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Concerning semi-quotient mappings

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Abstract. In 1989, N.V. Velichko [1] introduced a semi-quotient *ws*-mapping, and proved that a sequential space has a point-countable *k*-network if and only if it is a semi-quotient *ws*-image of a metric space. Recently, Shou Lin and Jinjin Li [2] introduced and studied the concept of *wks*-mappings, *wcs*-mappings, and proved that every sequential space with a point-countable *k*-network is preserved by a continuous closed mapping. In this article, we introduce a class of mappings named *wscc-mappings* and give some properties of *semi-quotient wscc-mappings*. Moreover, we also give a result stating that every sequential space with a point-countable *k*-network is preserved by a continuous closed compact mapping.

Keywords: semi-quotient *ws*-mappings; *wks*-mappings; *wcs*-mappings; *wscc*-mappings; semi-quotient *wscc*-mappings

1. Introduction

A study of images of topological spaces under certain semi-quotient mappings is an important question in general topology. In 2009, to characterize spaces with a point-countable *k*-network as images of metric spaces under "nice" mappings, Shou Lin and Jinjin Li introduced concepts of *wk*-mappings, *wc*-mappings, *wks*-mappings, *wcs*-mappings in order to modify semi-quotient mappings. In this article, we introduce a class of mappings named *wscc-mappings* and give some properties of *semi-quotient wscc-mappings.*

Throughout this article, all spaces are assumed to be *Hausdorff*, all mappings are assumed onto. For terms are not defined here, please refer to [3].

Definition 1.1 [1]. Suppose that a mapping $f: X \longrightarrow Y$, and X_0 is a subspace of X. the mapping f is called continuous about X_0 if for each $x \in X$ and any neighborhood V of $f(x)$ in Y there is a neighborhood W of x in X such that $f(W \cap X_0) \subset V$.

Denote $f_0 = f|_{X_0}: X_0 \longrightarrow Y$.

Lemma 1.2 [2]. Suppose that a mapping $f: X \longrightarrow Y$, and X_0 is a subspace of X. The following are equivalent:

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(1) f is continuous about X_0.

If a net $\{x_d\}_{d\in D}$ in X_0 converges to a point x in X, then a net $\{f(x_d)\}_{d\in D}$ converges to $f(x)$ in Y.

(2) If T is a subset of Y, then $\overline{f_0^{-1}(T)} \subset f^{-1}(\overline{T})$.

Remark 1.3 [2]. By Lemma 1.2, the restriction $f|_{\overline{X_0}} : \overline{X_0} \longrightarrow Y$ is continuous $\Longrightarrow f$ is continuous about $X_0 \Longrightarrow$ the striction $f_0 = f|_{X_0}: X_0 \longrightarrow Y$ is continuous.

Definition 1.4 [1]. A mapping $f : (X, X_0) \longrightarrow Y$ is called a *semi-quotient ws-mapping* if $X_0 \subset X$ and the following are satisfied:

(1) The restriction $f_0 = f|_{X_0}: X_0 \longrightarrow Y$ is an *s-mapping*, i. e., $f_0^{-1}(y)$ is a separable subspace of X_0 for each $y \in Y$.

(2) f is continuous about X_0.

(3) A subset T of Y is closed if and only if $\overline{f_0^{-1}(T)} \subset f^{-1}(T)$.

Definition 1.5 [2]

(1) $f: (X, X_0) \longrightarrow Y$ is called a *ws-mapping* if it satisfies the conditions (1) and (2) in Definition 1.4.

(2) $f : (X, X_0) \longrightarrow Y$ is called a *semi-quotient mapping* if it satisfies the condition (3) in Definition 1.4.

Definition 1.6 [2]. Suppose that a mapping $f : X \longrightarrow Y$ is continuous about X_0.

(1) $f: (X, X_0) \longrightarrow Y$ is called a *wk-mapping* if K is a compact subset of Y and T is a sequence in K, there is a sequence S in X_0 such that S has an accumulation in X and $f(S)$ is a subsequence of T.

(2) $f: (X, X_0) \longrightarrow Y$ is called a *wc-mapping* if T is a convergent sequence in Y, there is a sequence S in X_0 such that S has an accumulation in X and $f(S)$ is a subsequence of T.

(3) $f: (X, X_0) \longrightarrow Y$ is called a *wks-mapping* (*wcs-mapping*) if it is a *wk*-mapping (*wc*-mapping) and a *ws*-mapping.

Definition 1.7 [2],[4]. Suppose that $f : X \longrightarrow Y$ is a continuous mapping.

(1) f is called a *compact-covering mapping* if K is a compact subset of Y, there is a compact subset L of X with $f(L) = K$.

(2) f is called a *sequence-covering mapping* if T is a convergent sequence including the limit point in Y, there is a compact subset L in X with $f(L) = T$.

Definition 1.8. Suppose that a mapping $f : X \longrightarrow Y$ is continuous about X_0. Then, $f: (X, X_0) \longrightarrow Y$ is called a *wscc-mapping* if it is a compact-covering mapping and a *ws*-mapping.

Remark 1.9. The following statements hold.

(1) Compact-covering mappings $\Longrightarrow$ sequence-covering mappings [2].

(2) Compact-covering mappings $\Longrightarrow$ *wk*-mappings $\Longrightarrow$ *wc*-mappings [2].

(3) *wscc*-mappings $\Longrightarrow$ *wks*-mappings $\Longrightarrow$ *wcs*-mappings.

(4) *wscc*-mappings $\Longrightarrow$ sequence-covering *ws*-mappings.

Definition 1.10 [5]. A mapping $f : X \longrightarrow Y$ is called *weakly continuous* if $f^{-1}(V) \subset [f^{-1}(\overline{V})]^{\circ}$ for each open set V in Y. $f : X \longrightarrow Y$ is weakly continuous if and only if for each $x \in X$ and any neighborhood V of $f(x)$ in Y, there is a neighborhood W of x in X with $f(W) \subset \overline{V}$.

2. Main results

Theorem 2.1. Every continuous closed compact mapping is a semi-quotient wscc-mapping.

Proof. Suppose that $f: X \longrightarrow Y$ is a continuous closed compact mapping. For a compact subset K of a space Y, and we put $L = f^{-1}(K)$. Since f is closed compact mapping, L is a compact subset of X. This implies that there is a compact subset L of X with $f(L) = f(f^{-1}(K)) = K$. Therefore, f is a compact-covering mapping. On the other hand, for each $y \in Y$ take an $x_y \in f^{-1}(y)$, and put $X_0 = \{x_y : y \in Y\}$. It is obvious that, $f : (X, X_0) \longrightarrow Y$ is continuous about X_0 and is a *ws*-mapping. If T is a subset of Y, and $f_0{}^{-1}(T) \subset f^{-1}(T)$, then $\overline{T} = f(X_0 \cap f^{-1}(T)) = f(f_0{}^{-1}(T)) \subset T$, thus T is closed in Y. Therefore f is a semi-quotient mapping. This implies that f is a semi-quotient *wscc*-mapping. The proof is complete.

Remark 2.2. From Theorem 2.1 and Remark 1.9, the following holds.

(1) Continuous closed compact mappings $\Longrightarrow$ semi-quotient *wscc*-mappings $\Longrightarrow$ semi-quotient *wks*-mappings $\Longrightarrow$ semi-quotient *wcs*-mappings.

(2) Semi-quotient *wscc*-mappings $\Longrightarrow$ semi-quotient sequence-covering *ws*-mappings.

Theorem 2.3. Let f: X $\longrightarrow$ Y be a continuous closed compact mapping, and X be a sequential space, M_0 be a subspace of a metric space M. If g: (M, M_0) $\longrightarrow$ X is a weakly continuous semi-quotient wscc-mapping. Then the composition h = f_og is a semi-quotient wscc-mapping.

First, let us prove a lemma.

Lemma 2.4. If f : X $\longrightarrow$ Y and g : Y $\longrightarrow$ Z are compact-covering mappings, then h = g_o f is also a compact-covering mapping.

Proof. Suppose that K is a compact subset of Z, because g is compact-covering mapping, there exists a compact subset L_1 of Y such that $g(L_1) = K$. On the other hand, since L_1 is a compact subset in Y and f is compact-covering mapping, there is a compact subset L in X with $f(L) = L_1$. Therefore, there exists a compact subset L in X such that $h(L) = (g_o f)(L) = g(f(L)) = g(L_1) = K$. This shows that h is a compact-covering mapping.

Now, we give a proof of Theorem 2.3.

Firstly, for each $y \in Y$, take an $x_y \in f^{-1}(y)$, and put $M_1 = g_0{}^{-1}(\{x_y : y \in Y\})$, $h = f_o g: M \longrightarrow Y$ and $h_1 = h|_{M_1}$. Since $M_1 \subset M_0$, $h : (M, M_1) \longrightarrow Y$ is a *ws*-mapping and continuous about M_1. Now, we show that h is a semi-quotient mapping. Suppose that T is a non-closed subset of Y, thus there is a sequence $\{y_n\}$ in T such that the sequence $\{y_n\}$ converges to $y \notin T$ in Y. For each $n \in \mathbb{N}$, put $x_n = x_{y_n}$, and let $\mathcal{X} = \{x_n : n \in \mathbb{N}\}$. Since f is closed, $\mathcal{X}$ is not closed in X, and since X is a sequential space, the sequence $\{x_n\}$ has a convergent subsequence. We can assume that the sequence $\{x_n\}$ converges to a point x in X, then $f(x) = y$. Because g is semi-quotient and $\mathcal{X}$ is not closed in X, there exists a $m \in g_0{}^{-1}(\overline{\mathcal{X}}) \setminus g^{-1}(\mathcal{X})$. We shall prove that $g(m) = x$. Indeed, if $g(m) \neq x$, then there is a neighborhood V of $g(m)$ in X such that $\overline{V} \cap \overline{\mathcal{X}} = \emptyset$. Since g is weakly continuous, there exists a neighborhood W of m in M such that $g(W) \subset \overline{V}$, thus $W \cap g^{-1}(\mathcal{X}) = \emptyset$, and it implies that $m \notin g_0{}^{-1}(\overline{\mathcal{X}})$. This contradicts to $m \in g_0{}^{-1}(\overline{\mathcal{X}}) \setminus g^{-1}(\mathcal{X})$. Therefore, $g(m) = x$ and $h(m) = y \notin T$. For each open neighborhood U of m in M, $U \cap h_1{}^{-1}(T) \supset U \cap g^{-1}(\mathcal{X}) \cap M_1 = U \cap M_0 \cap g^{-1}(\mathcal{X}) \neq \emptyset$, thus $m \in h_1{}^{-1}(T) \setminus h^{-1}(T)$, hence $h_1{}^{-1}(T) \not\subset h^{-1}(T)$. It implies that h is a semi-quotient mapping. On the other hand, in view of the proof of Theorem 2.1, f is a compact-covering mapping. Finally, because g

is also a compact-covering mapping, by Lemma 2.4, h is a compact-covering mapping. Therefore, h is a semi-quotient *wscc*-mapping and so completes the proof.

Remark 2.5. In Theorem 2.3, if sequential space X is a sequential space with a point-countable k-network, then, because the closed mappings are quotient mappings and sequential spaces are preserved by quotient mappings [6], Y is a sequential space. By Corollary 16 in [2] and Theorem 2.3, we have the following corollary.

Corollary 2.6. Every sequential space with a point-countable k-network is preserved by a continuous closed compact mapping.

3. Conclusion

In this article, a class of mappings, called *wscc*-mappings is introduced. Besides, some theorems are obtained, which improve some results of Shou Lin and Jinjin Li.

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