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# Some Remarks on the Complex Stability Radius of Differential-Algebraic Equations

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**Abstract.** This paper is concerned with the robust stability of linear differential-algebraic equations (DAEs). A system of linear DAEs subjected to structured perturbation is considered. Computable formulas of the complex stability radius are given and analysed. A comparison of our formula to previous results is given.

*Keywords:* linear differential-algebraic equations, index of matrix pencil, asymptotic stability, structured perturbation, complex stability radius.

## 1. Introduction

Differential-algebraic equations (DAEs) play an important roles in mathematical modeling of real-life problems arising in a wide range of applications, for example, multibody mechanics, prescribed path control, eletrical design, biology, biomedicine, see [1, 2] and references therein. On the other hand, the robustness issue is a crucial problem for the application of control theory, for example, one of the basic goal of feedback control is to enhance system robustness. Robust stability is also an important topic in linear algebra as well as in numerical analysis.

Consider a linear DAE

$$Ey(x) = Ay(x), \tag{1.1}$$

where $A, E \in \mathbb{K}^{n\times n}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. The leading coefficient matrix $E$ is singular.

**Definition 1.1.** (see [1]) The matrix pencil $\{E, A\}$ is said to be regular if there exists $t \in \mathbb{C}$ such that the determinant of $(A - tE)$, denoted by $\det(A - tE)$, is different from zero. We also say that (1.1) is regular. Otherwise, if $\det(A - tE) = 0$, $\forall t \in \mathbb{C}$, we say that $\{E, A\}$ is irregular.

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If $\{E, A\}$ is regular, then a complex number t is called a (generalized finite) eigenvalue of $\{E, A\}$ if $\det(A - tE) = 0$. The set of all eigenvalues is called the spectrum of the pencil $\{E, A\}$ and denoted by $\sigma(E, A)$.

If $E$ is singular and $\{E, A\}$ is regular, then we say that $\{E, A\}$ has the eigenvalue $\infty$.

Suppose that the matrix pencil $\{E, A\}$ is regular. Then the pairs can be transformed to Kronecker canonical form i.e there exist nonsingular matrices $P$, $Q$ such that

$$PEQ = \begin{bmatrix} I_r & O \\ O & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} J & O \\ O & I_{n-r} \end{bmatrix}, \qquad (1.2)$$

where $N$ is a nilpotent matrix of index k (see [1, 2]). If $N$ is a zero matrix, then $k = 1$. Furthermore, we may assume without loss of generality, that $N$ and $J$ are upper triangular. If $\{E, A\}$ is regular, then the nilpotency index of $N$ in (1.2) is called the index of matrix pencil $\{E, A\}$ and we write $\text{index}\{E, A\} = k$.

In particular, a regular index-one system can be given by the form

$$E = \begin{bmatrix} E_{11} & E_{12} \\ O & O \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{22}$ and $E_{11} - E_{12}A_{22}^{-1}A_{21}$ are square and of full rank (or invertible matrices).

Now, we give the definition of asymptotic stability of the solution of (1.1), see [2].

**Definition 1.2.** Suppose that {E, A} is regular. Let Q be a projector onto the subspace of consistent initial conditions. Let P = I - Q. We say that the null solution of (1.1) is stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that for an arbitrary vector $y_0 \in \mathbb{C}^n$ satisfying $\| y_0 \| < \delta$, the solution of the initial value problem

$$\begin{cases} E\dot{y}(x) = Ay(x), & x \in [0, \infty), \\ P(y(0) - y_0) = 0 \end{cases}$$

exists uniquely and the estimate $\| y(x) \| < \epsilon$ holds for all $x \geq 0$.

The null solution is said to be asymptotically stable if it is stable and $\lim_{x \to \infty} \| y(x) \| = 0$ for solution $y$ of (1.1). If the null solution, of (1.1) is asymptotically stable, we say that system (1.1) is asymptotically stable.

**Theorem 1.1.** (see [3]) The null solution, of system (1.1) is asymptotically stable if and only if the eigenvalues of the matrix pencil {E, A} all have negative real part.

If the eigenvalues of the matrix pencil *{E , A }* all have negative real part, then the matrix pencil *{ E , A }* is said to be stable.

Now, let us suppose that system (1.1) is asymptotically stable and consider the perturbed system

$$(E + B_E A_E C)y' = (A + B_A A_A C)y, \qquad (1.3)$$

where $B_A \in \mathbb{K}^{n\times p_1}$, $B_E \in \mathbb{K}^{n\times p_2}$, $C \in \mathbb{K}^{q\times n}$ are given matrices, $\Delta_A \in \mathbb{K}^{p_1\times q}$, $\Delta_E \in \mathbb{K}^{p_2\times q}$ ($\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$) are uncertain perturbations. $B_A\Delta_A C$, $B_E\Delta_E C$ are called structured perturbations. Denote $\Delta = \begin{bmatrix}\Delta_A \\ \Delta_E\end{bmatrix}$, we define the set of "bad" (destabilizing) perturbations

$$\mathcal{V}_{\mathbb{K}} = \left\{ \Delta \in \mathbb{K}^{(p_1+p_2)\times q} \,\middle|\, \begin{array}{l}\text{the matrix pencil } \{(E+B_E\Delta_E C),(A+B_A\Delta_A C)\} \\ \text{is either irregular or unstable}\end{array} \right\}.$$

**Definition 1.3.** Let, the system (1.1) be asymptotically stable. The structured stability radius for (1.1) is defined by $r_{\mathbb{K}} = \inf\{\|\Delta\| \mid \Delta \in \mathcal{V}_{\mathbb{K}}\}$, where $\|\cdot\|$ is a matrix norm induced by vector norms in $\mathbb{K}^{(p_1+p_2)\times q}$.

Depending on $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$, we talk about the complex or the real stability radius, respectively. Obviously, we have the estimate $r_{\mathbb{C}} \le r_{\mathbb{R}}$ The problem of computing the stability radius for ODEs was introduced in [4-7]. Later, the result was extended to DAEs in [8-12], The aim of this paper is to compute a general formula of the complex stability radius. Moreover, a comparison of our formula to previous results is given.

The outline of this paper is as follows. Firstly, the complex structured stability radius for (1.1) is given. Furthermore, we show the details on stability and structure robustness of systems of index one. Finaly, we show that the formula of the complex stability radius by Byers and Nichols in [8] can be obtained as a special case of our result.

## 2. Main result

**Theorem 2.1**. The complex structured stability radius for (1.1) is given by

$$r_{\mathbb{C}} = \left\{ \sup_{t\in \mathrm{i}\mathbb{R}} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1}, \tag{2.4}$$

where $G_1(t) = -C(A-tE)^{-1}B_A$, $G_2(t) = tC(A-tE)^{-1}B_E$.

**Proof.** To prove this theorem, we use the technique that is same as in [9]. First, we prove that

$$r_{\mathbb{C}} = \left\{ \sup_{\mathrm{Re}\, t\ge 0} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1}.$$

To this end, we prove that

$$r_{\mathbb{C}} \ge \left\{ \sup_{\mathrm{Re}\, t\ge 0} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1}.$$

There are two cases in which $\{(E+B_E\Delta_E C),(A+B_A\Delta_A C)\}$ is either unstable or irregular.

**The first case.** Let $\{(E+B_E\Delta_E C),(A+B_A\Delta_A C)\}$ be unstable, then there exists $t\in\sigma((E+B_E\Delta_E C),(A+B_A\Delta_A C))$ and $\mathrm{Re}(t)\geq 0$. There exists $x\neq 0$ satisfying,

$$\begin{aligned}
&(A+B_A\Delta_A C)x = t(E+B_E\Delta_E C)x\\
\Leftrightarrow\quad &(A-tE)x \;=\; (-B_A\Delta_A C+tB_E\Delta_E C)x\\
\Leftrightarrow\quad &x \;=\; (A-tE)^{-1}\begin{bmatrix}-B_A & tB_E\end{bmatrix}\begin{bmatrix}\Delta_A\\ \Delta_E\end{bmatrix}Cx\\
\Rightarrow\quad &Cx \;=\; C(A-tE)^{-1}\begin{bmatrix}-B_A & tB_E\end{bmatrix}\Delta Cx.
\end{aligned}$$

Given $u=Cx$, we have, $u=\begin{bmatrix}G_1(t) & G_2(t)\end{bmatrix}\Delta u.$

Hence, $\|\Delta\|\geq\left\{\sup_{\mathrm{Re}t\geq 0}\left\|\begin{bmatrix}G_1(t) & G_2(t)\end{bmatrix}\right\|\right\}^{-1}$,

or

$$r_{\mathbb{C}}\geq\left\{\sup_{\mathrm{Re}t\geq 0}\left\|\begin{bmatrix}G_1(t) & G_2(t)\end{bmatrix}\right\|\right\}^{-1}.$$

The second case. Let $\{(E+B_E\Delta_E C),(A+B_A\Delta_A C)\}$ be irregular which means that for any $t\in\mathbb{C}$ we have $\det((A+B_A\Delta_A C)-t(E+B_E\Delta_E C))=0$. Given $t$ such that $\mathrm{Re}(t)\geq 0$, then there exists $x\neq 0$ satisfying

$$t(E+B_A\Delta_A C)x=(A+B_E\Delta_E C)x.$$

Similarly to the first case, we obtain

$$\|\Delta\|\geq\left\{\sup_{\mathrm{Re}t\geq 0}\left\|\begin{bmatrix}G_1(t) & G_2(t)\end{bmatrix}\right\|\right\}^{-1}.$$

It is clear that, in any case,

$$r_{\mathbb{C}}\geq\left\{\sup_{\mathrm{Re}t\geq 0}\left\|\begin{bmatrix}G_1(t) & G_2(t)\end{bmatrix}\right\|\right\}^{-1}. \tag{2.5}$$

Now, we prove the inverse inequality

$$r_{\mathbb{C}}\leq\left\{\sup_{\mathrm{Re}t\geq 0}\left\|\begin{bmatrix}G_1(t) & G_2(t)\end{bmatrix}\right\|\right\}^{-1}.$$

Indeed, for any $\epsilon>0$, there exists $t_0$ having $\mathrm{Re}(t_0)\geq 0$ such that

$$\left\|\begin{bmatrix}G_1(t_0) & G_2(t_0)\end{bmatrix}\right\|^{-1}\leq\left\{\sup_{\mathrm{Re}t\geq 0}\left\|\begin{bmatrix}G_1(t) & G_2(t)\end{bmatrix}\right\|\right\}^{-1}+\epsilon.$$

We construct a destabilizing perturbation $\Delta = \begin{bmatrix} \Delta_A \\ \Delta_E \end{bmatrix}$ such that $\|\Delta\| = \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|^{-1}$. There exists a vector $x \in \mathbb{C}^{p_1+p_2}$, $\|x\|=1$ such that $\left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix} x\right\| = \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|$. Invoking a corollary of the Hahn-Banach theorem, there exists a functional $y^* \in \mathbb{C}^q$, $\|y^*\|=1$ such that

$y^*\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix} x = \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix} x\right\| = \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|$.

Let us define $\Delta = \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|^{-1} xy^*$, then

$$\begin{aligned} \Delta\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix} x &= \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|^{-1} xy^*\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix} x \\ &= \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|^{-1} x \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|. \end{aligned}$$

We deduce $\|\Delta\| \ge \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|^{-1}$.

On the other hand, from the definition of $\Delta$, we have $\|\Delta\| \le \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|^{-1}$.

Thus, $\|\Delta\| = \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|^{-1}$.

We show that the perturbed system will be either unstable or irregular.

$$\begin{aligned} & \Delta\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix} x && = x \\ \Leftrightarrow\quad & \Delta\begin{bmatrix} -C(A-t_0E)^{-1}B_A & t_0C(A-t_0E)^{-1}B_E \end{bmatrix} x && = x \\ \Leftrightarrow\quad & \Delta C(A-t_0E)^{-1}\begin{bmatrix} -B_A & t_0B_E \end{bmatrix} x && = x. \end{aligned}$$

Multiplying both sides with $(A-t_0E)^{-1}\begin{bmatrix} -B_A & t_0B_E \end{bmatrix}$ from the left, denoting $u = (A-t_0E)^{-1}\begin{bmatrix} -B_A & t_0B_E \end{bmatrix} x$, we obtain

$$\begin{aligned} & u && = (A-t_0E)^{-1}\begin{bmatrix} -B_A & t_0B_E \end{bmatrix}\Delta Cu \\ \Leftrightarrow\quad & (A-t_0E)u && = \begin{bmatrix} -B_A & t_0B_E \end{bmatrix}\begin{bmatrix} \Delta_A \\ \Delta_E \end{bmatrix} Cu \\ \Leftrightarrow\quad & (A-t_0E)u && = -B_A\Delta_ACu + t_0B_E\Delta_ECu \\ \Leftrightarrow\quad & (A+B_A\Delta_AC)u && = t_0(E+B_E\Delta_EC)u. \end{aligned}$$

We have $u \ne 0$ because $x \ne 0$. It is obtained that either $t_0 \in \sigma((E+B_E\Delta_EC),(A+B_A\Delta_AC))$ or $\{(E+B_E\Delta_EC),(A+B_A\Delta_AC)\}$ is irregular.

Because $\mathrm{Re}(t_0) \ge 0$ then the perturbed system is either unstable or irregular. It is clear that

$r_{\mathbb{C}} \le \|\Delta\| = \left\|\begin{bmatrix} G_1(t_0) & G_2(t_0) \end{bmatrix}\right\|^{-1} \le \left\{\sup_{\mathrm{Re}\, t\ge 0}\left\|\begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix}\right\|\right\}^{-1} + \epsilon.$ Because $\epsilon$ is arbitrary, we deduce

$$r_{\mathbb{C}} \leq \left\{ \sup_{\mathrm{Re}\,t \geq 0} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1}. \tag{2.6}$$

From (2.5) and (2.6) we have

$$r_{\mathbb{C}} = \left\{ \sup_{\mathrm{Re}\,t \geq 0} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1}.$$

To complete this proof, we note that $\begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix}$ is analytic in $\mathbb{C} \setminus \mathbb{C}^-$, due to the maximum principle, their least upper bound is attained in $i\mathbb{R}$ (at a finite point or at infinity). Hence,

$$\left\{ \sup_{\mathrm{Re}\,t \geq 0} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1} = \left\{ \sup_{t \in i\mathbb{R}} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1}.$$

Finaly, we have $r_{\mathbb{C}} = \left\{ \sup_{t \in i\mathbb{R}} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1}$,

where $G_1(t) = -C(A - tE)^{-1} B_A$, $G_2(t) = tC(A - tE)^{-1} B_E$. □

**Remark 2.1.**

If $E$ is nonsingular matrix, then

$$\begin{aligned} \lim_{|t| \to +\infty} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| &= \lim_{|t| \to +\infty} \left\| \begin{bmatrix} -C(A - tE)^{-1} B_A & tC(A - tE)^{-1} B_E \end{bmatrix} \right\| \\ &= \lim_{|t| \to +\infty} \left\| \begin{bmatrix} -C(A - tE)^{-1} B_A & C(\frac{1}{t} A - E)^{-1} B_E \end{bmatrix} \right\| < +\infty. \end{aligned}$$

If $E$ is singular matrix, then

$$G_1(t) = -C(A - tE)^{-1} B_A = -CQ \begin{bmatrix} (J - tI_r)^{-1} & O \\ O & I_{n-r} + \sum_{i=1}^{k-1} (tN)^i \end{bmatrix} PB_A,$$

$$G_2(t) = tC(A - tE)^{-1} B_E = CQ \begin{bmatrix} (\frac{1}{t} J - I_r)^{-1} & O \\ O & t(I_{n-r} + \sum_{i=1}^{k-1} (tN)^i) \end{bmatrix} PB_E, t \neq 0.$$

Thus, the complex structured stability radius for (1.1) is strictly positive only if the $index\{E, A\} = 0$. Moreover, for $index\{E, A\} = 1$, the radius is positive when the structured perturbation of the leading term has influence only on the differential part. And from now on, we consider the system having index one with suitable structured perturbations of the leading term.

**Lemma 2.1.** Let, {E, A} be regular.

If $index\{E, A\} \leq 1$ then $\deg\left[\det(A - tE)\right] = \mathrm{rank} E$.

If $\text{index}\{E, A\} > 1$ then $\deg\left[\det(A-tE)\right] < rankE$.

**Proof.** Let pencil {E, A} have canonical form,

$$E = P^{-1}\begin{bmatrix} I_r & O \\ O & N \end{bmatrix}Q^{-1}, \quad A = P^{-1}\begin{bmatrix} J & O \\ O & I_{n-r} \end{bmatrix}Q^{-1}.$$

• If $k = 0$ then $E = P^{-1}I_nQ^{-1}$, $A = P^{-1}JQ^{-1}$. It means that $\text{rank}E = n$

$$\det(A-tE) = \det(P^{-1}Q^{-1})\det(J-tI_n).$$

Because $\det(J-tI_n)$ is polynomial of degree $n$, so $\deg\left[\det(A-tE)\right] = rankE$.

• If $k = 1$, then $N$ is null matrix, and $A - tE = P^{-1}\begin{bmatrix} J-tI_r & O \\ O & I_{n-r} \end{bmatrix}Q^{-1}.$

Hence, $\det(A-tE) = \det(P^{-1}Q^{-1})\det(J-tI_r).$

Because pencil $\{E, A\}$ is regular, so $\det(A-tE) \not\equiv 0$, or $\det(J-tI_r)$ is polynomial of degree $r$. On ther other hand, $\text{rank}E = r$, so $\deg\left[\det(A-tE)\right] = \text{rank}E$.

• If $k > 1$, then $N$ is in Jordan form, $N = \text{diag}\left[J_1, \cdots, J_l\right]$, and $A - tE = P^{-1}\begin{bmatrix} J-tI_r & O \\ O & MT \end{bmatrix}Q^{-1}$,

where MT is upper triangular matrix which have diagonal elements equal to one. We deduce

$$\det(A-tE) = \det(P^{-1}Q^{-1})\det(J-tI_r).$$

Because pencil $\{E, A\}$ is regular, so $\det(A-tE) \not\equiv 0$, or $\det(J-tI_r)$ is polynomial of degree $r$. On ther other hand, $\text{rank}E > r$, so $\deg\left[\det(A-tE)\right] < \text{rank}E$. □

To prove the following lemma, the technique is used in the proof of Theorem 2.1 can be applied. Thus, we obtain the result.

**Lemma 2.2.** Let $M \in \mathbb{C}^{n\times n}$ is nonsingular matrix. If $d_{\mathbb{C}} = \inf\left\{\|\Delta\|, M + H\Delta T \text{ is singular}\right\}$, where $H \in \mathbb{C}^{n\times p}$, $\Delta \in \mathbb{C}^{p\times q}$, $T \in \mathbb{C}^{q\times n}$ are given matrices, then $d_{\mathbb{C}} = \left\|TM^{-1}H\right\|^{-1}$.

Without loss of generality, the system having index one can be simplified as follows

$$\begin{bmatrix} I & O \\ O & O \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where $A_{22}$ is invertible. From the *Remark 2.1*, for $\text{index}\{E, A\} = 1$, the radius is positive when the structured perturbation of the leading term has influence only on the differential part. So that, we consider the structured perturbations with

$$B_A = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ B_E = \begin{bmatrix} B \\ O \end{bmatrix},\ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

**Theorem 2.2.** Let pencil $\{E, A\}$ be regular and index one. For any $\Delta = \begin{bmatrix} \Delta_A \\ \Delta_E \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times q}$ satisfies $\|\Delta\| < r_{\mathbb{C}}$, we have

$$\begin{aligned} \operatorname{index}\{(E + B_E\Delta_E C), (A + B_A\Delta_A C)\} &= \operatorname{index}\{E, A\}. \\ \deg\left[\det((A + B_A\Delta_A C) - t(E + B_E\Delta_E C))\right] &= \deg\left[\det(A - tE)\right]. \end{aligned}$$

**Proof.** Consider the perturbed system

$$\begin{bmatrix} I + B\Delta_E C_1 & B\Delta_E C_2 \\ O & O \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} + B_1\Delta_A C_1 & A_{12} + B_1\Delta_A C_2 \\ A_{21} + B_2\Delta_A C_1 & A_{22} + B_2\Delta_A C_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The perturbed system has index one if and only if $A_{22} + B_2\Delta_A C_2$, and $I + B\Delta_E C_1 - B\Delta_E C_2 (A_{22} + B_2\Delta_A C_2)^{-1}(A_{21} + B_2\Delta_A C_1)$ are invertible.

Denote $R_1 = \inf\left\{\|\Delta_A\|, A_{22} + B_2\Delta_A C_2 \text{ is singular}\right\}$. Using Lemma 2.2, we obtain $R_1 = \left\|C_2 A_{22}^{-1} B_2\right\|^{-1}$.

Next, we prove inequality

$$\left\{\sup_{t\in i\mathbb{R}} \|G_1(t)\|\right\}^{-1} \le \left\|C_2 A_{22}^{-1} B_2\right\|^{-1},$$

i.e

$$\lim_{|t|\to+\infty} \left\|-C(A - tE)^{-1} B_A\right\| = \left\|C_2 A_{22}^{-1} B_2\right\|.$$

We have (see [13,14])

$$A - tE = \begin{bmatrix} A_{11} - tI & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ O & I \end{bmatrix} \begin{bmatrix} A_{11} - tI - A_{12}A_{22}^{-1}A_{21} & O \\ O & A_{22} \end{bmatrix} \begin{bmatrix} I & O \\ A_{22}^{-1}A_{21} & I \end{bmatrix},$$

So

$$(A - tE)^{-1} = \begin{bmatrix} I & O \\ -A_{22}^{-1}A_{21} & I \end{bmatrix} \begin{bmatrix} (A_{11} - tI - A_{12}A_{22}^{-1}A_{21})^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ O & I \end{bmatrix}.$$

We deduce

$$-C(tE - A)^{-1} B_A = -(C_1 - C_2A_{22}^{-1}A_{21})(tI - A_{11} + A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2) - C_2A_{22}^{-1}B_2.$$

We have $\lim\limits_{|t|\to+\infty} \left\|-C(A - tE)^{-1} B_A\right\| = \left\|C_2 A_{22}^{-1} B_2\right\|$.

Hence,

$$r_{\mathbb{C}} \le \left\{ \sup_{t\in i\mathbb{R}} \|G_1(t)\| \right\}^{-1} \le R_1.$$

This means that $A_{22}+B_2\Delta_A C_2$ is nonsingular if $\|\Delta\| < r_{\mathbb{C}}$.

On the other hand, we have

$$\begin{bmatrix} I+B\Delta_E C_1 & B\Delta_E C_2 \\ A_{21}+B_2\Delta_A C_1 & A_{22}+B_2\Delta_A C_2 \end{bmatrix} = \begin{bmatrix} I & B\Delta_E C_2(A_{22}+B_2\Delta_A C_2)^{-1} \\ O & I \end{bmatrix}$$

$$\begin{bmatrix} I+B\Delta_E C_1 - B\Delta_E C_2(A_{22}+B_2\Delta_A C_2)^{-1}(A_{21}+B_2\Delta_A C_1) & O \\ O & A_{22}+B_2\Delta_A C_2 \end{bmatrix}$$
$$\begin{bmatrix} I & O \\ (A_{22}+B_2\Delta_A C_2)^{-1}(A_{21}+B_2\Delta_A C_1) & I \end{bmatrix}.$$

It is easy to see that $I+B\Delta_E C_1 - B\Delta_E C_2(A_{22}+B_2\Delta_A C_2)^{-1}(A_{21}+B_2\Delta_A C_1)$ is singular if and only if

$$\begin{bmatrix} I+B\Delta_E C_1 & B\Delta_E C_2 \\ A_{21}+B_2\Delta_A C_1 & A_{22}+B_2\Delta_A C_2 \end{bmatrix} \left( \text{or } \begin{bmatrix} I & O \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} O & B \\ B_2 & O \end{bmatrix} \begin{bmatrix} \Delta_A \\ \Delta_E \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right) \text{ is singular.}$$

Denote

$$R_2 = \inf \left\{ \left\| \begin{bmatrix} \Delta_A \\ \Delta_E \end{bmatrix} \right\| . \begin{bmatrix} I & O \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} O & B \\ B_2 & O \end{bmatrix} \begin{bmatrix} \Delta_A \\ \Delta_E \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} \text{ singular.} \right\}$$

Using Lemma 2.2, we have

$$R_2 = \left\| \begin{bmatrix} C_2A_{22}^{-1}B_2 & (C_1 - C_2A_{22}^{-1}A_{21})B \end{bmatrix} \right\|^{-1}.$$

Now, we need to prove

$$\lim_{|t|\to+\infty} \left\| -C(A-tE)^{-1}B_A \quad tC(A-tE)^{-1}B_E \right\| = \left\| C_2A_{22}^{-1}B_2 \quad (C_1 - C_2A_{22}^{-1}A_{21})B \right\|.$$

From $(A-tE)^{-1} = \begin{bmatrix} I & O \\ -A_{22}^{-1}A_{21} & I \end{bmatrix} \begin{bmatrix} (A_{11}-tI-A_{12}A_{22}^{-1}A_{21})^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ O & I \end{bmatrix}.$

We deduce

$$-C(A-tE)^{-1}B_A = -(C_1 - C_2A_{22}^{-1}A_{21})(tI - A_{11} + A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2) - C_2(-A_{22})^{-1}B_2,$$
$$tC(A-tE)^{-1}B_E = t(C_1 - C_2A_{22}^{-1}A_{21})(A_{11} - tI - A_{12}A_{22}^{-1}A_{21})^{-1}B.$$

We obtain

$$\lim_{|t|\to+\infty} \left\| -C(A-tE)^{-1}B_A \quad tC(A-tE)^{-1}B_E \right\| = \left\| C_2A_{22}^{-1}B_2 \quad (C_1 - C_2A_{22}^{-1}A_{21})B \right\|.$$

Thus, $r_{\mathbb{C}} = \left\{ \sup_{t\in i\mathbb{R}} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\| \right\}^{-1} \le R_2$.

In conclusion, if $\|\Delta\| < r_{\mathbb{C}}$, then $index\{(E+B_E\Delta_E C),(A+B_A\Delta_A C)\}=1$. Using Lemma 1.1, we have

$$\deg\left[\det((A+B_A\Delta_A C)-t(E+B_E\Delta_E C))\right]=\operatorname{rank}(E+B_E\Delta_E C)=\operatorname{rank}E. \qquad \square$$

The algebraic structure of an index one DAEs is characterized by the index and the number of the finite eigenvalues of the pencil. We denote

$$\widetilde{\mathcal{V}}_{\mathbb{C}}=\left\{\Delta\in\ \mathbb{C}^{(p_1+p_2)\times q}\ \middle|\ \begin{array}{l}\text{the pencil } \{(E+B_E\Delta_E C),(A+B_A\Delta_A C)\}\\ \text{is either irregular or unstable,}\\ \text{or its algebraic structure changes}\end{array}\right\}.$$

The algebraic structure of the pencil $\{(E+B_E\Delta_E C),(A+B_A\Delta_A C)\}$ changes if its index, or the number of finite singular values, or both change . Now, for a DAEs of index one, the complex structured stability radius can be redefined as following $\tilde{r}_{\mathbb{C}}=\inf\left\{\|\Delta\|\mid\Delta\in\ \widetilde{\mathcal{V}}_{\mathbb{C}}\right\}$.

By the Theorems 2.1 and 2.2, the following result immediately follows.

**Theorem 2.3.** Using the same assumption as in the Theorem. (2.1) and (2.2), we have

$$\tilde{r}_{\mathbb{C}}=r_{\mathbb{C}}=\left\{\sup_{t\in i\mathbb{R}}\left\|\left[G_1(t)\quad G_2(t)\right]\right\|\right\}^{-1}, \qquad (2.7)$$

where $G_1(t)=-C(A-tE)^{-1}B_A$, $G_2(t)=tC(A-tE)^{-1}B_E$.

**Example 2.1.** For sake of simplicity, we use the maximum norm as the matrix norm.

Consider

$$A=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad E=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}.$$

It is easy to verify that $index\{E,A\}=2$, $\sigma(E,A)=\{-1\}$. With $C=I$, $B_A=\begin{bmatrix}1&0&0\\-2&0&0\\0&0&0\end{bmatrix}$,

$B_E=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$, we have

$$G_1(t)=-C(A-tE)^{-1}B_A=\begin{bmatrix}\frac{1}{t+1}&0&0\\2&0&0\\0&0&0\end{bmatrix},\ G_2(t)=tC(A-tE)^{-1}B_E=\begin{bmatrix}\frac{-t}{t+1}&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

Hence, $\sup_{t\in i\mathbb{R}} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\|_\infty = \sup_{t\in i\mathbb{R}} \max \left\{ \left| \frac{1}{t+1} \right| + 2, \left| \frac{-t}{t+1} \right| \right\} = 3$.

The least upper bound is attained at $t = 0$. So, $r_{\mathbb{C}} = \frac{1}{3}$, we construct the destabilizing perturbations $\Delta_A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\Delta_E = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We deduce $\sigma((E + B_E\Delta_E C), (A + B_A\Delta_A C)) = \{0\}$ which means that the system is unstable.

In the case which the least upper bound attained at $\infty$, we consider arbitrary sequence $t_n$ proceeding to $\infty$. Then, for each $t_n$, we construct the destabilizing perturbations $\begin{bmatrix} \Delta_A^n \\ \Delta_E^n \end{bmatrix}$. The sequence $\left\| \begin{bmatrix} \Delta_A^n \\ \Delta_E^n \end{bmatrix} \right\|$ which have limits being the stability radius. But the perturbations that have norm equal to the stability radius only change the algebraic structured. To verify that, we consider following

**Example 2.2.** Consider

$$A = \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}.$$

It is easy to verify that $index\{E, A\} = 1$, $\sigma(E, A) = \left\{ -\frac{1}{2} \right\}$. With $C = I = B_A$ and $B_E = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$, we have $G_1(t) = -C(A - tE)^{-1} B_A = \begin{bmatrix} \frac{4t}{2t+1} & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$,

$$G_2(t) = tC(A - tE)^{-1} B_E = \begin{bmatrix} \frac{2t}{2t+1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $\sup_{t\in i\mathbb{R}} \left\| \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix} \right\|_\infty = \sup_{t\in i\mathbb{R}} \max \left\{ \left| \frac{2\text{t}}{2\text{t}+1} \right|, \left| \frac{4t}{2t+1} \right| + 1, \frac{3}{2} \right\} = 3,$

the least upper bound attained at $t = \infty$. Considering the sequence $t_n = in$, for each $t_n$, we construct the destabilizing perturbations $\Delta_A^n = \begin{bmatrix} \frac{1}{\sqrt{3^2 - \frac{1}{4n^2}}} & \frac{1}{\sqrt{3^2 - \frac{1}{4n^2}}} \\ 0 & 0 \end{bmatrix}$, $\Delta_E^n = O$. It is easy to see that $\begin{bmatrix} \Delta_A^n \\ \Delta_E^n \end{bmatrix}$ is a destabilizing perturbations i.e the pencil $\{(E + B_E \Delta_E^n C), (A + B_A \Delta_A^n C)\}$ is unstable. On the other hand, we have $\Delta_A^n \to \Delta_A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$, $\Delta_E^n \to \Delta_E = O$ when $n \to \infty$. But $\begin{bmatrix} \Delta_A \\ \Delta_E \end{bmatrix}$ only changes the algebraic structured because $\sigma((E + B_E \Delta_E^n C), (A + B_A \Delta_A^n C)) = \varnothing$. This can be explained that the least upper bound is not attained at a finite point so the greatest lower bound of the destabilizing perturbations is not attained. Althought, the limits of the sequence of perturbations is existence. Now, we will show that our result can be used to obtain the result of Ralph Byers, N.K. Nichols in [8]. Firstly, we can repeat some results from [8]

**Definition 2.1.** ( see 8] ) The radius of stability of the stable regular pencil $\{A, E\}$ is given by

$$\rho(A,E) = \inf \left\{ \| [\delta A \mid \delta E] \|_F \;\middle|\; \begin{array}{l} \{(E+\delta E),(A+\delta A)\} \text{ is either unstable or irregular} \\ \text{or algebraic structure changes} \end{array} \right\}$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

**Lemma 2.3.** ( see [8] ) If $\{E, A\}$ is regular and of index less than or equal to one, then there exists an orthogonal matrix $P$ and a permutation matrix $Q$ such that

$$PEQ = \begin{bmatrix} E_{11} & E_{12} \\ O & O \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $\text{rank}(E_{11}, E_{12}) = \text{rank}E = k$, $\text{rank}A_{22} = n - k$ and $\rho(A,E) = \rho\left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} E_{11} & E_{12} \\ O & O \end{bmatrix} \right)$.

Furthermore, $P\delta EQ = \begin{bmatrix} \delta E_{11} & \delta E_{12} \\ O & O \end{bmatrix}$, $P\delta AQ = \begin{bmatrix} \delta A_{11} & \delta A_{12} \\ \delta A_{21} & \delta A_{22} \end{bmatrix}$.

We define for $\theta$, $\omega \in \mathbb{R}$, $\theta^2 + \omega^2 = 1$, the matrix function $H(\theta,\omega) = \begin{bmatrix} \theta A_{11} & \theta A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} i\omega E_{11} & i\omega E_{12} \\ O & O \end{bmatrix}$.

**Theorem 2.4.** ( see [8] ) If $\{E, A\}$ is stable, regular, and of index less than or equal to one, then

$$\rho(A,E) = \inf_{\substack{\theta,\omega\in\mathbb{R}\\ \theta^2+\omega^2=1}} \sigma_{\min}\{H(\theta,\omega)\}, \quad (2.8)$$

To show (2.7) implies (2.8), we consider the pencil with index one in the form

$$E = \begin{bmatrix} E_{11} & E_{12} \\ O & O \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

and

$$B_A = \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \quad B_E = \begin{bmatrix} I & O \\ O & O \end{bmatrix}, \quad C = \begin{bmatrix} I & O \\ O & I \end{bmatrix}.$$

**Theorem 2.5.** If the matrix pencil $\{E, A\}$ is regular, stable, and has index one, and the matrix norm is Euclidean norm, then the complex structured stability radius is

$$r_{\mathbb{C}} = \inf_{\substack{\theta,\omega\in\mathbb{R}\\ \theta^2+\omega^2=1}} \sigma_{\min}\{H(\theta,\omega)\},$$

Proof. Using the formula of the complex structured stability radius from Theorem 2.1, we have

$$r_{\mathbb{C}} = \left\{ \sup_{t\in i\mathbb{R}} \left\| \begin{bmatrix} A_{11}-tE_{11} & A_{12}-tE_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} -I & O & tI & O \\ O & -I & O & O \end{bmatrix} \right\| \right\}^{-1}.$$

We shall consider two cases.

• **The first case.** If the least upper bound is attained at $t_0$, then we set $t_0 = i\dfrac{\omega}{\theta}$, where $\theta,\ \omega\in\mathbb{R}$, $\theta\neq 0$, and $\theta^2+\omega^2=1$. Thus,

$$r_{\mathbb{C}} = \left\{ \left\| \begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} -I & O & i\frac{\omega}{\theta}I & O \\ O & -I & O & O \end{bmatrix} \right\|_2 \right\}^{-1}.$$

On the other hand,

$$\left\| \begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} -I & O & i\frac{\omega}{\theta}I & O \\ O & -I & O & O \end{bmatrix} \right\|_2^2 =$$

$$= \lambda_{\max}\left\{ \begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} -I & O & i\frac{\omega}{\theta}I & O \\ O & -I & O & O \end{bmatrix} \right.$$

$$\left.\begin{bmatrix} -I & O \\ O & -I \\ -i\frac{\omega}{\theta}I & O \\ O & O \end{bmatrix}\left(\begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\right)^*\right\}$$

where $\lambda_{\max}\{\cdot\}$ is the largest eigenvalue. Hence,

$$\left\|\begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\begin{bmatrix} -I & O & i\frac{\omega}{\theta}I & O \\ O & -I & O & O \end{bmatrix}\right\|_2^2 =$$

$$= \lambda_{\max}\left\{\begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{\theta^2}I & O \\ O & I \end{bmatrix}\left(\begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\right)^*\right\}$$

$$= \lambda_{\max}\left\{\begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{\theta}I & O \\ O & I \end{bmatrix}\begin{bmatrix} \frac{1}{\theta}I & O \\ O & I \end{bmatrix}\left(\begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\right)^*\right\}$$

$$= \lambda_{\max}\left\{\left(\begin{bmatrix} \theta I & O \\ O & I \end{bmatrix}\begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)^{-1}\left(\left(\begin{bmatrix} \theta I & O \\ O & I \end{bmatrix}\begin{bmatrix} A_{11}-i\frac{\omega}{\theta}E_{11} & A_{12}-i\frac{\omega}{\theta}E_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)^{-1}\right)^*\right\}$$

$$= \lambda_{\max}\left\{\begin{bmatrix} \theta A_{11}-i\omega E_{11} & \theta A_{12}-i\omega E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\left(\begin{bmatrix} \theta A_{11}-i\omega E_{11} & \theta A_{12}-i\omega E_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\right)^*\right\}.$$

We deduce

$$r_{\mathbb{C}} = \left\{\sigma_{\max}\left\{(H(\theta,\omega))^{-1}\right\}\right\}^{-1}$$

where $\sigma_{\max}\{\cdot\}$ is largest singular value.

• The second case. If the least upper bound is attained at $\infty$, then we set $t = i\frac{1}{\theta}$, $\theta \in \mathbb{R}$, and $|t| \to +\infty$ where $|\theta| \to 0$. Similarary with the previous case, we also have

$$\lim_{|t|\to\infty}\left\|\begin{bmatrix} A_{11}-tE_{11} & A_{12}-tE_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\begin{bmatrix} -I & O & tI & O \\ O & -I & O & O \end{bmatrix}\right\|$$

$$= \lim_{|\theta|\to 0}\left\|\begin{bmatrix} \theta A_{11}-iE_{11} & \theta A_{12}-iE_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\right\| = \left\|\begin{bmatrix} -iE_{11} & -iE_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}\right\| = \sigma_{\max}\left\{(H(0,1))^{-1}\right\}.$$

Hence,

$$r_{\mathbb{C}} = \left\{\sigma_{\max}\left\{(H(0,1))^{-1}\right\}\right\}^{-1}$$

where $\sigma_{\max}\{\cdot\}$ is largest singular value.

From the both cases, we deduce $r_{\mathbb{C}} = \left\{ \sup\limits_{\substack{\theta,\omega\in\mathbb{R} \\ \theta^2+\omega^2=1}} \sigma_{\max}\left\{(H(\theta,\omega))^{-1}\right\} \right\}^{-1}$

i.e

$$r_{\mathbb{C}} = \inf_{\substack{\theta,\omega\in\mathbb{R} \\ \theta^2+\omega^2=1}} \sigma_{\min}\left\{H(\theta,\omega)\right\}$$

where $\sigma_{\min}\{\cdot\}$ is smallest singular value.

**Remark 2.2.** Using the fact that a rank-one destabilizing perturbation can be constructed, an alternative proof for Theorem. 2.5 can be given. The Frobenius norm gives a upper bound for the Euclidean, norm. Note that, for a rank one matrix, the Euclidean, and the Frobenius norms are equal. Then, we can. show that, the formula of the complex stability radius given, in [8] and in Theorem 2.3 are the same.

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