite time ruin one. After, authors generalize Lundbergs upper bound for the infinite time ruin probability. Later, X. Wei and Y. Hu consider a more general model

$$U_k = (U_{k-1} + X_n)(1+I_k) - Y_k, \qquad k = 1,2,... \tag{1.2}$$

* Tel.: 84- 984032636
E-mail: hienvcu83@gmail.com

Where, X_n is the total amount of premiums; $\{X_k,\ k=1,2,...\}$; $\{Y_k,\ k=1,2,...\}$ are two sequences independent and identically distributed random variables and the interest rate $\{I_k,\ k=0,1,...\}$ is a sequence of random variables and independent of $\{(X_k, Y_k),\ k=1,2,...\}$. They claim that the time of ruin, denoted by $\tau_u = \inf\{n \geq 1; U_n < 0\}$, has an estimate $P\{\tau_u < \infty\} \approx u^{-\lambda}$ as $u \to \infty$, where λ is a specific positive parameter. See [2], [3].

In this paper, we continue to consider the model described by (1.2) by studying a recursive equation for the finite ruin probabilities and an integral equation for infinite time ruin one. We also investigate the asymptotic formula for ruin probability.

The organization of this paper is as follows: Section 2 derives recursive and integral equation for ruin probability. In Section 3 we concern with asymptotic formulas for ruin probabilities.

2. Recursive and integral equation for ruin probpability

Consider the risk model

$$U_k = (U_{k-1} + X_n)(1 + I_k) - Y_k, \quad k = 1,2,... \tag{2.1}$$

With the sequences of random variable $(X_n);(Y_n)$ and (I_n) satisfy the assumptions in the section 1. Denote by H(x) the common distribution function of the i.i.d sequence $\{X_k,\ k=1,2,...\}$, that is

$$H(x) = P\{X_1 \leq x\},$$

And by F(x) the common loss distribution function of $\{Y_k,\ k=1,2,...\}$, i.e.,

$$F(y) = P\{Y_1 \leq y\}, \text{ with } F(0) = 0.$$

It is easy to see that the solution of (2.1) has the expression

$$U_k = u\prod_{j=1}^{k}(1+I_k) + \sum_{j=1}^{k}\left(\left(X_j(1+I_j) - Y_j\right)\prod_{t=j+1}^{k}(1+I_t)\right) \quad k = 1,2,... \tag{2.2}$$

Assume that the interest rates $\{I_n,\ \mathrm{n}=0,1,...\}$ is a homogeneous Markov chain with state space $\mathcal{I} = \{i_0, i_1, ..., i_N\}$ and the transition probabilities $p_{ij}, i, j \in \mathcal{I}$. That is for all $n = 0,1,...$and all states $i_s, i_t, i_{t_o}, ..., i_{t_{n-1}}$,

$$P\{I_{n+1} = i_t \mid I_n = i_s, I_{n-1} = i_{t_{n-1}}, ..., I_1 = i_{t_1}, I_0 = i_{t_0}\}$$
$$= P\{I_{n+1} = i_t \mid I_n = i_s\} = p_{st} \geq 0; s,t = 0,1,2,...,N,$$

where $\sum_{t=0}^{N} p_{st} = 1, s = 0,1,2,...,N.$

We define the finite and infinite time ruin probabilities in risk model (2.1) with the initial surplus u and $I_0 = i_s$ given, respectively, by

$$\psi_n(u,i_s) = P\left\{\bigcup_{k=1}^{n}(U_k < 0) \mid I_0 = i_s\right\}$$
$$= P\left\{\bigcup_{k=1}^{n}\left(u\prod_{j=1}^{k}(1+I_j) + \sum_{j=1}^{k}\left(X_j(1+I_j)-Y_j\right)\prod_{t=j+1}^{k}(1+I_t) < 0\right) \mid I_0 = i_s\right\},$$

and

$$\psi(u,i_s) = P\left\{\bigcup_{k=1}^{\infty}(U_k < 0) \mid I_0 = i_s\right\}$$
$$= P\left\{\bigcup_{k=1}^{\infty}\left(u\prod_{j=1}^{k}(1+I_j) + \sum_{j=1}^{k}\left(X_j(1+I_j)-Y_j\right)\prod_{t=j+1}^{k}(1+I_t) < 0\right) \mid I_0 = i_s\right\},$$

where U_k is given by (2.2). It is clear that

$$\psi_1(u,i_s) \le \psi_2(u,i_s) \le \psi_3(u,i_s) \le ...,$$

and

$$\lim_{n\to\infty}\psi_n(u,i_s) = \psi(u,i_s). \qquad (2.3)$$

Throughout this paper, if B is a distribution function then the function $\overline{B}(x) = 1 - B(x)$ is called tail of the distribution $B(x)$. We first give a recursive equation for $\psi_n(u,i_s)$ and an integral equation for $\psi(u,i_s)$.

Lemma 2.1. For $n = 1,2,...$ *and any* $u \ge 0$,

$$\psi_{n+1}(u,i_s) = \sum_{t=0}^{N} p_{st}\int_0^{\infty}\overline{F}\left((u+x)(1+i_t)\right)dH(x)$$
$$+ \sum_{t=0}^{N} p_{st}\int_0^{\infty}\int_0^{(u+x)(1+i_t)}\psi_n\left((u+x)(1+i_t)-y,i_t\right)dF(y)dH(x), \quad (2.4)$$

with

$$\psi_1(u,i_s) = \sum_{t=0}^{N} p_{st}\int_0^{\infty}\overline{F}\left((u+x)(1+i_t)\right)dH(x)$$

and

$$\psi(u,i_s) = \sum_{t=0}^{N} p_{st}\int_0^{\infty}\overline{F}\left((u+x)(1+i_t)\right)dH(x)$$
$$+ \sum_{t=0}^{N} p_{st}\int_0^{\infty}\int_0^{(u+x)(1+i_t)}\psi\left((u+x)(1+i_t)-y,i_t\right)dF(y)dH(x). \quad (2.5)$$

Proof. Given $X_1 = x, Y_1 = y$ and $I_1 = i_t$, from (2.1), we have

$$U_1 = (u+X_1)(1+I_1) - Y_1 = (u+x)(1+i_t) - y = h - y,$$

where $h = (u+x)(1+i_t)$. Thus, for $y > h$ it yields

$$P\{U_1 < 0 \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s\} = 1$$

Which implies that for $y > h,\ P\left\{\bigcup_{k=1}^{n+1}(U_k < 0) \mid Y_1 = y, X_1 = x, I_t = i_t, I_0 = i_s\right\} = 1.$

In case $0 \le y \le h$ one has $P\{U_1 < 0 \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s\} = 0.$ (2.6)

Let $\{\tilde{X}, n = 1,2,...\};\{\tilde{Y}, n = 1,2,...\}$ and $\{\tilde{I}_n, n = 0,1,2,...\}$ be independent copies of $\{X_n, n = 1,2,...\}$, $\{Y_n, n = 1,2,...\}$, and $\{I_n, n = 0,1,2,...\}$, respectively. From (2.6) and (2.2) it follows that for $0 \le y \le h$

$$P\left\{\bigcup_{k=1}^{n+1}(U_k < 0) \mid Y_1 = y, X_1 = x, I_t = i_t, I_0 = i_s\right\}$$
$$= P\left\{\bigcup_{k=2}^{n+1}(U_k < 0) \mid Y_1 = y, X_1 = x, I_t = i_t, I_0 = i_s\right\}$$
$$= P\left\{\bigcup_{k=2}^{n+1}\left(u\prod_{j=1}^{k}(1+I_j) + \sum_{j=1}^{k}\left(X_j(1+I_j) - Y_j\right)\prod_{t=j+1}^{k}(1+I_t) < 0\right) \mid I_1 = i_t\right\}$$
$$= P\left\{\bigcup_{k=2}^{n+1}\left((h-y)\prod_{j=2}^{k}(1+I_j) + \sum_{j=2}^{k}\left(X_j(1+I_j) - Y_j\right)\prod_{t=j+1}^{k}(1+I_t) < 0\right) \mid I_1 = i_t\right\}$$
$$= P\left\{\bigcup_{k=1}^{n+1}\left((h-y)\prod_{j=1}^{k}(1+\tilde{I}_j) + \sum_{j=1}^{k}\left(\tilde{X}_j(1+\tilde{I}_j) - \tilde{Y}_j\right)\prod_{t=j+1}^{k}(1+\tilde{I}_t) < 0\right) \mid \tilde{I}_0 = i_t\right\}$$
$$= \psi_n(h-y, i_t) = \psi_n\left(u(1+i_t) + x - y, i_t\right).$$

Therefore, by conditioning on Y_1, X_1 and I_1 , we get

$$\psi_{n+1}(u, i_s) = P\left\{\bigcup_{k=1}^{n+1}(U_k < 0) \mid I_0 = i_s\right\}$$
$$= \sum_{t=0}^{N} p_{st}\int_0^\infty\int_0^\infty P\left\{\bigcup_{k=1}^{n+1}(U_k < 0) \mid Y_1 = y, X_1 = x, I_t = i_t, I_0 = i_s\right\} dF(y)dH(x)$$
$$= \sum_{t=0}^{N} p_{st}\int_0^\infty\int_{(u+x)(1+i_t)}^\infty dF(y)dH(x)$$
$$+ \sum_{t=0}^{N} p_{st}\int_0^\infty\int_0^{(u+x)(1+i_t)} \psi_n\left((u+x)(1+i_t) - y, i_t\right) dF(y)dH(x)$$
$$= \sum_{t=0}^{N} p_{st}\int_0^\infty \overline{F}\left((u+x)(1+i_t)\right) dH(x)$$
$$+ \sum_{t=0}^{N} p_{st}\int_0^\infty\int_0^{(u+x)(1+i_t)} \psi_n\left((u+x)(1+i_t) - y, i_t\right) dF(y)dH(x).$$

Which yields the recursive equation (2.4) for the finite ruin probabilities $\psi_n(u,i_s)$ in lemma 2.1. By using the Lesbegue dominated convergence theorem and letting $n \to \infty$ in (2.4) we obtain (2.5). The proof is complete.

We remark that the techniques used in this proof are similar in [4]

3. Asymptotic formula for ruin probability

A distribution F on $(-\infty,+\infty)$ is said to have a regularly varying tail, or simply says $F \in \mathcal{R}_{-\beta}$, if there exists some constant $\beta \geq 0$ such that for any $y > 0$,

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\beta}.$$

We say that $F \in \mathcal{D}$ is dominant – tailed if for any $0 < y < 1$ one has

$$\lim_{x\to\infty} \sup \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

Similarly, we say $F \in \mathcal{L}$ to have long – tailed if for any $y > 0$

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1.$$

In case

$$\lim_{x\to\infty} e^{\lambda x}\overline{F}(x) = \infty,\ \forall \lambda > 0,$$

F is called heavy – tailed.

It is easy to see that

- $\mathcal{R}_{-\beta} \subset \mathcal{D} \subset \mathcal{L}.$
- All three classes of distributions $\mathcal{R}_{-\beta}$, $\mathcal{D}$ and $\mathcal{L}$ are heavy - tailed.
- The set $\mathcal{R}_{-\beta}$ b is stable under convolution operator, that is if $F_1 \in \mathcal{R}_{-\beta}$ and $F_2 \in \mathcal{R}_{-\beta}$ then

  $F_1 * F_2 \in \mathcal{R}_{-\beta}$ Moreover,

$$\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x) \quad \text{as } x \to \infty. \tag{3.1}$$

  See in [5] and [6].
- In addition, the class $\mathcal{R}_{-\beta}$ is closed under tail – equivalences, i.e., for two distributions F_1 and

F_2, if $F_1 \in \mathcal{R}_{-\beta}$ and $\overline{F_1}(x) \sim c\overline{F_2}(x)$ for some $c > 0$, then $F_2 \in \mathcal{R}_{-\beta}$

- Further, it is easy to see that if $F \in \mathcal{D}$ then for any constant $c > 0$, the function $\overline{F}(cu) / \overline{F}(u)$ is uniformly bounded in $u \in (-\infty, \infty)$.

We consider asymptotic formula for the finite time ruin probability $\psi_n(u, i_s)$ when the loss distribution F is heavy – tailed. We now allow that the interest rates may be negative, so that we can think of interest rates $I_n, n = 0,1,...$ as the rates of return on a risky investment satisfying:

$$1 + I_n > 0, n = 0,1,..., \text{ or equivalently, } 1 + i_s > 0, s = 0,1,...,N. \qquad (3.2)$$

We rewrite (2.1) under the form

$$U_k = U_{k-1}(1 + I_k) + X_k(1 + X_k) - Y_k, k = 1,2,...$$

Define $Q_s(z) = P(Y_k - X_k(1 + i_s) < z)$ for s=0, 1, 2 ,…,N. Since Y_k and X_k are independent,

$$\begin{aligned} Q_s(z) &= P(Y_k - X_k(1 + i_s) < z) = E\left(I_{[Y_k - X_k(1+i_s) < z]}\right) \\ &= \int_0^\infty E\left(I_{[Y_k - X_k(1+i_s) < z]} \mid X_k = x\right) dH(x) \\ &= \int_0^\infty P\left(Y_k < x(1 + i_s) + z\right) dH(x), \qquad (3.3) \end{aligned}$$

Which implies that

$$\overline{Q}_s(z) = 1 - Q_s(z) = \int_0^\infty \overline{F}\left(x(1 + i_s) + z\right) dH(x).$$

By lemma 2.1 we obtain

$$\Phi_1(u, i_s) = \sum_{t=0}^{N} \int_0^\infty \overline{F}\left((u + x)(1 + i_t)\right) dH(x) = \sum_{t=0}^{N} p_{st} \overline{Q}_t\left(u(1 + i_t)\right). \qquad (3.4)$$

Denote

$$B_{n,i_s}(u) = 1 - \Phi_n(u, i_s) \text{ and } \overline{B}_{n,i_s}(u) = \Phi_n(u, i_s), u \geq 0$$

For n=1, 2, … and s=1, 2, ….,N. All distributions $B_{n,i_s}(u) = 1 - \Phi_n(u, i_s), n = 1,2,...$ are supported on $[0, +\infty)$.

By lemma 2.1 and equation (3.3) we have

$$\begin{aligned}\Phi_{n+1}(u,i_s) &= \sum_{t=0}^{N} p_{st}\int_0^{\infty}\overline{F}\left((u+x)(1+i_t)\right)dH(x)\\ &+\sum_{t=0}^{N} p_{st}\int_0^{\infty}\int_0^{(u+x)(1+i_t)}\Phi_n\left((u+x)(1+i_t)-y,i_t\right)dF(y)\,dH(x),\\ &=\sum_{t=0}^{N} p_{st}\overline{Q}_t\left(u(1+i_t)\right)+\sum_{t=0}^{N} p_{st}\int_0^{u(1+i_t)}\Phi_n\left(u(1+i_t)-z,i_t\right)dQ_t(z).\end{aligned}$$

Further, for any distribution F_1 on $[0,\infty)$ and a distribution F_2 on $(-\infty,\infty)$, the tail of convolution F_1*F_2 satisfies

$$\overline{F_1*F_2}(x)=\overline{F_2}(x)+\int_{-\infty}^{x}\overline{F_1}(x-y)\,dF_2(y)\quad\text{for } -\infty<x<\infty.$$

Therefore,

$$\begin{aligned}\overline{B}_{n+1,i_s}(u) &= \Phi_{n+1}(u,i_s)\\ &=\sum_{t=0}^{N} p_{st}\left(\overline{Q_t}\left(u(1+i_t)\right)+\int_0^{u(1+i_t)}\Phi_n\left(u(1+i_t)-z,i_t\right)dQ_t(z)\right)\\ &=\sum_{t=0}^{N} p_{st}\overline{B_{n,i_s}*Q_t}\left(u(1+i_t)\right). \qquad (3.5)\end{aligned}$$

Theorem 3.1. Let $F\in\mathcal{R}_{-\beta}$ for some $\beta>0$ then, for any n=1,2,… and $i_s\in\mathcal{I}$

$$\Phi_n(u,i_s)\sim D_n(i_s)\overline{F}(u)\ \text{as } u\to\infty, \qquad (3.6)$$

where $D_n(i_s)$ is given recursively by

$$D_n(i_s)=E[(1+D_{n-1}(I_1))(1+I_1)^{-\beta}\mid I_0=i_s], \qquad (3.7)$$

with $D_0(i_s)=0.$

Proof. Since F is increasing function, $\overline{F}(z+x)\le\overline{F}(z)$. Therefore, for any $F\in\mathcal{R}_{-\beta}$.

$$0\le\frac{\overline{F}(z+x)}{\overline{F}(z)}\le 1 \text{ and } \lim_{z\to\infty}\frac{\overline{F}(z+x)}{\overline{F}(z)}=1. \qquad (3.8)$$

Using Lebesgue dominated convergent theorem we obtain

$$\lim_{z\to\infty}\frac{\overline{Q_t}(z)}{\overline{F}(z)}=\lim_{z\to\infty}\int_0^{\infty}\frac{\overline{F}\left(x(1+i_t)+z\right)}{\overline{F}(z)}dH(x)=1. \qquad (3.9)$$

This means that $\overline{Q_t}(z)\sim\overline{F}(z)$ as $z\to\infty$ for t =1, 2, …, N. Further, with $F\in\mathcal{R}_{-\beta}$ we have

$$\lim_{u\to\infty}\frac{\overline{F}\left(u\left(1+i_t\right)\right)}{\overline{F}\left(u\right)}=\left(1+i_t\right)^{-\beta}.$$

By (3.4) and (3.9) we get

$$\lim_{u\to\infty}\frac{\Phi_1\left(u,i_s\right)}{\overline{F}\left(u\right)}=\lim_{u\to\infty}\sum_{t=0}^{N}p_{st}\frac{\overline{Q_t}\left(u\left(1+i_t\right)\right)}{\overline{F}\left(u\left(1+i_t\right)\right)}.\frac{\overline{F}\left(u\left(1+i_t\right)\right)}{\overline{F}\left(u\right)}$$
$$=\sum_{t=0}^{N}p_{st}\left(1+i_t\right)^{-\beta}=E[\left(1+I_1\right)^{-\beta}\mid I_0=i_s]=D_1\left(i_s\right).$$

Hence, with any $i_s\in\mathcal{I}$ we have

$$\Phi_1\left(u,i_s\right)=\overline{B}_{1,i_s}\left(u\right)\sim D_1(i_s)\overline{F}\left(u\right)\ as\ u\to\infty.$$

Assume in induction that for any $i_s\in\mathcal{I}$

$$\Phi_n\left(u,i_s\right)=\overline{B}_{n,i_s}\left(u\right)\sim D_n\left(i_s\right)\overline{F}\left(u\right)\text{ as }u\to\infty. \tag{3.10}$$

We need to prove that

$$\Phi_{n+1}\left(u,i_s\right)=\overline{B}_{n+1,i_s}\left(u\right)\sim D_{n+1}\left(i_s\right)\overline{F}\left(u\right)\text{ as }u\to\infty.$$

From (3.10) it follows that B_{n,i_s} and F are tail – equivalent. Thus, $B_{n,i_s}\in\mathcal{R}_{-\beta}$. Hence, by (3.1) and (3.9) we have for any $i_s\in\mathcal{I}$,

$$\overline{B_{n,i_s}*Q_t}\left(u\right)\sim\overline{B}_{n,i_s}+\overline{Q_t}\left(u\right)\sim\left(1+D_n\left(i_s\right)\right)\overline{F}\left(u\right)\ u\to\infty. \tag{3.11}$$

Hence,

$$\lim_{u\to\infty}\frac{\overline{B_{n,i_t}*Q_t}\left(u\left(1+i_t\right)\right)}{\overline{F}\left(u\right)}=\lim_{u\to\infty}\frac{\overline{B_{n,i_t}*Q_t}\left(u\left(1+i_t\right)\right)}{\overline{F}\left(u(1+i_t)\right)}.\lim_{u\to\infty}\frac{\overline{F}\left(u\left(1+i_t\right)\right)}{\overline{F}\left(u\right)}$$
$$=\left(1+D_n\left(i_t\right)\right)\left(1+i_t\right)^{-\beta}.$$

Thus, by (3.5) we get

$$\lim_{u\to\infty}\frac{\Phi_{n+1}\left(u,i_s\right)}{\overline{F}\left(u\right)}=\lim_{u\to\infty}\sum_{t=0}^{N}p_{st}\frac{\overline{B_{n,i_t}*Q_t}\left(u\left(1+i_t\right)\right)}{\overline{F}\left(u\right)}$$
$$=\sum_{t=0}^{N}p_{st}\left(1+D_n\left(i_t\right)\right)\left(1+i_t\right)^{-\beta}=\mathrm{D}_{n+1}\left(i_s\right).$$

Hence, (3.6) holds for all n=1, 2, … This completes the proof of theorem.

4. Conclusion

The article gave an asymptotic formula for the finite time ruin probability in case of possibly negative interest rates. It is seen that the rate of convergence of ruin probability does not depend on the distribution of premiums X_n when the initial capital tends to infinity. This result confirmed a capital injection at each period whether there is premium or without premium does not play a role in the ruin probability of the model. This is similar to the results already known in game theory, when the player has an infinite amount of capital, the ruin probability gradually to 0 wich does not depend on the person's capacity to play. We think this is an important discovery in the field of financial mathematics.

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