ON AN INTEGRAL TRANSFORM WITH THE HUMBERT CONFLUENT HYPERGEOMETRIC FUNCTION IN THE KERNEL

Dinh Thanh Duc

Faculty of Mathematics, Pedagogical Institute of Quynhon

Abstract. *In this paper, we establish an integral transform with the Humbert confluent hypergeometric function of two variables Φ_1 in the kernel and its inverse formula.*

I. Introduction

It is well known that the theory of the integral transform plays an important role in classical analysis. One of the problems of integral transform is to study the inversion formulas. There exist some inversion formulas, which are of more interest from the theoretical point of view.

In recent years, classical integral transforms as well as multidimensional integral transforms have been studied by Srivastava, Buschman [6], Vu Kim Tuan [2], H.J.Glaeske, M.Saigo [7], [8], also the table of integral transforms in A.P.Prudnikov, Yu.A.Brychkov et al [3], are mostly considered in L_p and other spaces. In 1996, Vu Kim Tuan, M.Saigo and Dinh Thanh Duc established some integral transforms with the Humbert confluent hypergeometric function of two variables Φ_1 in the kernel in the space of entire functions of exponential type [5].

In this paper, we will establish an inverse formula of the following new integral transform

$$g(x) = \int_{-1}^{1} e^{ixy}\Phi_1\left(\alpha, \beta, \alpha+\gamma+1; \frac{y-1}{y+1}, (1-y)(a+ix)\right) f(y)dy,$$

where Φ_1 is the Humbert confluent hypergeometric function.

II. Some preliminary results and notations

In this section we will recall some notations and necessary results

Definition 2.1. *The Humbert confluent hypergeometric function of two variables is defined by*

$$\Phi_1(\alpha, \beta, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m!n!} \tag{2.1}$$

where $(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+m-1)$ *is the Pochhammer symbol.*

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We recall the integral representation formula of the function $\Phi_1(\alpha, \beta, \gamma; x, y)$ [9].

$$\Phi_1(\alpha, \beta, \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}e^{yt}\,dt \tag{2.2}$$

with $\Re\gamma > \Re\beta > 0$.

Definition 2.2. [1] *Let $\varphi(x) \in L_1(a,b)$. Then the integrals*

$$(I^{\alpha}_{a_+}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, \qquad x > a \tag{2.3}$$

$$(I^{\alpha}_{b_-}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}}dt, \qquad x < b \tag{2.4}$$

where $\alpha > 0$, are called integrals of fractional order α.

Definition 2.3. [1] *Let $f(x)$ be defined on $[a,b]$. Then the operators*

$$(\mathcal{D}^{\alpha}_{a_+}f)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}} \tag{2.5}$$

$$(\mathcal{D}^{\alpha}_{b_-}f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_x^b \frac{f(t)dt}{(t-x)^{\alpha-n+1}}, \tag{2.6}$$

where $n = [\alpha]+1$, $[\alpha]$ is integer part of α, are called derivative of fractional order $\alpha > 0$.

Remark 1. *Definitions 2.2 and 2.3 remain valid if α is the complex number with $\Re\alpha > 0$.*

Remark 2. *We have the following formula*

$$\mathcal{D}^{\alpha}_{a_+}I^{\alpha}_{a_+}\varphi = \varphi(x), \quad \Re\alpha > 0 \tag{2.7}$$

where $\varphi(x)$ is the arbitrary integrable function.

Definition 2.2. *Let E^{σ} $(\sigma > 0)$ be the class of entire functions of type at most σ, that means $f \in E^{\sigma}$ if and only if $f(z) = O\left(e^{(\sigma+\epsilon)|z|}\right)$ as $|z| \to \infty$ for every $\epsilon > 0$ [4]. The intersection of the restriction of E^{σ} on $\mathbf{R}$ with $L_2(\mathbf{R})$ is denoted by M^{σ}*

Theorem 2.1. (Paley-Wiener Theorem) [4]. *The function $f \in M^{\sigma}$ if and only if f is the Fourier transform of a function $\tilde{f} \in L_2(\mathbf{R})$ with compact support from $[-\sigma,\sigma]$:*

$$f(x) = \int_{-\sigma}^{\sigma} \tilde{f}(y)e^{ixy}dy, \quad \tilde{f}(y) \in L_2(-\sigma,\sigma)$$

III. The inversion formula

Now, we consider the following integral transform

$$g(x) = \int_{-1}^{1} e^{ixy}\Phi_1\left(\alpha, \beta, \alpha+\gamma+1; \frac{y-1}{y+1}, (1-y)(a+ix)\right) f(y)dy, \tag{3.1}$$

where $a = const > 0$.

Applying the integral representation of the function $\Phi_1(\alpha, \beta, \gamma; x, y)$ we obtain

$$(1-y)^{\alpha+\gamma}(1+y)^{-\beta}e^{ay+ixy}\,\Phi_1\left(\alpha, \beta, \alpha+\gamma+1; \frac{y-1}{y+1}, (1-y)(a+ix)\right) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha)\Gamma(\gamma+1)}$$

$$\times \int_0^1 (1-y)^{\alpha+\gamma}\tau^{\alpha-1}(1-\tau)^{\gamma}\left(y(1-\tau)+(1+\tau)\right)^{-\beta} e^{(a+ix)(y+(1-y)\tau}d\tau, \tag{3.2}$$

where $\Re(\alpha+\gamma+1) > \Re\beta > 0$.

Putting $t = y + (1-y)\tau$, from (3.2) we get

$$\int_y^1 (t-y)^{\alpha-1}(1-t)^{\gamma}(1+t)^{-\beta}e^{t(a+ix)}dt = \frac{\Gamma(\alpha)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(1-y)^{\alpha+\gamma}(1+y)^{-\beta}$$

$$\times e^{ay+ixy}\,\Phi_1\left(\alpha, \beta, \alpha+\gamma+1; \frac{y-1}{y+1}, (1-y)(a+ix)\right) \tag{3.3}$$

Hence, the equation (3.1) can be represented in the following form

$$g(x) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha)\Gamma(\gamma+1)}\int_{-1}^{1} f(y)(1-y)^{-\alpha-\gamma}(1+y)^{\beta}e^{-ay}$$

$$\times \int_y^1 (t-y)^{\alpha-1}(1-t)^{\gamma}e^{t(a+ix)}dt\,dy \tag{3.4}$$

Changing the order of integration we obtain

$$g(x) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha)\Gamma(\gamma+1)}\int_{-1}^{1} e^{ixt}(1-t)^{\gamma}(1+t)^{-\beta}e^{at}$$

$$\times \int_{-1}^{t} (t-y)^{\alpha-1}(1-y)^{-\alpha-\gamma}(1+y)^{\beta}e^{-ay}f(y)dy\,dt \tag{3.5}$$

From the formula of the inverse Fourier transform [4] we derive

$$\int_{-1}^{t} \frac{(t-y)^{\alpha-1}}{\Gamma(\alpha)}(1-y)^{-\alpha-\gamma}(1+y)^{\beta}e^{-ay}f(y)dy$$

$$= \frac{1}{2\pi}\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\int_{-\infty}^{\infty} g(x)e^{-ixt}(1-t)^{-\gamma}(1+t)^{\beta}e^{-at}dx. \tag{3.6}$$

Taking the fractional derivative of (3.6) and using (2.7) we obtain

$$(1-y)^{-\alpha-\gamma}(1+y)^{\beta}e^{-ay}f(y)$$

$$= \frac{1}{2\pi}\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \times \frac{d^n}{dy^n}\int_{-\infty}^{+\infty} g(x)\int_{-1}^{y}\frac{(y-t)^{n-\alpha-1}}{\Gamma(n-\alpha)}(1-t)^{-\gamma}(1+t)^{\beta}e^{-t(a+ix)}dt\,dx \tag{3.7}$$

Putting $t = \tau(y+1) - 1$, then we get

$$f(y) = \frac{1}{2\pi}\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(1-y)^{\alpha+\gamma}(1+y)^{-\beta}e^{ay}$$
$$\times \frac{d^n}{dy^n}\int_{-\infty}^{\infty} g(x)(y+1)^{n-\alpha+\beta}2^{-\gamma}$$
$$\times e^{a+ix}\int_0^1 \frac{\tau^\beta(1-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left(1-\frac{y+1}{2}\tau\right)^{-\gamma} e^{-(y+1)(a+ix)\tau}d\tau\, dx. \quad (3.8)$$

Finally, applying (2.2) we get the inverse formula

$$f(y) = \frac{e^a}{2^{\gamma+1}}\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{\Gamma(\alpha+\gamma+1)\Gamma(n-\alpha+\beta+1)}$$
$$\times (1-y)^{\alpha+\gamma}(1+y)^{-\beta}e^{ay}\frac{d^n}{dy^n}\Bigg[(y+1)^{n-\alpha+\beta}$$
$$\times\int_{-\infty}^{+\infty} g(x)e^{ix}\,\Phi_1\!\left(\beta+1,\gamma,n-\alpha+\beta+1;\frac{y+1}{2},-(y+1)(a+ix)\right)\Bigg]dx \quad (3.9)$$

From (3.4) we derive $g(x)$ is the entire function of exponential type. Now we consider the following function

$$(1-t)^\gamma(1+t)^{-\beta}e^{at}\int_{-1}^{t}(t-y)^{\alpha-1}(1-y)^{-\alpha-\gamma}(1+y)^\beta e^{-ay}f(y)dy. \quad (3.10)$$

Suppose $(1-y)^{-\alpha-\gamma}(1+y)^\beta f(y)\in L(-1,1)$, $\Re\,\alpha>0$, then $\int_{-1}^t(t-y)^{\alpha-1}(1-y)^{-\alpha-\gamma}(1+y)^\beta e^{-ay}f(y)dy$ is a continuous function on [-1,1]. Hence, if $\Re\,\gamma>-\frac{1}{2}$ and $\Re\,(\alpha-\beta)>-\frac{1}{2}$, then we derive the function (3.10) belongs to $L_2[-1,1]$, it follows that $g(x)\in L_2[-1,1]$ (according to the Paley-Wiener theorem). Thus we have

Theorem 3.1. *Let $a\in\mathbb{R}_+$, $\alpha,\beta,\gamma\in\mathbb{C}$ and $\Re\,(\alpha+\beta+1)>Re\,\beta>0$. Then integral transform (3.1) will have the inversion formula (3.9). Moreover, suppose*

$$(1-y)^{-\alpha-\gamma}(1+y)^\beta f(y)\in L(-1,1), \Re\,\alpha>0, \Re\,\gamma>-\frac{1}{2}\quad and\quad \Re\,(\alpha-\beta)>-\frac{1}{2}$$

then we obtain $g(x)\in L_2[-1,1]$.

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VỀ PHÉP BIẾN ĐỔI TÍCH PHÂN VỚI NHÂN LÀ HÀM SIÊU BỘI SUY BIẾN

Đinh Thanh Đức
Khoa Toán, Đại học Sư phạm Quy Nhơn

Trong bài báo này, chúng tôi đã thiết lập được công thức ngược của một lớp biến đổi tích phân mới với nhân là hàm siêu bội suy biến trong nhân.