

Buckling analysis of eccentrically stiffened functionally graded circular cylindrical thin shells under mechanical load

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Abstract: An analytical approach is presented to investigate the linear buckling of eccentrically stiffened functionally graded thin circular cylindrical shells subjected to axial compression, external pressure and torsional load. Based on the classical thin shell theory and the smeared stiffeners technique, the governing equations of buckling of eccentrically stiffened functionally graded circular cylindrical shells are derived. The functionally graded cylindrical shells with simply supported edges are reinforced by ring and stringer stiffeners system on internal and (or) external surface. The resulting equations in the case of compressive and pressure loads are solved directly, while in the case of torsional load is solved by the Galerkin procedure to obtain the explicit expression of static critical buckling load. The obtained results show the effects of stiffeners and input factors on the buckling behavior of these structures.

Keywords: Functionally graded material; Cylindrical shells; Stiffeners; Buckling loads; Axial compression; External pressure; Torsional load.

1. Introduction

The static and dynamic behavior of FGM cylindrical shell attracts special attention of a lot of authors in the world.

In static analysis of FGM cylindrical shells, many studies have been focused on the buckling and postbuckling of shells under mechanic and thermal loading. Shen [1] presented the nonlinear postbuckling of perfect and imperfect FGM cylindrical thin shells in thermal environments under lateral pressure by using the classical shell theory with the geometrical nonlinearity in von Karman–Donnell sense. By using higher order shear deformation theory; this author [2] continued to investigate the postbuckling of FGM hybrid cylindrical shells in thermal environments under axial loading. Bahtui and Eslami [3] investigated the coupled thermo-elasticity of FGM cylindrical shells. Huang and Han [4-7] studied the buckling and postbuckling of un-stiffened FGM cylindrical shells under axial

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compression, radial pressure and combined axial compression and radial pressure based on the Donnell shell theory and the nonlinear strain-displacement relations of large deformation. The postbuckling of shear deformable FGM cylindrical shells surrounded by an elastic medium was studied by Shen [8]. Sofiyev [9] analyzed the buckling of FGM circular shells under combined loads and resting on the Pasternak type elastic foundation. Zozulya and Zhang [10] studied the behavior of functionally graded axisymmetric cylindrical shells based on the high order theory.

For dynamic analysis of FGM cylindrical shells, Ng et al. [11] and Darabi et al. [12] presented respectively linear and nonlinear parametric resonance analyses for un-stiffened FGM cylindrical shells. Three-dimensional vibration analysis of fluid-filled orthotropic FGM cylindrical shells was investigated by Chen et al. [13]. Sofiyev and Schnack [14] and Sofiyev [15] obtained critical parameters for un-stiffened cylindrical thin shells under linearly increasing dynamic torsional loading and under a periodic axial impulsive loading by using the Galerkin technique together with Ritz type variation method. Shariyat [16] and [17] investigated the nonlinear dynamic buckling problems of axially and laterally preloaded FGM cylindrical shells under transient thermal shocks and dynamic buckling analysis for un-stiffened FGM cylindrical shells under complex combinations of thermo-electro-mechanical loads. Geometrical imperfection effects were also included in his research. Li et al. [18] studied the free vibration of three-layer circular cylindrical shells with functionally graded middle layer. Huang and Han [19] presented the nonlinear dynamic buckling problems of un-stiffened functionally graded cylindrical shells subjected to time-dependent axial load by using the Budiansky-Roth dynamic buckling criterion [20]. Various effects of the inhomogeneous parameter, loading speed, dimension parameters; environmental temperature rise and initial geometrical imperfection on nonlinear dynamic buckling were discussed. Shariyat [21] analyzed the nonlinear transient stress and wave propagation analyses of the FGM thick cylinders, employing a unified generalized thermo-elasticity theory

Recently, idea of eccentrically stiffened FGM structures has been proposed by Najafizadeh et al. [22] and Bich et al. [23 and 24]. Najafizadeh et al. [22] have studied linear static buckling of FGM axially loaded cylindrical shell reinforced by ring and stringer FGM stiffeners. In order to provide material continuity and easily to manufacture, the FGM shells are reinforced by an eccentrically homogeneous stiffener system; Bich et al. have investigated the nonlinear static postbuckling of functionally graded plates and shallow shells [23] and nonlinear dynamic buckling of functionally graded cylindrical panels [24].

This paper presented an analytical approach to investigated the linear buckling of eccentrically stiffened FGM cylindrical shell subjected to axial compression, external pressure and torsional load. Effects of stiffeners and input factors on the static buckling behavior of these structures are also considered.

2. Governing equations

2.1. Functionally graded material (FGM)

FGMs are microscopically inhomogeneous materials, in which material properties vary smoothly and continuously from one surface of the material to the other surface. These materials are made from

a mixture of ceramic and metal, or a combination of different materials. A such mixture of ceramic and metal with a continuously varying volume fraction can be manufactured. Especially FGM thin – walled structures with ceramic in inner surface and metal in outer surface are widely used in practice. Assume that the modulus of elasticity E changes in the thickness direction z , while the Poisson ratio ν is assumed to be constant. Denote V_m and V_c being volume – fractions of metal and ceramic phases respectively, which are related by $V_m + V_c = 1$ and V_c is expressed as $V_c(z) = \left(\frac{2z+h}{2h}\right)^k$, where h is the thickness of thin-walled structure, k is the volume – fraction exponent ($k \geq 0$). Then the elasticity modulus and the Poisson ratio of functionally graded material can be evaluated as following

$$E(z) = E_m V_m + E_c V_c = E_m + (E_c - E_m) \left(\frac{2z+h}{2h}\right)^k,$$

$$\nu(z) = \nu = \text{const.}$$

The values with subscripts m and c belong to metal and ceramic respectively.

2.2. Eccentrically stiffened functionally graded cylindrical shells.

Consider a cylindrical shell of thickness h , length L , radius R and reinforced by internal and external stiffeners. The shell is referred to a coordinate system (x, y, z) , in which x and y are in the axial and circumferential directions of the shell and z is in the direction of the inward normal to the middle surface.

In the present study, the classical shell theory and the Lekhnitsky smeared stiffeners technique are used to obtain the equilibrium and compatibility equations as well as expressions of buckling loads and nonlinear load – deflection curves of eccentrically stiffened FGM cylindrical shells.

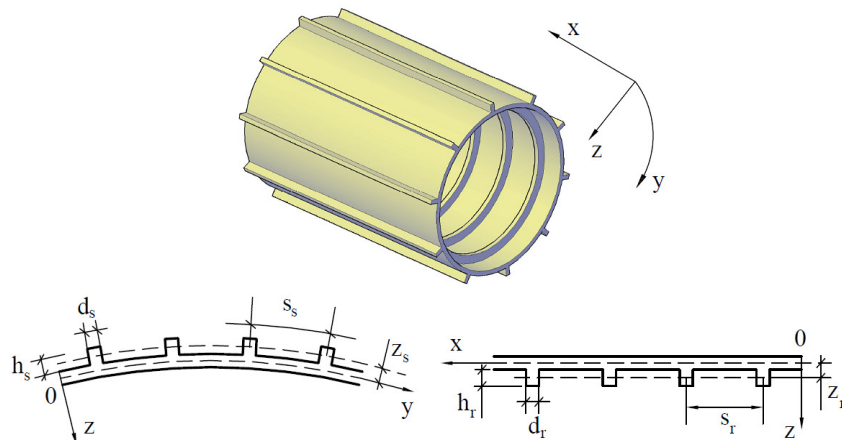


Fig.1. Configuration of an eccentrically stiffened cylindrical shells.

The strains across the shell thickness at a distance z from the mid-surface are

$$\varepsilon_x = \varepsilon_x^0 - z\chi_x, \quad \varepsilon_y = \varepsilon_y^0 - z\chi_y, \quad \gamma_{xy} = \gamma_{xy}^0 - 2z\chi_{xy}, \quad (1)$$

where ε_x^0 and ε_y^0 are normal strains, γ_{xy}^0 is the shear strain at the middle surface of the shell and χ_{ij} are the curvatures.

According to the classical shell theory the strains at the middle surface and curvatures are related to the displacement components u , v , w in the x , y , z coordinate directions as [25].

$$\begin{aligned} \varepsilon_x^0 &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, & \chi_x &= \frac{\partial^2 w}{\partial x^2}, \\ \varepsilon_y^0 &= \frac{\partial v}{\partial y} - \frac{1}{R} w + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, & \chi_y &= \frac{\partial^2 w}{\partial y^2}, \\ \gamma_{xy}^0 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}, & \chi_{xy} &= \frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \quad (2)$$

From Eqs.(2) the strain must satisfy in the deformation compatibility equation

$$\frac{\partial^2 \varepsilon_x^0}{\partial y^2} + \frac{\partial^2 \varepsilon_y^0}{\partial x^2} - \frac{\partial^2 \gamma_{xy}^0}{\partial x \partial y} = -\frac{1}{R} \frac{\partial^2 w}{\partial x^2}. \quad (3)$$

The constitutive stress – strain equations by Hooke law for the shell material are omitted here for brevity. The contribution of stiffeners can be accounted for using the Lekhnitsky smeared stiffeners technique. Then integrating the stress – strain equations and their moments through the thickness of the shell, the expressions for force and moment resultants of an eccentrically stiffened FGM cylindrical shell are obtained.

$$\begin{aligned} N_x &= \left(A_{11} + \frac{EA_s}{s_s} \right) \varepsilon_x^0 + A_{12} \varepsilon_y^0 - (B_{11} + C_s) \chi_x - B_{12} \chi_y, \\ N_y &= A_{12} \varepsilon_x^0 + \left(A_{22} + \frac{EA_r}{s_r} \right) \varepsilon_y^0 - B_{12} \chi_x - (B_{22} + C_r) \chi_y, \end{aligned} \quad (4)$$

$$N_{xy} = A_{66} \gamma_{xy}^0 - 2B_{66} \chi_{xy},$$

$$\begin{aligned} M_x &= (B_{11} + C_s) \varepsilon_x^0 + B_{12} \varepsilon_y^0 - \left(D_{11} + \frac{EI_s}{s_s} \right) \chi_x - D_{12} \chi_y, \\ M_y &= B_{12} \varepsilon_x^0 + (B_{22} + C_r) \varepsilon_y^0 - D_{12} \chi_x - \left(D_{22} + \frac{EI_r}{s_r} \right) \chi_y, \end{aligned} \quad (5)$$

$$M_{xy} = B_{66} \gamma_{xy}^0 - 2D_{66} \chi_{xy},$$

where A_{ij} , B_{ij} , D_{ij} ($i, j = 1, 2, 6$) are extensional, coupling and bending stiffnesses of the shell without stiffeners.

$$\begin{aligned} A_{11} = A_{22} &= \frac{E_1}{1-\nu^2}, & A_{12} &= \frac{E_1\nu}{1-\nu^2}, & A_{66} &= \frac{E_1}{2(1+\nu)}, \\ B_{11} = B_{22} &= \frac{E_2}{1-\nu^2}, & B_{12} &= \frac{E_2\nu}{1-\nu^2}, & B_{66} &= \frac{E_2}{2(1+\nu)}, \\ D_{11} = D_{22} &= \frac{E_3}{1-\nu^2}, & D_{12} &= \frac{E_3\nu}{1-\nu^2}, & D_{66} &= \frac{E_3}{2(1+\nu)}, \end{aligned} \tag{6}$$

with

$$\begin{aligned} E_1 &= \left(E_m + \frac{E_c - E_m}{k+1} \right) h, & E_2 &= \frac{(E_c - E_m)kh^2}{2(k+1)(k+2)}, \\ E_3 &= \left[\frac{E_m}{12} + (E_c - E_m) \left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4k+4} \right) \right] h^3, \end{aligned}$$

and

$$C_s = \pm \frac{EA_s z_s}{s_s}, \quad C_r = \pm \frac{EA_r z_r}{s_r}. \tag{7}$$

In above relations (4), (5) and (7) E is the elasticity modulus of the corresponding stiffener which is assumed identical for both types of stiffeners. The spacings of the longitudinal and transversal stiffeners are denoted by s_1 and s_2 respectively. The quantities A_s , A_r are the cross section areas of the stiffeners and I_s , I_r , z_s , z_r are the second moments of cross section areas and eccentricities of the stiffeners with respect to the middle surface of the shell respectively. The sign plus or minus of C_s , C_r dependent on internal or external stiffeners.

Important remark. In order to provide continuity between the shell and stiffeners, thus stiffeners are made of full metal if putting them at the metal – rich side of the shell and conversely full ceramic stiffeners at the ceramic-rich side of the shell, consequently $E = E_m$ for full metal stiffeners and $E = E_c$ for full ceramic ones.

The nonlinear equilibrium equations of a cylindrical shell based on the classical shell theory are given by

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0, \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0, \\ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{N_y}{R} &= q. \end{aligned} \tag{8}$$

Stability equations of eccentrically stiffened functionally graded shell may be established by the adjacent equilibrium criterion. It is assumed that equilibrium state of the eccentrically stiffened functionally graded shell under applied load is presented by displacement component u_0, v_0, w_0 . The state of adjacent equilibrium differs that of stable equilibrium by u_1, v_1 , and w_1 , and the total displacement component of a neighboring configuration are

$$u = u_0 + u_1, \quad v = v_0 + v_1, \quad w = w_0 + w_1. \quad (9)$$

Similar, the force and moment resultants of a neighboring state are represented by

$$\begin{aligned} N_x &= N_x^0 + N_x^1, & N_y &= N_y^0 + N_y^1, & N_{xy} &= N_{xy}^0 + N_{xy}^1, \\ M_x &= M_x^0 + M_x^1, & M_y &= M_y^0 + M_y^1, & M_{xy} &= M_{xy}^0 + M_{xy}^1, \end{aligned} \quad (10)$$

where terms 0 subscripts correspond to the u_0, v_0, w_0 displacements and those with 1 subscription represents the portions of the increments of force and moment resultants that are linear in u_1, v_1, w_1 . Subsequently, introduction of Eqs. (9) and Eq.(10) into (8) and subtracting from the resulting equations term relating to stable equilibrium state, neglecting nonlinear term in u_1, v_1, w_1 or their counterparts in the form of N_x^1, N_y^1, N_{xy}^1 , etc... and prebuckling rotations yield stability equations.

$$\begin{aligned} \frac{\partial N_x^1}{\partial x} + \frac{\partial N_{xy}^1}{\partial y} &= 0, \\ \frac{\partial N_{xy}^1}{\partial x} + \frac{\partial N_y^1}{\partial y} &= 0, \\ \frac{\partial^2 M_x^1}{\partial x^2} + 2 \frac{\partial^2 M_{xy}^1}{\partial x \partial y} + \frac{\partial^2 M_y^1}{\partial y^2} + N_x^0 \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^0 \frac{\partial^2 w}{\partial x \partial y} + N_y^0 \frac{\partial^2 w}{\partial y^2} + \frac{N_y^1}{R} &= 0. \end{aligned} \quad (11)$$

Considering the first two of Eqs.(11), a stress function may be defined as

$$N_x^1 = \frac{\partial^2 \varphi}{\partial y^2}, \quad N_y^1 = \frac{\partial^2 \varphi}{\partial x^2}, \quad N_{xy}^1 = -\frac{\partial^2 \varphi}{\partial x \partial y}. \quad (12)$$

For using later, the reverse relations are obtained from Eqs.(4)

$$\begin{aligned} \varepsilon_x^0 &= A_{22}^* N_x^1 - A_{12}^* N_y^1 + B_{11}^* \chi_x + B_{12}^* \chi_y, \\ \varepsilon_y^0 &= A_{11}^* N_y^1 - A_{12}^* N_x^1 + B_{21}^* \chi_x + B_{22}^* \chi_y, \\ \gamma_{xy}^0 &= A_{66}^* + 2B_{66}^* \chi_{xy}, \end{aligned} \quad (13)$$

where

$$\begin{aligned}
 A_{11}^* &= \frac{1}{\Delta} \left(A_{11} + \frac{EA_1}{s_1} \right), & A_{22}^* &= \frac{1}{\Delta} \left(A_{22} + \frac{EA_2}{s_2} \right), & A_{12}^* &= \frac{A_{12}}{\Delta}, & A_{66}^* &= \frac{1}{A_{66}}, \\
 \Delta &= \left(A_{11} + \frac{EA_1}{s_1} \right) \left(A_{22} + \frac{EA_2}{s_2} \right) - A_{12}^2; \\
 B_{11}^* &= A_{22}^* (B_{11} + C_1) - A_{12}^* B_{12}, & B_{22}^* &= A_{11}^* (B_{22} + C_2) - A_{12}^* B_{12}, \\
 B_{12}^* &= A_{22}^* B_{12} - A_{12}^* (B_{22} + C_2), & B_{21}^* &= A_{11}^* B_{12} - A_{12}^* (B_{11} + C_1), & B_{66}^* &= \frac{B_{66}}{A_{66}}.
 \end{aligned}$$

Substituting Eqs. (13) into Eqs.(5) yields

$$\begin{aligned}
 M_x^1 &= B_{11}^* N_x^1 + B_{21}^* N_y^1 - D_{11}^* \chi_x - D_{12}^* \chi_y, \\
 M_y^1 &= B_{12}^* N_x^1 + B_{22}^* N_y^1 - D_{21}^* \chi_x - D_{22}^* \chi_y, \\
 M_{xy}^1 &= B_{66}^* N_{xy}^1 - 2D_{66}^* \chi_{xy},
 \end{aligned} \tag{14}$$

where

$$\begin{aligned}
 D_{11}^* &= D_{11} + \frac{EI_1}{s_1} - (B_{11} + C_1) B_{11}^* - B_{12} B_{21}^*, \\
 D_{22}^* &= D_{22} + \frac{EI_2}{s_2} - B_{12} B_{12}^* - (B_{22} + C_2) B_{22}^*, \\
 D_{12}^* &= D_{12} - (B_{11} + C_1) B_{12}^* - B_{12} B_{22}^*, \\
 D_{21}^* &= D_{12} - B_{12} B_{11}^* - (B_{22} + C_2) B_{21}^*, \\
 D_{66}^* &= D_{66} - B_{66} B_{66}^*.
 \end{aligned}$$

The substitution of Eqs.(13) into the compatibility Eqs.(3) and Eqs.(14) into the third of Eqs.(11), taking into account expressions (2) and (12), yields a system of equations

$$\begin{aligned}
 A_{11}^* \frac{\partial^4 \varphi}{\partial x^4} + (A_{66}^* - 2A_{12}^*) \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + A_{22}^* \frac{\partial^4 \varphi}{\partial y^4} + B_{21}^* \frac{\partial^4 w_1}{\partial x^4} + \\
 + (B_{11}^* + B_{22}^* - 2B_{66}^*) \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + B_{12}^* \frac{\partial^4 w_1}{\partial y^4} + \frac{1}{R} \frac{\partial^2 w_1}{\partial x^2} = 0,
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 D_{11}^* \frac{\partial^4 w_1}{\partial x^4} + (D_{12}^* + D_{21}^* + 4D_{66}^*) \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + D_{22}^* \frac{\partial^4 w_1}{\partial y^4} - B_{21}^* \frac{\partial^4 \varphi}{\partial x^4} - \\
 - (B_{11}^* + B_{22}^* - 2B_{66}^*) \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} - B_{12}^* \frac{\partial^4 \varphi}{\partial y^4} - \frac{1}{R} \frac{\partial^2 \varphi}{\partial x^2} - N_x^0 \frac{\partial^2 w_1}{\partial x^2} - 2N_{xy}^0 \frac{\partial^2 w_1}{\partial x \partial y} - N_y^0 \frac{\partial^2 w_1}{\partial y^2} = 0.
 \end{aligned} \tag{16}$$

Eqs.(15) and (16) are the basic equations used to investigate the stability of eccentrically stiffened functionally graded cylindrical shells. They are linear equations in terms of two dependent unknowns w_1 and φ .

2.3. Buckling analysis of eccentrically stiffened functionally graded cylindrical shells subjected to axial compressive load and external pressure.

In the present study, the eccentrically stiffened FGM shell to be free simply supported at all edges and subjected to axial compression load p uniformly distributed on the two end edges of the shell and external pressure q uniform distributed on the surface. By solving the membrane form of equilibrium equations, prebuckling force resultants are determined

$$N_x^0 = -ph, \quad N_y^0 = -qR, \quad N_{xy}^0 = 0. \quad (17)$$

The boundary conditions considered in the current study are

$$w_1 = 0, \quad \frac{\partial^2 w_1}{\partial x^2} = 0, \quad N_x^1 = 0, \quad N_{xy}^1 = 0, \quad \text{at } x = 0; L. \quad (18)$$

where L are the lengths of in-plane edges of the cylindrical shell.

The mentioned conditions (18) can be satisfied if the buckling mode shape is represented by

$$w_1 = \sum_m \sum_n W_{mn} \sin \frac{m\pi x}{L} \sin \frac{ny}{R}, \quad (19)$$

where W_{mn} is a maximum deflection, m is the number of axis half waves and n is the number of circumferential waves. Substituting Eq.(19) into Eq.(15) and solving obtained equation for unknown φ leads to

$$\varphi = \sum_m \sum_n \phi_{mn} \sin \frac{m\pi x}{L} \sin \frac{ny}{R} \quad (20)$$

where

$$\phi_{mn} = - \frac{\left[B_{21}^* m^4 \pi^4 + (B_{11}^* + B_{22}^* - 2B_{66}^*) m^2 n^2 \pi^2 \lambda^2 + B_{12}^* n^4 \lambda^4 - R m^2 \pi^2 \lambda^2 \right]}{A_{11}^* m^4 \pi^4 + (A_{66}^* - 2A_{12}^*) m^2 n^2 \pi^2 \lambda^2 + A_{22}^* n^4 \lambda^4} W_{mn}. \quad (21)$$

Introduction of expressions (19) and (20) into Eqs.(16) leads to

$$\sum_m \sum_n \left[D + \frac{B^2}{A} + (N_x^0 m^2 \pi^2 + N_y^0 n^2 \lambda^2) L^2 \right] W_{mn} \sin \frac{m\pi x}{L} \sin \frac{ny}{R} = 0, \quad (22)$$

where denote

$$\begin{aligned}
 A &= A_{11}^* m^4 \pi^4 + (A_{66}^* - 2A_{12}^*) m^2 n^2 \pi^2 \lambda^2 + A_{22}^* n^4 \lambda^4, \quad \lambda = \frac{L}{R}, \\
 B &= B_{21}^* m^4 \pi^4 + (B_{11}^* + B_{22}^* - 2B_{66}^*) m^2 n^2 \pi^2 \lambda^2 + B_{12}^* n^4 \lambda^4 - R m^2 \pi^2 \lambda^2, \\
 D &= D_{11}^* m^4 \pi^4 + (D_{12}^* + D_{21}^* + 4D_{66}^*) m^2 n^2 \pi^2 \lambda^2 + D_{22}^* n^4 \lambda^4.
 \end{aligned}$$

Eq.(22) satisfies for all x, y if

$$D + \frac{B^2}{A} + (N_x^0 m^2 \pi^2 + N_y^0 n^2 \lambda^2) L^2 = 0. \tag{23}$$

Now investigate the linear buckling of reinforced FGM cylindrical shells in some cases of active load.

Consider the cylindrical shell subjected the axial compression ($q = 0$), Eq. (23) becomes:

$$D + \frac{B^2}{A} - p h m^2 \pi^2 L^2 = 0 \tag{24}$$

Introduction parameters:

$$\bar{D} = \frac{D}{h^3}, \quad \bar{B} = \frac{B}{h}, \quad \bar{A} = A.h, \tag{25}$$

from Eq.(24) the compressive buckling load can be obtained

$$p = \frac{h^2}{m^2 \pi^2 L^2} \left(\bar{D} + \frac{\bar{B}^2}{\bar{A}} \right). \tag{26}$$

The critical axial compression load of eccentrically stiffened FGM cylindrical shell is determined by condition $p_{cr} = \min p$ vs. (m, n) .

Consider the cylindrical shell subjected the external pressure ($p = 0$), the Eq. (23) becomes:

$$D + \frac{B^2}{A} - q R n^2 \lambda^2 L^2 = 0$$

The pressure buckling load can be determined :

$$q = \frac{1}{R n^2 \lambda^2 L^2} \left(D + \frac{B^2}{A} \right) = \frac{1}{\left(\frac{R}{h} \right)^3 n^2 \lambda^4} \left(\bar{D} + \frac{\bar{B}^2}{\bar{A}} \right) \tag{27}$$

The critical external pressure of eccentrically stiffened FGM cylindrical shell are determined by condition $q_{cr} = \min q$ vs. (m, n) .

2.4. Buckling analysis of eccentrically stiffened functionally graded cylindrical shells subjected to torsional load

The eccentrically stiffened FGM shell to be free simply supported at all edges and subjected to torsional load τ . By solving the membrane form of equilibrium equations, prebuckling force resultants are determined

$$N_x^0 = 0, \quad N_y^0 = 0, \quad N_{xy}^0 = \tau h = \frac{M_s}{2\pi R^2}. \quad (28)$$

The buckling mode shape is represented in the form

$$w_1 = W \sin \frac{\pi x}{L} \sin \frac{n(y - \gamma x)}{R}, \quad (29)$$

where W is a maximum deflection. At the edges $x = 0$, $x = L$ the simple supported condition of shell is satisfied. The deflection is vanished along the straight lines $y = \gamma x$ repeated n times at each shell cross-section, where γ is tangent of slope angle between these lines and the shell genetic. Substituting (29) into Eq.(15) and solving obtained equation for unknown ϕ leads to

$$\phi = \phi_1 \sin \frac{\pi x}{L} \sin \frac{n(y - \gamma x)}{R} + \phi_2 \cos \frac{\pi x}{L} \cos \frac{n(y - \gamma x)}{R}, \quad (30)$$

where

$$\begin{aligned} \phi_1 &= \frac{MH - NK}{H^2 - K^2} W, \quad \phi_2 = \frac{MK - NH}{K^2 - H^2} W, \\ K &= A_{11}^* \left\{ \left(\frac{\pi}{L} \right)^4 + 6 \left(\frac{\pi}{L} \right)^2 \left(\frac{n\gamma}{R} \right)^2 + \left(\frac{n\gamma}{R} \right)^4 \right\} + (A_{66}^* - 2A_{12}^*) \left\{ \left(\frac{\pi}{L} \right)^2 + \left(\frac{n\gamma}{R} \right)^2 \right\} \left(\frac{n}{R} \right)^2 + A_{22}^* \left(\frac{n}{R} \right)^4, \\ H &= 4A_{11}^* \left\{ \left(\frac{\pi}{L} \right)^3 \frac{n\gamma}{R} + \left(\frac{\pi}{L} \right) \left(\frac{n\gamma}{R} \right)^3 \right\} + 2(A_{66}^* - 2A_{12}^*) \frac{\pi n\gamma}{L R} \left(\frac{n}{R} \right)^2, \\ M &= - \left[B_{21}^* \left\{ \left(\frac{\pi}{L} \right)^4 + 6 \left(\frac{\pi}{L} \right)^2 \left(\frac{n\gamma}{R} \right)^2 + \left(\frac{n\gamma}{R} \right)^4 \right\} + \right. \\ &\quad \left. + (B_{11}^* + B_{22}^* - 2B_{66}^*) \left\{ \left(\frac{\pi}{L} \right)^2 + \left(\frac{n\gamma}{R} \right)^2 \right\} \left(\frac{n}{R} \right)^2 + B_{12}^* \left(\frac{n}{R} \right)^4 - \frac{1}{R} \left\{ \left(\frac{\pi}{L} \right)^2 + \left(\frac{n\gamma}{R} \right)^2 \right\} \right], \\ N &= - \left[4B_{21}^* \left\{ \left(\frac{\pi}{L} \right)^3 \frac{n\gamma}{R} + \left(\frac{\pi}{L} \right) \left(\frac{n\gamma}{R} \right)^3 \right\} + 2(B_{11}^* + B_{22}^* - 2B_{66}^*) \frac{\pi n\gamma}{L R} \left(\frac{n}{R} \right)^2 - 2 \frac{1}{R} \frac{\pi n\gamma}{L R} \right], \end{aligned}$$

Introduction of expressions (29) and (30) into Eqs.(16) leads to

$$\begin{aligned} & \left[D_1 W + M\phi_1 + N\phi_2 - 2N_{xy}^0 \frac{n\gamma}{R} \frac{n}{R} W \right] \sin \frac{\pi x}{L} \sin \frac{n(y-\gamma x)}{R} + \\ & + \left[D_2 W + N\phi_1 + M\phi_2 - 2N_{xy}^0 \frac{n\gamma}{R} \frac{n}{R} W \right] \cos \frac{\pi x}{L} \cos \frac{n(y-\gamma x)}{R} = 0. \end{aligned} \tag{31}$$

where

$$\begin{aligned} D_1 &= D_{11}^* \left\{ \left(\frac{\pi}{L} \right)^4 + 6 \left(\frac{\pi}{L} \right)^2 \left(\frac{n\gamma}{R} \right)^2 + \left(\frac{n\gamma}{R} \right)^4 \right\} + \\ & + (D_{12}^* + D_{21}^* + 4D_{66}^*) \left\{ \left(\frac{\pi}{L} \right)^2 + \left(\frac{n\gamma}{R} \right)^2 \right\} \left(\frac{n}{R} \right)^2 + D_{22}^* \left(\frac{n}{R} \right)^4, \\ D_2 &= 4D_{11}^* \left\{ \left(\frac{\pi}{L} \right)^3 \frac{n\gamma}{R} + \left(\frac{\pi}{L} \right) \left(\frac{n\gamma}{R} \right)^3 \right\} + 2(D_{12}^* + D_{21}^* + 4D_{66}^*) \frac{\pi}{L} \frac{n\gamma}{R} \left(\frac{n}{R} \right)^2. \end{aligned}$$

Application of Garlerkin method for the Eq.(31) yields

$$\left[U.P + V.Q - 2 \left\{ \frac{n^2\gamma}{R^2} P + \frac{\pi}{L} \frac{n}{R} Q \right\} N_{xy}^0 \right] W = 0, \tag{32}$$

where

$$U = D_1 W + M\phi_1 + N\phi_2,$$

$$V = D_2 W + N\phi_1 + M\phi_2,$$

$$P = 2\pi L - \frac{1}{4} \left\{ \frac{R^2 \gamma L^2}{\pi^2 R^2 - n^2 \gamma^2 L^2} + \frac{R^2}{n^2 \gamma} \right\} \left\{ \sin \frac{4n\pi}{R} \sin \frac{2n\gamma L}{R} + 4 \sin^2 \frac{2n\pi}{R} \sin^2 \frac{n\gamma L}{R} \right\},$$

$$Q = \frac{R^3 \pi L}{4n(\pi^2 R^2 - n^2 \gamma^2 L^2)} \left\{ \sin \frac{4n\pi}{R} \sin \frac{2n\gamma L}{R} + 4 \sin^2 \frac{2n\pi}{R} \sin^2 \frac{n\gamma L}{R} \right\}.$$

By substitution $N_{xy}^0 = \tau h$ into Eq.(32), the buckling torsional load is obtained as

$$\tau = \frac{U.P + V.Q}{2h \left\{ \frac{n^2\gamma}{R^2} P + \frac{\pi}{L} \frac{n}{R} Q \right\}}, \quad M_s = 2\pi R^2 h \tau. \tag{33}$$

The critical torsion load of eccentrically stiffened FGM cylindrical shell are determined by condition $\tau_{cr} = \min \tau$ vs. (n, γ) .

3. Numerical examples

To validate the present formulation in buckling of stiffened FGM cylindrical shells under mechanical loads, the linear response of un-stiffened and stiffened FGM cylindrical shell under

mechanical load are analyzed. The results shown in the Table 1- 4. As can be seen, the very good agreements are obtained.

Table 1. Comparison of the present critical buckling load p_{cr} (MPa) with theoretical results reported by Huang and Han [19] ($T_0 = 300^0 K$, $L/R = 2$)

	Huang and Han ($\sigma_{scr} = \sigma_{der}/\tau_{cr}$)	Present	Difference (%)
Critical load versus k			
$R/h = 500$			
k=0.2	189.262 (2, 11)	189.324 (2, 11)	0.033
k= 1.0	164.352 (2, 11)	164.386 (2, 11)	0.021
k= 5.0	144.471 (2, 11)	144.504 (2, 11)	0.023
Critical load versus R/h			
k=0.2			
$R/h = 400$	236.578 (5, 15)	236.464 (5, 15)	-0.048
$R/h = 600$	157.984 (3, 14)	158.022 (3, 14)	0.024
$R/h = 800$	118.849 (2, 12)	118.898 (2, 12)	0.041

Table 2. Comparisons of critical buckling load of internal stiffened isotropic cylindrical shells under external pressure (Psi)

	Barush and Singer [27]	Shen [28]	Present
Un-stiffened	102	100.7 (1, 4)	103.327 (1, 4)
Stringer stiffened	103	102.2 (1, 4)	104.494 (1, 4)
Ring stiffened	370	368.3 (1, 3)	379.694 (1, 3)
Orthogonal stiffened	377	374.1 (1, 3)	387.192 (1, 3)

Table 3. Comparisons of critical torsion load τ_{cr} (psi) of un-stiffened isotropic cylindrical shell ($E = 29 \times 10^6$ Psi, $L = 19,85$ in, $R = 3$ in, $h = 0,0075$ in, $\nu = 0,3$)

Eksrom [30]	Theory	Shen [29]	Present
Experiment			
4800	5500	4997 (1, 3)	4831.57 (7, 0.14)

Table 4: Comparisons of critical buckling load per unit length $\bar{p}_{cr} = p_{cr} \cdot h$ ($10^6 N/m$) of stiffened homogeneous cylindrical shell under axial compression

	Present	Brush and Almoth [25]	Difference (%)
50 rings, 50 stringers, $L=1m$, $R=0.5m$, $E = 70 \times 10^9 (N/m^2)$, $\nu = 0.3$, $d_r = d_s = 0.0025m$, $h_r = h_s = 0.01m$,			
Internal stiffeners			
$R/h = 100$	3.0725 (6, 7)	3.0906 (6, 7)	0.59
$R/h = 200$	1.4147 (6, 7)	1.4328 (6, 7)	1.28
$R/h = 500$	0.6924 (5, 6)	0.7057 (5, 6)	1.92

External stiffeners			
$R/h = 100$	3.9529 (9,3)	3.9551 (9, 2)	0.06
$R/h = 200$	2.1410 (9, 4)	2.1369 (9,4)	0.28
$R/h = 500$	1.2764 (6, 6)	1.2897 (6, 6)	1.04

To illustrate the proposed approach of eccentrically stiffened FGM cylindrical shells, the stiffened and un-stiffened FGM cylindrical shells are made by the combination of materials consists of Aluminum $E_m = 7 \times 10^{10} \text{ N/m}^2$ and Alumina $E_c = 38 \times 10^{10} \text{ N/m}^2$. The Poisson's ratio ν is chosen to be 0.3 for simplicity. The height of stiffeners is equal to 0.005 m, its width 0.002 m. The material properties are $E_s = E_c$ and $E_r = E_c$ with internal stringer stiffeners and internal ring stiffeners; $E_s = E_m$, $E_r = E_m$ with external stringer stiffeners and external ring stiffeners, respectively. The stiffener system includes 10 ring stiffeners and 10 stringer stiffeners distributed regularly in the axial and circumferential directions, respectively.

Table 5: Critical buckling load of stiffened FGM cylindrical shell under axial and pressure load

($L/R=2, h=0.002m, d_r=d_s=0.002m, h_r=h_s=0.005m, n_r=n_s=10$).

R/h^k	$p_{cr} \text{ (GPa)}$			$q_{cr} \text{ (MPa)}$		
	Un-stiffened	External stiffeners	Internal stiffeners	Un-stiffened	External stiffeners	Internal stiffeners
100						
0.2	1.936 (7, 9)	2.245 (10, 5)	2.740 (6, 7)	1.548 (1, 6)	2.658 (1, 6)	5.848 (1, 5)
1	1.249 (8, 9)	1.584 (10, 5)	1.961 (6, 7)	0.970 (1, 6)	2.064 (1, 5)	4.729 (1, 5)
5	0.746 (6, 9)	1.051 (9, 5)	1.280 (5, 6)	0.610 (1, 6)	1.561 (1, 5)	3.623 (1, 4)
10	0.640 (11, 2)	0.921 (9, 4)	1.120 (5, 6)	0.541 (1, 6)	1.420 (1, 5)	3.293 (1, 4)
200						
0.2	0.968 (8, 13)	1.047 (13, 10)	1.197 (10,11)	0.270 (1, 7)	0.364 (1, 7)	0.712 (1, 6)
1	0.625 (17, 2)	0.712 (14, 9)	0.837 (10,11)	0.170 (1, 7)	0.272 (1, 7)	0.559 (1, 6)
5	0.373 (4, 11)	0.454 (14, 8)	0.537 (9,10)	0.106 (1, 7)	0.203 (1, 6)	0.438 (1, 6)
10	0.320 (6, 12)	0.394 (13, 7)	0.471 (8, 9)	0.093 (1, 7)	0.182 (1, 6)	0.420 (1, 6)
300						
0.2	0.645 (15,14)	0.681 (17, 11)	0.753 (13,13)	0.097 (1, 8)	0.121 (1, 8)	0.211 (1, 7)
1	0.416 (16,14)	0.456 (17, 12)	0.517 (13,13)	0.060 (1, 8)	0.087 (1, 8)	0.164 (1, 7)
5	0.249 (17,11)	0.285 (16,11)	0.329 (11,12)	0.038 (1, 8)	0.062 (1, 7)	0.128 (1, 7)
10	0.213 (19, 4)	0.247 (16, 9)	0.287 (11,12)	0.034 (1, 8)	0.056 (1, 7)	0.121 (1, 6)

^aThe numbers in brackets indicate the buckling mode (m, n).

Table 6: Critical buckling load τ_{cr} (GPa) of stiffened FGM cylindrical shell under torsion load
 ($L/R=2$, $h=0.002m$, $d_r=d_s=0.002m$, $h_r=h_s=0.005m$, $n_r=n_s=10$)

R/h	k	Un-stiffened	External stiffeners	Internal stiffeners
100	0.2	0.548 (8, 0.367) ^b	0.784 (8, 0.646)	1.128 (7, 0.925)
	1	0.348 (8, 0.349)	0.577 (8, 0.873)	0.825 (6, 1.047)
	5	0.213 (8, 0.384)	0.407 (7, 0.873)	0.566 (6, 0.925)
	10	0.186 (8, 0.401)	0.363 (7, 0.873)	0.516 (6, 0.908)
150	0.2	0.329 (9, 0.332)	0.434 (9, 0.436)	0.599 (8, 0.995)
	1	0.209 (9, 0.314)	0.317 (9, 0.960)	0.436 (8, 0.995)
	5	0.128 (9, 0.332)	0.216 (8, 1.117)	0.299 (7, 1.012)
	10	0.112 (9, 0.349)	0.191 (8, 1.065)	0.269 (7, 0.960)
200	0.2	0.229 (10, 0.314)	0.288 (10, 0.384)	0.392 (9, 1.030)
	1	0.146 (10, 0.297)	0.208 (10, 0.436)	0.280 (9, 1.030)
	5	0.089 (10, 0.332)	0.141 (10, 0.873)	0.192 (8, 1.065)
	10	0.078 (10, 0.349)	0.125 (10, 0.855)	0.172 (8, 0.995)

^bThe numbers in brackets indicate the buckling mode (n, γ) .

Critical buckling load of FGM cylindrical shell under axial compression, external pressure and torsion load are considered in table 5 and 6. The results show that the critical buckling load of stiffened shells is larger than one of un-stiffened shells. Table 5 and 6 also show effects of R/h ratio and k index to the critical buckling load of shells. Clearly, the critical buckling load of shell increases when R/h ratio or k index decreases. It seems that, effect of stiffeners on the external pressure case is the greatest than one of axial compression. Effects of stiffeners increase when R/h ratio or k index increases.

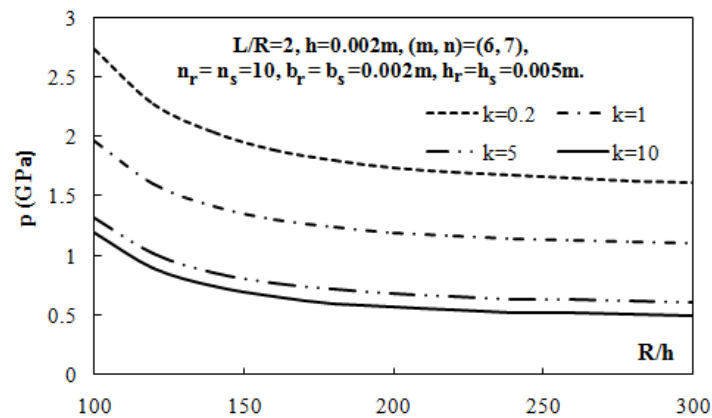


Fig.2. Effect of ratio R/h on the buckling load of internal stiffened FGM cylindrical shells under axial compression.

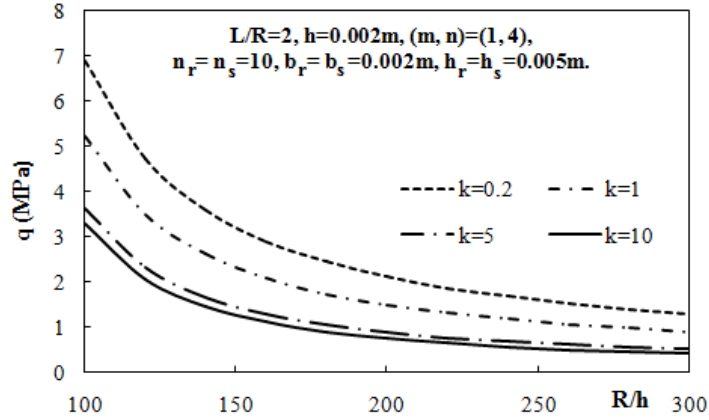


Fig.3. Effect of ratio R/h on the buckling load of internal stiffened FGM cylindrical shells under external pressure.

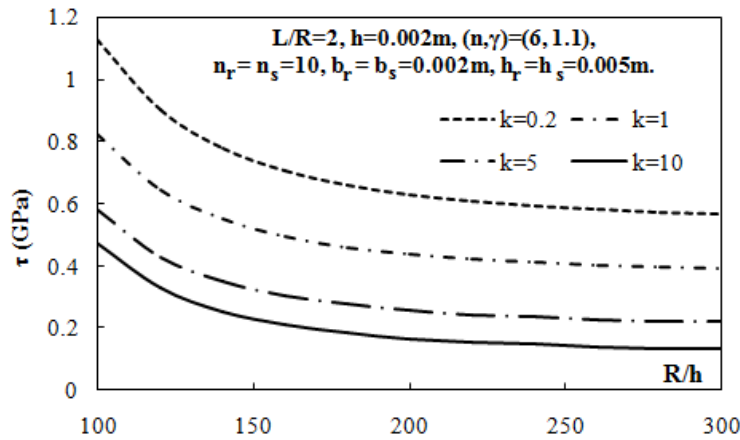


Fig.4. Effect of ratio R/h on the buckling load of internal stiffened FGM cylindrical shells under torsional load.

Effects of ratio R/h on the buckling load of internal stiffened FGM cylindrical shells under axial compression, external pressure and torsion load are investigated in Figs. 2-4, respectively. The obtained results show that for various values of k index, decreasing tendency of axial and torsion buckling loads versus R/h ratio is quite similar (Figs. 2 and 4). Conversely, the unsimilar tendency is obtained for external pressure case. A considerable difference between buckling loads curve as R/h is small and this difference becomes small when R/h ratio to be larger.

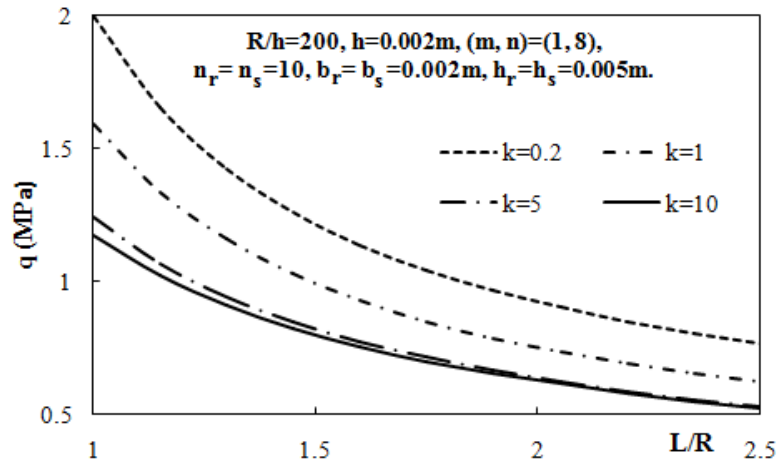


Fig.5. Effect of ratio L/R on the buckling load of internal stiffened FGM cylindrical shells under external pressure.

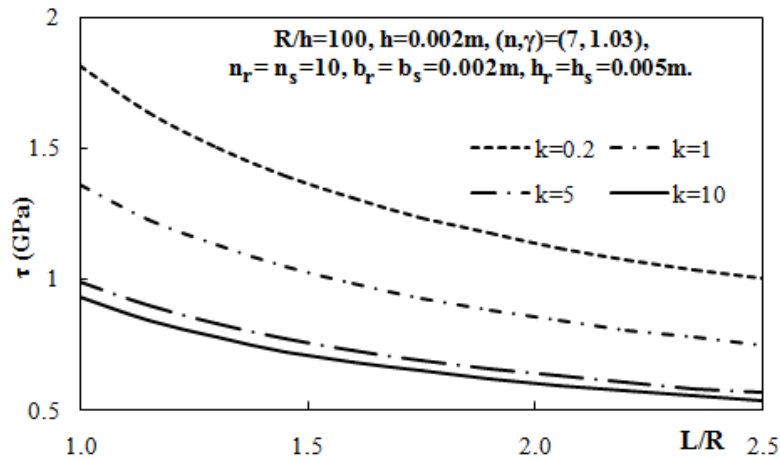


Fig.6. Effect of ratio L/R on the buckling load of internal stiffened FGM cylindrical shells under torsional load.

Finally, the variation of external pressure buckling and torsion buckling versus L/R ratio is separately illustrated in Figs. 5 and 6. As can be observed, there is a large difference between buckling loads curves as L/R is small. In contrast, this difference becomes larger when L/R ratio to be larger.

5. Conclusion

A formulation of governing equations of eccentrically stiffened functionally graded circular cylindrical thin shells subjected to axial compression, external pressure and torsion load based upon the classical shell theory and the smeared stiffeners technique is presented in this paper. By using the Galerkin method the explicit expressions of buckling torsion load. The obtained results show that stiffeners enhance the static stability and load-carrying capacity of FGM circular cylindrical shells. Effects of R/h ratio, L/R ratio and k index to the buckling curve and critical buckling load of shells were considered.

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