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Stability of S-FGM Annular Spherical Shells with Ceramic – Metal – Ceramic Layers under External Pressure

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Abstract: This paper presents an analytical approach to investigate the nonlinear stability of thin annular spherical shells made of functionally graded materials (FGM) with ceramic – metal – ceramic layers (S-FGM) under uniform external pressure and resting on elastic foundations. Material properties are graded in the thickness direction according to a Sigmoi power law distribution in terms of the volume fractions of constituents (S-FGM). Equilibrium and compatibility equations for annular spherical shells are derived by using the classical thin shell theory in terms of the shell deflection and the stress function. Approximate analytical solutions are assumed to satisfy simply supported boundary condition and Galerkin method is applied to obtain closed – form of load – deflection paths. An analysis is carried out to show the effects of material and geometrical properties and combination of loads on the stability of S-FGM annular spherical shells.

Keywords: Nonlinear stability, S-FGM annular spherical shells, elastic foundations, external pressure.

1. Introduction

The problems relating to the thermo-elastic, dynamic, buckling and post-buckling analyses of structures made of FGMs have attracted attention of many researchers. This is mainly due to the increasing use of FGM as the components of structures in the advanced engineering. FGM consisting of metal and ceramic constituents have received remarkable attention in structural applications. Smooth and continuous change in material properties enable FGM to avoid interface problems and unexpected thermal stress concentrations. By high performance heat resistance capacity, FGM is now chosen to use as structural components exposed to severe temperature conditions such as aircraft, aerospace structures, nuclear plants and other engineering applications. Furthermore, with the development of aesthetics, architectures and designs are becoming diversified and abundant. In response to these factors, the structure has special shape also increasingly more popular in life, thus, it requires study of shape and material of structures to be cared.

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There has been recently a few of publications on the structures made of FGMs with ceramic-metal-ceramic layer or metal-ceramic-metal layer. For examples, Duc et al. [1] studied the nonlinear buckling of higher deformable S-FGM thick circular cylindrical shells with metal-ceramic-metal layers surrounded on elastic foundations in thermal environment, Duc and Thang analyzed the nonlinear response of imperfect eccentrically stiffened ceramic-metal-ceramic S-FGM thin circular cylindrical shells surrounded on elastic foundations under uniform radial load [2] and the nonlinear buckling of imperfect eccentrically stiffened metal-ceramic-metal S-FGM thin circular cylindrical shells with temperature-dependent properties in thermal environments [3] with material properties are graded in the thickness direction according to a Sigmoid power law distribution in terms of the volume fractions of constituents (S-FGM), in addition to several other studies [4, 5] of these authors' group.

However, until now, the number of these researches is still very limited, because according to the classical distribution of materials FGM, considerable researches have focused only on the type of distribution, where one surface is rich metal and one is rich ceramic, and very few studies on other types of distributions. Eslami and Kiani [6] studied an exact solution for thermal buckling of annular FGM plates on an elastic medium, Eslami and Bagri generalized coupled thermo-elasticity of functionally graded annular disk considering the Lord – Shulman theory [7], Duc et al. [8] studied the nonlinear buckling analysis of thin FGM annular spherical shells on elastic foundations under external pressure and thermal loads, and [9] analyzed the nonlinear post-buckling of thin FGM annular spherical shells with metal – ceramic layer under mechanical loads and resting on elastic foundations.

The present study investigates is the nonlinear stability of thin S-FGM annular spherical shells with ceramic-metal-ceramic layers on elastic foundations under external pressure.

2. Theoretical formulations

We consider an annular spherical shell made of FGM, resting on elastic foundations with radius of curvature *R*, base radii r_1, r_0 and thickness h. The FGM annular spherical shell is subjected to external pressure q uniformly distributed on the outer surface as shown in Fig.1 [9]. It is defined in coordinate system (φ, θ, z), where φ and θ are in the meridional and circumferential direction of the shells, respectively and z is perpendicular to the middle surface positive inwards.

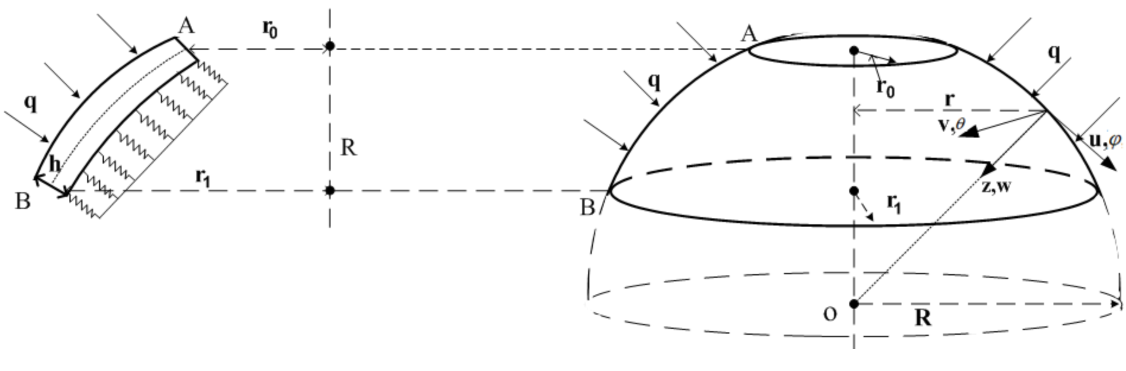

Fig. 1. Configuration of S-FGM annular spherical shell.

Asume that the shell is made from a mixture of ceramic and metal constituents, with ceramic-metal-ceramic layer. Not the same as the P- FGM, where one surface is rich metal and one is rich ceramic, in this study, suppose that the material composition of the shell varies smoothly along the thickness by a Sigmoi power law in terms of the volume fractions of the constituents as

$$V_m(z)=\begin{cases}\left(\dfrac{2z+h}{2h}\right)^k, -\dfrac{h}{2}\le z\le 0,\\ \left(\dfrac{-2z+h}{2h}\right)^k, 0\le z\le \dfrac{h}{2},\end{cases}\qquad (1)$$
$$V_c(\mathrm{z})=1-V_m(z).$$

in here, k (volume fraction index) is a non-negative number that defines the material distribution, subscripts m and c represent the metal and ceramic constituents, respectively. So the effective properties of S-FGM annular spherical shell such as modulus of elasticity of FGM annular spherical shell can be defined as

$$\left(E(z),\alpha(z)\right)=\begin{cases}\left(E_m,\alpha_m\right)+\left(E_{mc},\alpha_{mc}\right)\left(\dfrac{2z+h}{2h}\right)^k, -\dfrac{h}{2}\le z\le 0,\\ \left(E_m,\alpha_m\right)+\left(E_{mc},\alpha_{mc}\right)\left(\dfrac{-2z+h}{2h}\right)^k, 0\le z\le \dfrac{h}{2}.\end{cases}\qquad (2)$$

The Poisson ratio ν is assumed to be constant $\nu(z)=const$ and $E_{mc}=E_m-E_c, \alpha_{mc}=\alpha_m-\alpha_c$.

The reaction-deflection relation of Pasternak foundation is given by [9, 10] $q_e=k_1w-k_2\Delta w$ where $\Delta w=\dfrac{\partial^2 w}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial w}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}$ is a Laplace's operator, w is the deflection of the annular spherical shell, k_1 is Winkler foundation modulus and k_2 is the shear layer foundation stiffness of Pasternak model.

In the present study, the classical shell theory is used to obtain the equilibrium and compatibility equations as well as expressions of buckling loads and nonlinear load–deflection curves of thin S-FGM annular spherical shells. For a thin annular spherical shell it is convenient to introduce a variable r, referred as the radius of parallel circle with the base of shell and defined by $r=R\sin\varphi$. Moreover, due to shallowness of the shell it is approximately assumed that $\cos\varphi=1, Rd\varphi=dr$.

Taking into account Von Karman – Donnell nonlinear terms as [9] and under the classical shell theory, the strains at the middle surface and the change of curvatures and twist are related to the displacement components u,v,w in the $\varphi,\theta,\mathrm{z}$ coordinate directions, respectively

$$\varepsilon_r^0=\frac{\partial u}{\partial r}-\frac{w}{R}+\frac{1}{2}(\frac{\partial w}{\partial r})^2,\ \varepsilon_\theta^0=\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{R}-\frac{w}{R}+\frac{1}{2r^2}(\frac{\partial w}{\partial\theta})^2,\ \gamma_r^\theta=\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}-\frac{v}{r}+\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial\theta},$$
$$\chi_r=\frac{\partial^2 w}{\partial r^2},\ \chi_\theta=\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2},\ \chi_{r\theta}=\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial w}{\partial\theta}.\qquad (3)$$

with ε_r^0 and ε_θ^0 are the normal strains, γ_r^θ is the shear strain at the middle surface of the spherical shell, $\chi_r,\chi_\theta,\chi_{r\theta}$ are the changes of curvatures and twist.

The strains across the shell thickness at a distance z from the mid-plane are:

$$\varepsilon_r = \varepsilon_r^0 - z\chi_r \ ; \ \varepsilon_\theta = \varepsilon_\theta^0 - z\chi_\theta \ ; \quad \gamma_{r\theta} = \gamma_r^\theta - z\chi_{r\theta}. \tag{4}$$

By using Eqs. (3), (4), the geometrical compatibility equation of the shell is written as

$$\frac{1}{r^2}\frac{\partial^2 \varepsilon_r^0}{\partial \theta^2} - \frac{1}{r}\frac{\partial \varepsilon_r^0}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial \varepsilon_\theta^0}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial r \partial \theta}(r\gamma_r^\theta) = -\frac{\Delta w}{r} + \chi_{r\theta}^2 - \chi_r\chi_\theta, \tag{5}$$

The stress – strain relationships for the shell are defined by the Hooke law

$$(\sigma_r, \sigma_\theta) = \frac{E(z)}{1-\nu^2}\left[(\varepsilon_r, \varepsilon_\theta) + \nu(\varepsilon_\theta, \varepsilon_r)\right]; \ \sigma_{r\theta} = \frac{E(z)}{2(1+\nu)}\gamma_{r\theta}. \tag{6}$$

where σ_r and σ_θ are the normal stress, $\sigma_{r\theta}$ is the shear stress at the middle surface of the spherical shell in spherical system coordinate.

The force and moment resultants of the shell are expressed in terms of the stress components through the thickness as

$$(N_{ij}, M_{ij}) = \int_{-h/2}^{h/2} \sigma_{ij}(1, z)dz, ij = (rr, \theta\theta, r\theta) \tag{7}$$

with $(i = j = r)$ or $(i = j = \theta)$, for simplicity denoted $N_{rr} = N_r$, $N_{\theta\theta} = N_\theta$, $M_{rr} = M_r$, $M_{\theta\theta} = M_\theta$.

Using Eqs. (4), (6), and (7), the constitutive relations can be given as

$$\begin{aligned}
(N_r, M_r) &= \frac{(E_1, E_2)}{1-\nu^2}(\varepsilon_r^0 + \nu\varepsilon_\theta^0) + \frac{(E_2, E_3)}{1-\nu^2}(\chi_r + \nu\chi_\theta),\\
(N_\theta, M_\theta) &= \frac{(E_1, E_2)}{1-\nu^2}(\varepsilon_\theta^0 + \nu\varepsilon_r^0) + \frac{(E_2, E_3)}{1-\nu^2}(\chi_\theta + \nu\chi_r),\\
(N_{r\theta}, M_{r\theta}) &= \frac{(E_1, E_2)}{2(1+\nu)}\gamma_{r\theta}^0 + \frac{(E_2, E_3)}{1+\nu}\chi_{r\theta}.
\end{aligned} \tag{8}$$

From the relations one can write

$$\varepsilon_r^0 = \frac{1}{E_1}(N_r - \nu N_\theta); \ \varepsilon_\theta^0 = \frac{1}{E_1}(N_\theta - \nu N_r); \ \gamma_{r\theta}^0 = \frac{2(1+\nu)}{E_1}N_{r\theta}, \tag{9}$$

$$M_r = -D(\chi_r + \nu\chi_\theta); \ M_\theta = -D(\chi_\theta + \nu\chi_r); \ M_{r\theta} = -D(1-\nu)\chi_{r\theta}. \tag{10}$$

where

$$E_1 = hE_c + \frac{E_{mc}h}{k+1}; \ E_2 = 0; \ E_3 = \frac{E_c h^3}{12} + \frac{E_{mc}h^3}{2(k+1)(k+2)(k+3)}, D = \frac{E_1 E_3}{E_1(1-\nu^2)}. \tag{11}$$

Based on the classical shell theory, the nonlinear equilibrium equations of a the shell [9]

$$\begin{aligned}
&\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{N_r}{r} - \frac{N_\theta}{r} = 0,\\
&\frac{\partial N_\theta}{r\partial \theta} + \frac{\partial N_{r\theta}}{\partial r} + \frac{2N_{r\theta}}{r} = 0,
\end{aligned} \tag{12}$$

(13)

$$\frac{\partial^2 M_r}{\partial r^2}+\frac{2}{r}\frac{\partial M_r}{\partial r}+2\Big(\frac{\partial^2 M_{r\theta}}{r\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial M_{r\theta}}{\partial\theta}\Big)+\frac{1}{r^2}\frac{\partial^2 M_\theta}{\partial\theta^2}-\frac{1}{r}\frac{\partial M_\theta}{\partial r}+\frac{1}{R}(N_r+N_\theta)$$
$$+\frac{1}{r}\frac{\partial}{\partial r}\Big(rN_r\frac{\partial w}{\partial r}+N_{r\theta}\frac{\partial w}{\partial\theta}\Big)+\frac{1}{r}\frac{\partial}{\partial\theta}\Big(N_{r\theta}\frac{\partial w}{\partial r}+\frac{N_\theta}{r}\frac{\partial w}{\partial\theta}\Big)+q-k_1w+k_2\Delta w=0. \tag{14}$$

The Eqs. (12), (13) are identically satisfied by introducing a stress function F as

$$N_r=\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2},\quad N_\theta=\frac{\partial^2 F}{\partial r^2},\quad N_{r\theta}=-\frac{1}{r}\frac{\partial^2 F}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial F}{\partial\theta}. \tag{15}$$

Substituting Eqs. (3), (9), (15) into the Eqs. (5) and substituting Eqs. (3), (10), (15) into Eq. (14) leads to

$$\frac{1}{E_1}\Delta\Delta F=-\frac{\Delta w}{R}+\Big(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial w}{\partial\theta}\Big)^2-\frac{\partial^2 w}{\partial r^2}\Big(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\Big), \tag{16}$$

$$D\Delta\Delta w-\frac{\Delta F}{R}-\Big(\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial^2\theta}\Big)\frac{\partial^2 w}{\partial r^2}-\Big(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\Big)\frac{\partial^2 F}{\partial r^2}$$
$$+2\Big(\frac{1}{r}\frac{\partial^2 F}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial F}{\partial\theta}\Big)\Big(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial w}{\partial\theta}\Big)=q-k_1w+k_2\Delta w. \tag{17}$$

The stress function F should be determined by the substitution of deflection function w into compatibility equation (16) and solving resulting equation. Therefore one should find a transformation to lead Eqs. (16), (17) into constant coefficient differential equations. Suppose such a transformation [9]

$$w=w(\varsigma),F=F_0(\varsigma)\mathrm{e}^{2\varsigma}\text{ , where } r=r_0\,\mathrm{e}^{\varsigma};\varsigma=\ln\frac{r}{r_0} \tag{18}$$

Substituting Eq. (18) into Eqs. (16), (17) and establishing a lot of calculations lead to the transformed equations [9]

$$\frac{1}{E_1}\left(\frac{\partial^4 F_0}{\partial\varsigma^4}+4\frac{\partial^3 F_0}{\partial\varsigma^3}+4\frac{\partial^2 F_0}{\partial\varsigma^2}+4\frac{\partial^3 F_0}{\partial\varsigma\partial\theta^2}+2\frac{\partial^4 F_0}{\partial\varsigma^2\partial\theta^2}+4\frac{\partial^2 F_0}{\partial\theta^2}+\frac{\partial^4 F_0}{\partial\theta^4}\right)$$
$$=-\frac{r_0^2}{R}\Big(\frac{\partial^2 w}{\partial\varsigma^2}+\frac{\partial^2 w}{\partial\theta^2}\Big)+\frac{1}{e^{4\varsigma}}\Big(\frac{\partial^2 w}{\partial\varsigma\partial\theta}-\frac{\partial w}{\partial\theta}\Big)^2+\frac{1}{e^{4\varsigma}}\Big(\frac{\partial^2 w}{\partial\varsigma^2}-\frac{\partial w}{\partial\varsigma}\Big)\Big(\frac{\partial w}{\partial\varsigma}+\frac{\partial^2 w}{\partial\theta^2}\Big); \tag{19}$$

$$D\Big(\frac{\partial^4 w}{\partial\varsigma^4}-4\frac{\partial^3 w}{\partial\varsigma^3}+4\frac{\partial^2 w}{\partial\varsigma^2}-4\frac{\partial^3 w}{\partial\varsigma\partial\theta^2}+2\frac{\partial^4 w}{\partial\varsigma^2\partial\theta^2}+4\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial^4 w}{\partial\theta^4}\Big)+$$
$$-\frac{r_0^2e^{4\varsigma}}{R}\Big(\frac{\partial^2 F_0}{\partial\varsigma^2}+4\frac{\partial F_0}{\partial\varsigma}+4F_0+\frac{\partial^2 F_0}{\partial\theta^2}\Big)-\Big(\frac{\partial F_0}{\partial\varsigma}+2F_0+\frac{\partial^2 F_0}{\partial\theta^2}\Big)\Big(\frac{\partial^2 w}{\partial\varsigma^2}-\frac{\partial w}{\partial\varsigma}\Big)e^{2\varsigma}+ \tag{20}$$

$$-\Big(\frac{\partial^2 F_0}{\partial\varsigma^2}+2F_0+3\frac{\partial F_0}{\partial\varsigma}\Big)\Big(\frac{\partial w}{\partial\varsigma}+\frac{\partial^2 w}{\partial\varsigma\partial\theta}\Big)e^{2\varsigma}+2\Big(\frac{\partial^2 F_0}{\partial\varsigma\partial\theta}+\frac{\partial F_0}{\partial\theta}\Big)\Big(\frac{\partial^2 w}{\partial\varsigma\partial\theta}-\frac{\partial w}{\partial\theta}\Big)e^{2\varsigma}+$$

$$-qr_0^4e^{4\varsigma}+k_1wr_0^4e^{4\varsigma}-k_2(\frac{\partial^2 w}{\partial\varsigma^2}+\frac{\partial^2 w}{\partial\theta^2})r_0^2e^{2\varsigma}=0.$$

Eqs. (19) and (20) are the basic equations used to investigate the nonlinear buckling of FGM annular spherical shells. These are nonlinear equations in terms of two dependent unknowns $w(\varsigma)$ and $F_0(\varsigma)$.

3. Nonlinear mechanical stability analysis and numerical results.

The nonlinear mechanical and numerical analysis of S-FGM annular spherical shell is analyzed in this section. The shell consists of aluminum (metal) and alumina (ceramic) with the following properties

$$E_m=70GPa,\ \alpha_m=23\times10^{-6}\ {}^o\mathrm{C}^{-1},\ K_m=204\,{}^{W}\!/_{mK};\ E_c=380GPa\ \ \alpha_c=7.4\times10^{-6}\ {}^o\mathrm{C}^{-1},\ K_m=10.4\,{}^{W}\!/_{mK}.$$

and Poisson's ratio is chosen to be $v=0.3$.

3.1. Nonlinear mechanical stability analysis

The S-FGM annular spherical shell is assumed to be simply supported along the periphery and subjected to mechanical loads uniformly distributed on the outer surface and the base edges of the shell. Depending on the in-plane behavior at the edge of boundary conditions will be considered in case the edges are simply supported and immovable. For this case, the boundary conditions are

$$u=0, w=0, \frac{\partial^2 w}{\partial\varsigma^2}-\frac{\partial w}{\partial\varsigma}=0, N_r=N_0, N_{r\theta}=0, \quad \text{with } \varsigma=0 \text{ (i.e at } r=r_0\text{)} \tag{21}$$

where N_0 is the fictitious compressive load rendering the immovable edges.

The boundary conditions (21) can be satisfied when the deflection w is approximately assumed as follows [9, 11, 12]

$$w=We^{\varsigma}\sin(\beta_1\varsigma)\sin(n\theta), \beta_1=\frac{m\pi}{a}, a=\ln\frac{r_1}{r_0} \tag{22}$$

Where W is the maximum amplitude of deflection and m,n are the numbers of half waves in meridional and circumferential direction, respectively. The form of this approximate solution was proposed by Agamirov [12] and it was used by Sofiyev [11] for FGM truncated conical shells.

Introduction of Eqs. (22) into Eq. (19) gives

$$\frac{1}{E_1}(\frac{\partial^4 F_0}{\partial\varsigma^4}+4\frac{\partial^3 F_0}{\partial\varsigma^3}+4\frac{\partial^2 F_0}{\partial\varsigma^2}+4\frac{\partial^3 F_0}{\partial\varsigma\partial\theta^2}+2\frac{\partial^4 F_0}{\partial\varsigma^2\partial\theta^2}+4\frac{\partial^2 F_0}{\partial\theta^2}+\frac{\partial^4 F_0}{\partial\theta^4})$$
$$=-\frac{r_0^2e^{\varsigma}W}{R}[(1-\beta_1^2-n^2)\sin(\beta_1\varsigma)+2\beta_1\cos(\beta_1\varsigma)]\sin(n\theta)+ \tag{23}$$

$$+\frac{W^2\beta_1^2}{2}(n^2-1)\cos(2\beta_1\varsigma)-\frac{W^2}{4}(\beta_1-\beta_1 n-\beta_1^3)\sin(2\beta_1\varsigma)+\frac{W^2}{2}\beta_1^2 n^2\cos(2n\theta)+$$
$$+\frac{W^2}{4}(\beta_1-\beta_1 n^2-\beta_1^3)\sin(2\beta_1\varsigma)\cos(2n\theta)+\frac{W^2}{2}\beta_1^2\cos(2\beta_1\varsigma)\cos(2n\theta).$$

Solving this obtained equation with the boundary conditions (21) for the stress function F_0 yields [9]

$$F_0=f_1 e^{\varsigma}\sin(\beta_1\varsigma)\sin(n\theta)+f_2 e^{\varsigma}\cos(\beta_1\varsigma)\sin(n\theta)+f_3\cos(2\beta_1\varsigma)+f_4\cos(2n\theta)$$
$$+f_5\cos(2\beta_1\varsigma)\cos(2\beta_2\theta)+f_6\sin(2\beta_1\varsigma)\cos(2n\theta)+f_7\sin(2\beta_1\varsigma)+\frac{1}{2}N_0 r_0^2. \tag{24}$$

with $f_1=-\dfrac{r_0^2E_1W[A(1-\beta_1^2-n^2)+2B\beta_1]}{(A^2+B^2)\mathrm{R}}, f_2=-\dfrac{r_0^2E_1W[B(1-\beta_1^2-n^2)+2A\beta_1]}{(A^2+B^2)\mathrm{R}},$

$$f_3=E_1l_3W^2, f_4=E_1l_4W^2, f_5=E_1l_4W^2, f_6=E_1l_6W^2, f_7=E_1l_7W^2, \tag{25}$$

and

$$A=9-22\beta_1^2+\beta_1^4+n^4-10n^2+2\beta_1^2n^2, B=8\beta_1^3-24\beta_1+8\beta_1 n^2,$$
$$l_3=\frac{b_6c_3+c_4a_6}{a_3b_6+a_6b_3}, l_4=\frac{\beta_1^2}{32(\beta_2^2-1)}, l_5=\frac{b_5c_5+a_5c_6}{a_4b_5+a_5b_4}, l_6=\frac{a_4c_6-b_4c_5}{a_4b_5+a_5b_4}, l_7=\frac{a_3c_4-b_3c_3}{a_3b_6+a_6b_3}, \tag{26}$$

The remaining constants are given in Appendix.

After substitution Eqs. (22) and (24) in Eq. (20), for simplicity, the left hand side of the last obtained equation is denoted by Φ. Applying Galerkin method with the limits of integral is given by the formula [9]

$$\int_0^{\ln\frac{r_1}{r_0}}\int_0^{2\pi}\Phi e^{\varsigma}\sin(\beta_1\varsigma)\sin(n\theta)d\varsigma d\theta=0, \tag{27}$$

we obtain the following equation

$$q-\frac{N_0A_5}{B_1r_0^2}W-\frac{A_0N_0}{RB_1}=\left(\frac{DA_3}{B_1r_0^4}+\frac{E_1A_4}{B_1R^2}+\frac{k_1A_6}{B_1}+\frac{k_2A_7}{B_1r_0^2}\right)W+\frac{E_1A_2}{RB_1r_0^2}W^2+\frac{E_1A_1}{B_1r_0^4}W^3. \tag{28}$$

Where the constants $B_1, A_i, i=0,1,..7$ are given in Appendix.

Eq. (28) is used to determine the buckling loads and nonlinear equilibrium paths of S-FGM annular spherical shell under uniform external pressure.

The simply supported S-FGM annular spherical shell with freely movable edge is assumed to be subjected to external pressure q (in Pascals) uniformly distributed on the outer surface of the shell in the absence of temperature conditions. In this case $N_0=0$ and Eq. (28) reduces to

$$q=\left(\frac{D^*R_h^4A_3}{B_1R_0^4}+\frac{E_1^*A_4R_h^2}{B_1}+\frac{K_1D^*A_6}{B_1}+\frac{K_2D^*A_7R_h^2}{B_1R_0^2}\right)W^*+\frac{E_1^*A_2R_h^3}{B_1R_0^2}(W^*)^2+\frac{E_1^*A_1R_h^4}{B_1R_0^4}(W^*)^3 \tag{29}$$

where, by putting

$$E_1^* = \frac{E_1}{h}, W^* = \frac{W}{h}, R_h = \frac{h}{R}; \mathrm{R}_0 = \frac{r_0}{R}; D^* = \frac{D}{h^3}; K_1 = \frac{k_1 h^4}{D}; K_2 = \frac{k_2 h^2}{D}.$$

If the S-FGM annular spherical shell does not rest on elastic foundations $(K_1 = K_2 = 0)$ we received

$$q = \left(\frac{D^* R_h^4 A_3}{B_1 R_0^4} + \frac{E_1^* A_4 R_h^2}{B_1} \right) W^* + \frac{E_1^* A_2 R_h^3}{B_1 R_0^2} (W^*)^2 + \frac{E_1^* A_1 R_h^4}{B_1 R_0^4} (W^*)^3. \tag{30}$$

Eq. (30) may be used to find static critical buckling load and trace post-buckling load – deflection curves of S-FGM annular spherical shell. It is evident $q(W^*)$ curves originate from the coordinate origin. In addition, Eq. (29) indicates that there is no bifurcation-type buckling for pressure loaded annular spherical shell and extremum-type buckling only occurs under definite conditions.

3.2. Numerical results and discussion

In this section, it is noted that in all figures W/h denotes the dimensionless maximum deflection of the shell.

Figure 2 examines the dependence of the nonlinear response of FGM annular spherical shells on the mode (m,n). It is easily recognized that with $m=1$, the more increased the value of n the higher increasing of the value of extreme point, corresponding to the higher load capacity of the shells. Note that, when m is even or $m \geq 3$, the graphic consists of symmetric curves through the origin of the coordinate system and the extreme point does not exist in the load-deflection curves.

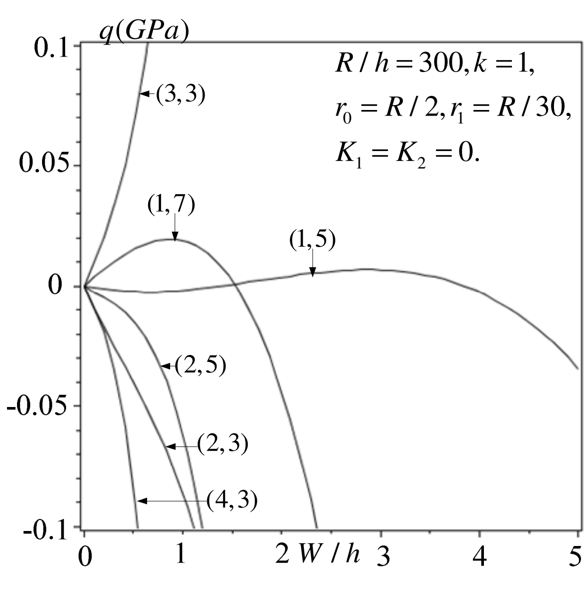

Fig. 2. Effects of mode (m,n) on the nonlinear response of S-FGM annular spherical shells.

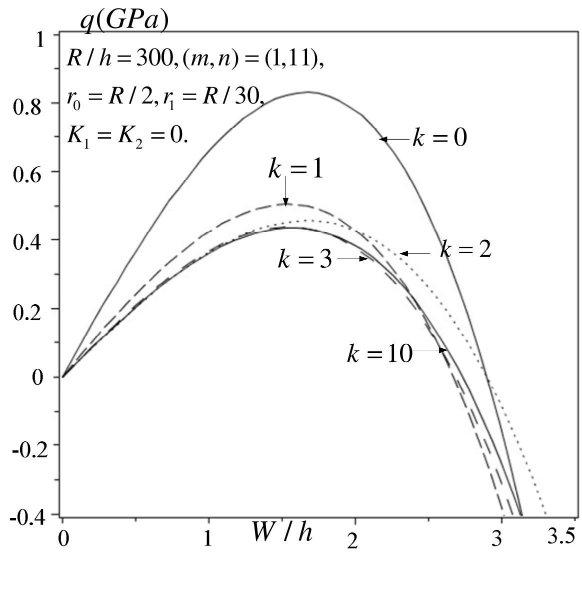

Fig. 3. Effects of volume fraction index k on the nonlinear response of the S-FGM annular spherical shell.

Fig.3. shows the effects of volume fraction index $k(0,1,2,3,10)$ on the nonlinear response of the FGM annular spherical shell subjected to external pressure (mode $(m,n)=(1,11)$). As can be seen, the load–deflection curves become lower when k increased. This is expected because the volume percentage of ceramic constituent, which has higher elasticity modulus, is dropped with increasing values of k .

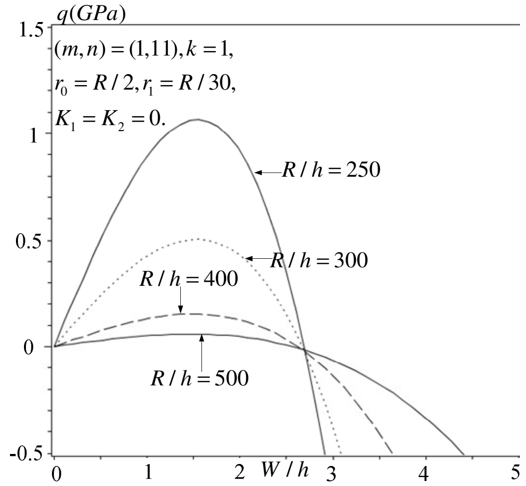

Fig. 4. Effects of curvature radius-thickness ratio on the nonlinear response of the shells under external pressure.

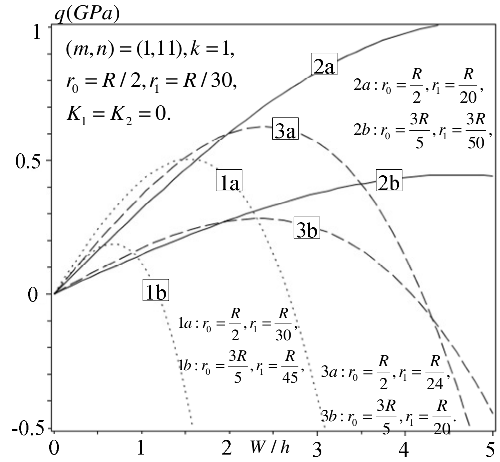

Fig.5. Effects of radius of base-curvature radius ratio r_1/r_0 on the nonlinear response of the shells.

Fig. 4 depicts the effects of curvature radius - thickness ratio R/h (250, 300, 400, and 500) on the nonlinear behavior of the external pressure of the S-FGM annular spherical shells (mode $(m,n)=(1,11)$). From Fig. 4 we can conclude that when the annular spherical shells get thinner - corresponding with R/h getting bigger, the critical buckling loads will get smaller.

Fig. 5 analyzes the effects of 2 base-curvature radius ratio r_1/r_0 on the nonlinear response of the shells subjected to uniform external pressure. It is shown that the nonlinear response of the shells is very sensitive with change of r_1/r_0 ratio characterizing the shallowness of annular spherical shell. Specifically, the enhancement of the upper buckling loads and the load carrying capacity in small range of deflection as r_1/r_0 increases is followed by a very severe snap - through behaviors. In other words, in spite of possessing higher limit buckling loads, deeper spherical shells exhibit a very unstable response from the post-buckling point of view. Furthermore, in the same effects of base-curvature radius ratio r_1/r_0 the load of the nonlinear response of the shells is higher when the shallowness of the shell (H) is smaller, where H is the distance between two radius r_1, r_0, and calculated by $H(r_1,r_0)=\sqrt{R^2-r_0^2}-\sqrt{R^2-r_1^2}$

Effects of the elastic foundations (K_1, K_2) on the nonlinear response of S-FGM annular spherical shells are shown in Fig 6. Obviously, elastic foundations played positive role on nonlinear static response of the S-FGM annular spherical shell: the large K_1 and K_2 coefficients are, the larger loading capacity of the shells is. It is clear that the elastic foundations can enhance the mechanical loading capacity for the S-FGM annular spherical shells.

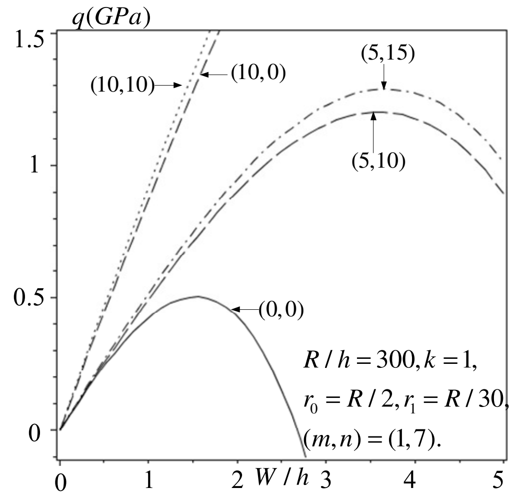

Fig. 6. Effects of the elastic foundations (K_1, K_2) on the nonlinear response of S-FGM annular spherical shells.

4. Conclusion

The present paper aims to propose an analytical approach to study the problem of nonlinear stability analysis of S-FGM thin annular spherical shells with ceramic-metal-ceramic layers on elastic foundations under uniform external pressure. Based on the classical shell theory, the equilibrium and compatibility equations are derived in terms of the shell deflection and the stress function. This system of equations has been transformed into another system of more simple equations. Galerkin method is used to get the explicit expression of post-buckling load – deflection curves of the shells. The effects of material, geometrical properties, elastic foundations and combination of external pressure on the nonlinear buckling and post-buckling of the S-FGM annular spherical shells are analyzed and discussed.

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References

[1] N. D. Duc, P. T. Thang, N. T. Dao, H. V. Tac. "Nonlinear buckling of higher deformable S-FGM thick circular cylindrical shells with metal-ceramic-metal layers surrounded on elastic foundations in thermal environment". J. Compos. Struct., Accepted for publication (2015).

[2] N. D. Duc, P. T. Thang. "Nonlinear response of imperfect eccentrically stiffened ceramic-metal-ceramic S-FGM thin circular cylindrical shells surrounded on elastic foundations under uniform radial load". J. Mech. of Advan. Mater. and Struct., Accepted for publication (2015).

[3] N. D. Duc, P. T. Thang. Nonlinear buckling of imperfect eccentrically stiffened metal-ceramic-metal S-FGM thin circular cylindrical shells with temperature-dependent properties in thermal environments. Int. J. of Mech. of Scien. 2014; 81: 17-25.

[4] N. D. Duc, P. T. Thang. Nonlinear response of imperfect eccentrically stiffened ceramic-metal-ceramic FGM circular cylindrical shells surrounded on elastic foundations and subjected to axial compression. J. Compos. Struct. 2014, 110: 200-206.

[5] N. D. Duc, P. H. Cong (2013). Nonlinear dynamic response of imperfect symmetric thin S-FGM plate with metal- ceramic-metal layers on elastic foundation. Journal of Vibration and Control, First published on July 8, 2013 as doi: 10.1177/1077546313489717.

[6] M. R. Eslami, Y. Kiani, "An exact solution for thermal buckling of annular FGM plates on an elastic medium", Composite Part B: Engineering 2013; 45: 101-110.

[7] A. Bagri, M. R. Eslami, "Generalized coupled thermoelasticity of functionally graded annular disk considering the Lord–Shulman theory". Composite Structures 2008; 83:168-79

[8] V. T. T Anh, D. H. Bich, N. D. Duc, "Nonlinear buckling analysis of thin FGM annular spherical shells on elastic foundations under external pressure and thermal loads", European Journal of Mechanics A/Solids 2014; 50: 28-38.

[9] N. D. Duc, V. T. T. Anh, D. H. Bich, "Nonlinear post-buckling of thin FGM annular spherical shells under mechanical loads and resting on elastic foundations", Vietnam Journal of Mechanics ISSN: 0866 7136, 2014; 36(4): 283 – 290.

[10] P. L. Pasternak, On a new method of analysis of an elastic foundation by means of two foundation constants, USSR: Gos Izd Lit po Stroit I Arkh; . Moscow (1954).

[11] A. H. Sofiyev, "The buckling of FGM truncated conical shells subjected to axial compressive load and resting on Winkler-Pasternak foundations", Int. J. of Pressure Vesels and Piping 2010; 87: 753-761.

[12] V. L. Agamirov, "Problem of Nonlinear shells theory", [in Russian] Science Edition, Moscow (1990).

Appendix

$$a_3 = 16(\beta_1^4 - \beta_1^2);\;\; a_4 = 16(\beta_1^4 - \beta_1^2 + 2\beta_1^2 n^2 - n^2 + n^4); a_5 = 32(\beta_1^3 + \beta_1 n^2);\;\; \mathrm{a}_6 = 32\beta_1^3,$$

$$b_3 = 32\beta_1^3;\;\; b_4 = 32(\beta_1^3 + \beta_1 n^2);\;\; b_5 = 16(\beta_1 - \beta_1^2 + 2\beta_1^2 n^2 - n^2 + n^4);\;\; b_6 = 16(\beta_1^4 - \beta_1^2),$$

$$c_3 = 0.5(\beta_1^2 n^2 - \beta_1^2);\;\; c_4 = 0.25(\beta_1 n^2 - \beta_1 + \beta_1^3);\;\; c_5 = 0.5\beta_1^2;\;\; c_6 = -\mathrm{c}_4,$$

$$C_1 = \mathrm{a}_4, C_2 = b_5,\; B_1 = \frac{-2m\pi a(-1 + e^{5a}(-1)^m)}{\mathrm{n}(25a^2 + m^2\pi^2)},\; A_0 = \frac{m^2\pi^3\left(1 - e^{6a}\right)}{\left(9a^2 + m^2\pi^2\right)};$$

$$A_1 = h_{11} + h_{12}\left(\frac{m\pi}{a}t_3 + t_4 - 2t_4 n^2\right) + h_{13}\left(\frac{m\pi}{a}t_6 + t_5\right) + h_{14}\left(\frac{m\pi}{a}t_5 + t_6\right) + h_{15}\left(\frac{m\pi}{a}t_4 - t_3 - 2t_3 n^2\right) +$$
$$+ h_{16}\left(\frac{2m^2\pi^2}{a^2}t_3 + \frac{3m\pi}{a}t_4 - t_3\right) + h_{17}\left(\frac{-2m^2\pi^2}{a^2}t_4 + \frac{3m\pi}{a}t_3 + t_4\right) + h_{18}\left(\frac{2m^2\pi^2}{a^2}t_5 + \frac{3m\pi}{a}t_6 - t_5\right) +$$
$$+ h_{19}\left(\frac{-2m^2\pi^2}{a^2}t_6 + \frac{3m\pi}{a}t_5 + t_6\right) + h_{110}\left(t_3 - \frac{2m\pi}{a}t_4\right) + h_{111}\left(t_4 + \frac{2m\pi}{a}t_3\right),$$

$$A_2 = h_{21}\left(-6t_1 + \frac{m^2\pi^2}{a^2}t_1 + \frac{5m\pi}{a}t_2\right) + h_{22}\left(6t_2 - \frac{m^2\pi^2}{a^2}t_2 + \frac{5m\pi}{a}t_1\right) + h_{23}\left(-t_3 + \frac{2m\pi}{a}t_4 + \frac{m^2\pi^2}{a^2}t_3 + n^2t_3\right) +$$
$$+ h_{24}\left(t_3 + \frac{2m\pi}{a}t_3 - \frac{m^2\pi^2}{a^2}t_4 + n^2t_4\right) + h_{25}\left(\frac{m^2\pi^2}{a^2}t_5 + \frac{2m\pi}{a}t_6 - t_5\right) + h_{26}\left(\frac{-m^2\pi^2}{a^2}t_6 + \frac{2m\pi}{a}t_5 + t_6\right) +$$
$$+ h_{27} + h_{28}\left(2nt_1 - \frac{mn\pi}{a}t_2\right) + h_{29}\left(2nt_2 + \frac{mn\pi}{a}t_1\right) + h_{210}\left(-3t_1 + \frac{m\pi}{a}t_2 + n^2t_1\right) + h_{211}\left(3t_2 + \frac{m\pi}{a}t_1 - n^2t_2\right),$$

$$A_3 = \frac{1}{8}\frac{m^2\pi^3\left(e^{2a}-1\right)\left(a^4 + 2m^2\pi^2a^2 + m^4\pi^4 - 2n^2a^4 + 2m^2n^2\pi^2a^2 + n^4a^4\right)}{a^4(a^2+m^2\pi^2)},$$

$$A_4 = \frac{-m^2\pi^3\left(e^{6a}-1\right)\left(-9t_1a^2 + 6t_2m\pi a + t_1m^2\pi^2 + t_1n^2a^2\right)}{24a^2(9a^2+m^2\pi^2)} + \frac{m\pi^2\left(e^{6a}-1\right)\left(-9t_2a^2 - 6t_1m\pi a + t_2m^2\pi^2 + t_2n^2a^2\right)}{8a(9a^2+m^2\pi^2)},$$

$$A_5 = \frac{-8m^3\pi^3\left(-3m^2\pi^2 - 7a^2 + 3e^{5a}m^2\pi^2(-1)^m + 7e^{5a}(-1)^m a^2\right)}{3an(625a^4 + 250m^2\pi^2a^2 + 9m^4\pi^4)},\ A_6 = \frac{m^2\pi^3\left(e^{6a}-1\right)}{24\left(9a^2+m^2\pi^2\right)};$$

$$A_7 = \frac{m^2\pi^3\left(-m^2\pi^2 + 3a^2e^{4a} - 3a^2 + e^{4a}m^2\pi^2\right)}{16a^2\left(4a^2+m^2\pi^2\right)};\ A_8 = \left[\begin{array}{l}\dfrac{m^2\pi^2}{16a^2} - \dfrac{\left(t_6a^2 + t_5m\pi a - vt_6a^2 + 2vt_6m^2\pi^2 - 3vt_5m\pi a\right)}{a^2+m^2\pi^2} + \\ + \dfrac{m\pi\left(t_5a^2 - t_6m\pi a - vt_5a^2 + 2vt_5m^2\pi^2 + 3vt_6m\pi a\right)}{a\left(a^2+m^2\pi^2\right)}\end{array}\right];$$

$$A_9 = \frac{-\pi m\left(-1+e^a(-1)^m\right)}{an};$$

$$A_{10} = \left[\begin{array}{l}\dfrac{a\pi m\left(-1+e^{3a}(-1)^m\right)}{n\left(9a^2+m^2\pi^2\right)} - \dfrac{m\pi\left(-1+e^{3a}(-1)^m\right)\left(-t_2m\pi a + 3t_1a^2 - n^2t_1a^2 + vm^2\pi^2t_1 + 5vt_2m\pi a - 6vt_1a^2\right)}{an\left(9a^2+m^2\pi^2\right)} + \\ + \dfrac{3\left(-1+e^{3a}(-1)^m\right)\left(m\pi t_1a + 3t_2a^2 - n^2t_2a^2 - 5vm\pi t_1a + vt_2m^2\pi^2 - 6vt_2a^2\right)}{n\left(9a^2+m^2\pi^2\right)}\end{array}\right]$$

$$h_{11} = \frac{m^3\pi^4\left(e^{4a}-1\right)\left[4a^2\pi m\left(3n^2-1\right) + \left(2n^2-1\right)\left(m^3\pi^3 - 2a^3\right)\right]}{512a^4(n^2-1)(4a^2+m^2\pi^2)} + \frac{\pi^4m(e^{4a}-1)\left(a^3 - \pi m^3\right)}{256a^2(n^2-1)(a^2+m^2\pi^2)};$$

$$h_{12} = \frac{-\pi^2m\left(e^{4a}-1\right)\left(2\pi m - a\right)}{32\left(a^2+m^2\pi^2\right)} + \frac{\pi^5m^4\left(e^{4a}-1\right)\left(\pi^2m^2 - 2a^2\right)}{32a^2(4a^4 + 5a^2m^2\pi^2 + m^4\pi^4)};$$

$$h_{13} = \frac{\pi^3m^2\left(e^{4a}-1\right)\left(2\pi m - a\right)}{16a\left(a^2+m^2\pi^2\right)} + \frac{9m^7n\pi^{10}(-1+e^{4a})^2}{16a(4a^4+5a^2m^2\pi^2+m^4\pi^4)};\ h_{14} = -2h_{12};\ h_{15} = -\frac{1}{2}h_{13};$$

$$h_{16} = \frac{-3a\pi^4m^3(-1+e^{2a})}{8\left(a^4+5a^2m^2\pi^2+4m^4\pi^4\right)} + \frac{\pi^3m^2(-1+e^{2a})}{4\left(a^2+4m^2\pi^2\right)};\ h_{17} = \frac{\pi^2m\left(e^{2a}-1\right)\left[2\pi^3m^3 - a^2\pi m + a^3 + am^2\pi^2\right]}{8\left(a^2+4m^2\pi^2\right)\left(a^2+m^2\pi^2\right)};$$

$$h_{18} = \frac{\pi^3m^2(-1+e^{2a})\left(3a\pi m - 2a^2 - 2\pi^2m^2\right)}{4\left(a^2+4m^2\pi^2\right)\left(a^2+m^2\pi^2\right)};\ h_{19} = \frac{\pi^2m\left(e^{2a}-1\right)\left[-4\pi^3m^3 + 2a^2\pi m + a^3 + am^2\pi^2\right]}{8\left(a^2+4m^2\pi^2\right)\left(a^2+m^2\pi^2\right)};$$

$$h_{110} = \frac{-\pi^4 nm^3(-1+e^{4a})}{16a(a^2+m^2\pi^2)};\ h_{111} = \frac{n\pi^3 m^2(-1+e^{4a})}{16(a^2+m^2\pi^2)};\ h_{21} = \frac{8am^2\pi^2\left(a-m\pi\right)\left(-1+(-1)^m e^{3a}\right)}{9n\left(9a^4+10a^2m^2\pi^2+m^4\pi^4\right)},$$

$$h_{22} = \frac{4am\pi\left(-1+(-1)^m e^{3a}\right)\left(-2am\pi+m^2\pi^2+3a^2\right)}{9n\left(9a^4+10a^2m^2\pi^2+m^4\pi^4\right)},\ h_{23} = \frac{160a^2m^2\pi^2(-1+(-1)^m e^{5a})}{3n(625a^4+250a^2m^2\pi^2+9m^4\pi^4)},$$

$$h_{24} = \frac{-8a\pi m\left(-1+(-1)^m e^{5a}\right)\left(25a^2-3m^2\pi^2\right)}{3n(625a^4+250a^2m^2\pi^2+9m^4\pi^4)},\ h_{25} = \frac{160a^2m^2\pi^2}{3n(625a^4+250a^2m^2\pi^2+9m^4\pi^4)},\ h_{26} = -h_{24},$$

$$h_{27} = \frac{am\pi^3\left(-1+(-1)^m e^{5a}\right)}{12n(25a^2+m^2\pi^2)},\ h_{28} = \frac{-40\pi^3m^3a\left(-1+(-1)^m e^{5a}\right)}{3(625a^4+250a^2m^2\pi^2+9m^4\pi^4)},$$

$$h_{29} = \frac{4m^2\pi^2\left(-1+(-1)^m e^{5a}\right)\left(3m^2\pi^2+25a^2\right)}{3(625a^4+250a^2m^2\pi^2+9m^4\pi^4)},\ h_{210} = \frac{-8m^2\pi^2\left(-1+(-1)^m e^{5a}\right)\left(-5a^3+10a^2m\pi+3m^3\pi^3\right)}{3an(625a^4+250a^2m^2\pi^2+9m^4\pi^4)},$$

$$h_{211} = \frac{4\pi m\left(-1+(-1)^m e^{5a}\right)\left(50a^2m\pi-25a^3-3am^2\pi^2+16m^3\pi^3\right)}{3n(625a^4+250a^2m^2\pi^2+9m^4\pi^4)},$$

$$t_1 = \frac{\left\{A(1-\beta_1^2-n^2)+2B\beta_1\right\}}{(A^2+B^2)},\qquad t_4 = \frac{\left\{C_2\beta_1^2+16(\beta_1^3+\beta_1n^2)(-\beta_1n^2+\beta_1-\beta_1^3)\right\}}{2\left[C_1C_2+(32\beta_1^3+32\beta_1n^2)^2\right]},$$

$$t_2 = \frac{\left\{B(1-\beta_1^2-n^2)+2A\beta_1\right\}}{(A^2+B^2)},\qquad t_5 = \frac{4\left\{(\beta_1^4-\beta_1^2)(\beta_1n^2-\beta_1+\beta_1^3)-4\beta_1^3(\beta_1^2n^2-\beta_1^2)\right\}}{(16\beta_1^4-16\beta_1^2)^2+1024\beta_1^6},$$

$$t_3 = \frac{\left\{C_1(-\beta_1n^2+\beta_1-\beta_1^3)-64(\beta_1^3+\beta_1n^2)\beta_1^2\right\}}{4\left[C_1C_2+(32\beta_1^3+32\beta_1n^2)^2\right]}\qquad t_6 = \frac{8\left\{(\beta_1^4-\beta_1^2)(\beta_1^2n^2-\beta_1^2)+(\beta_1n^2-\beta_1+\beta_1^3)\beta_1^3\right\}}{(16\beta_1^4-16\beta_1^2)^2+1024\beta_1^6},$$