VNU. JOURNAL OF SCIENCE, Nat. Sci., t.XVII, n^{0}4 - 2001

ON MODULES HAVING $pf(M) \leq 0$

Nguyen Thai Hoa
Faculty of Mathematics, Pedagogical Institute of Quynhon

I. Introduction

Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring and M be a finitely generated A-module with $\dim M = d$. We denote $Q_M(\underline{x})$ the submodule of M defined by

$$Q_M(\underline{x}) = \bigcup_{n>0} \Big((x_1^{n+1}, ..., x_d^{n+1})M : x_1^n \cdots x_d^n \Big),$$

where $\underline{x} = (x_1, ..., x_d)$ is a system of parameters on M (for short s.o.p). We consider the difference

$$J_{M,\underline{x}}(\underline{n}) = n_1 \cdots n_d e(\underline{x}; M) - \ell_A(M/Q_M(\underline{x}(\underline{n})))$$

as a function of $\underline{n}$, where $\underline{x}(\underline{n}) = (x_1^{n_1}, ..., x_d^{n_d})$ and $\underline{n} = (n_1, ..., n_d)$ is a d-tuple of positive integers. It is known by [5] that $\ell_A(M/Q_M(\underline{x}(\underline{n})))$ is just the length of generalized fractions in [12]. Therefore we can recall Question 1.2 of [12] as follows: is $J_{M,\underline{x}}(\underline{n})$ a polynomial for enough large $\underline{n}$? We do not know the complete answer to this question at the time of writing. But as mentioned above for the function $J_{M,\underline{x}}(\underline{n})$, the authors in [11], [6] and [4] have shown that the least degree of all polynomials of $\underline{n}$ bounding above the function $J_{M,\underline{x}}(\underline{n})$ is independent of the choice of $\underline{x}$. In [6], this invariant for the module M is denoted by $pf(M)$. The aim of this paper is to study some properties of modules having $pf(M) \leq 0$.

In Section 2 we will give a characterization of modules with $pf(M) \leq 0$. In Section 3, we will give some relations between the invariant $pf(M)$ with Cohen-Macaulay filtered modules (for short CMF modules) which are defined in [15]. The last section is devoted to study some properties of modules having $pf(M) = -\infty$.

II. A characterization of modules with $pf(M) \leq 0$

Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring and a finitely generated with $\dim M = d \geq 1$. Let $\underline{x} = (x_1, ..., x_d)$ be a system of parameters on M and $\underline{n} = (n_1, ..., n_d)$ a d-tuple of positive integers. We set

$$Q_M(\underline{x}) = \bigcup_{t>0} \Big((x_1^{t+1}, ..., x_d^{t+1})M : x_1^t \cdots x_d^t \Big),$$

and $\underline{x}(\underline{n}) = (x_1^{n_1}, ..., x_d^{n_d})$. By [5; 2.3] and [11], [6], we denote

$$J_{M,\underline{x}}(\underline{n}) = n_1 \cdots n_d e(\underline{x}; M) - \ell\Big(M/Q_M(\underline{x}(\underline{n}))\Big).$$

For simplicity, we write $J_{M,\underline{x}}(\underline{n}) = J_M(\underline{x})$ when $n_1 = \cdots = n_d = 1$. The difference $J_{M,\underline{x}}(\underline{n})$ which is non-negative by [6], can be considered as a function of $\underline{n}$ and it is bounded above by the polynomial $n_1 \cdots n_d\, J_M(\underline{x})$.

Theorem 2.1. (see [11; 1.1], [6; 3.2]). *The least degree of all polynomials of* $\underline{n}$ *bounding above the function* $J_{M,\underline{x}}(\underline{n})$ *is independent of the choice of* $\underline{x}$.

Remark 2.2. (see [6]) The numerical invariant of M by Theorem 2.1 is denoted by $pf(M)$. For convenience, we stipulate that the degree of the zero-polynomial is $-\infty$. The following results on $pf(M)$ have been proved in [11].

(i) Let $\widehat{M}$ be the $\mathfrak{m}$-adic completion of M. Then

$$pf_A(M) = pf_A\Big(M/H^0_{\mathfrak{m}}(M)\Big) = pf_{\widehat{A}}(\widehat{M}) = pf_{A/\operatorname{Ann}(M)}(M).$$

(ii) Let $\underline{x}$ be an s.o.p on M with $\dim(0 : x_1) < d - 1$. Then

$$pf(M/x_1M) \leq pf(M) \leq pf(M/x_1M) + 1.$$

Theorem 2.3. *With the notations as above. Then,* $pf(M) \leq 0$ *if and only if there exist a constant* K *such that* $J_{M,\underline{x}}(\underline{n}) \leq K$ *for every* $\underline{x}$ *on* M *and for all* $\underline{n} = (n_1, ..., n_d)$.

Proof. ($\Leftarrow$): It is clearly by Theorem 2.1.

($\Rightarrow$): Assume that $pf(M) \leq 0$. By Remark 2.2.(i), we have $pf_A(M) = pf_A(M/H^0_{\mathfrak{m}}(M)) = pf_{A/\operatorname{Ann}(M)}(M)$. Therefore we may assume without any loss of generality that $\operatorname{Ann}_A(M) = 0$ and depth $M > 0$. Let $\underline{x} = (x_1, ..., x_d)$ be an s.o.p on M, then there is a constant $K(\underline{x})$ such that $J_{M,\underline{x}}(\underline{n}) \leq K(\underline{x})$ for all $\underline{n}$. Assume that $\underline{y} = (y_1, ..., y_d)$ is any s.o.p on M. By [16; 8.2.5], there exist an s.o.p $\underline{z} = (z_1, ..., z_d)$ on M and positive integers $r_1, ..., r_d$ such that

$$\begin{aligned}(x_1^{r_1}, ..., x_d^{r_d})A &\subseteq (z_1, x_2^{r_2}, ..., x_d^{r_d})A \subseteq \cdots \subseteq (z_1, ..., z_d)A \\ &\subseteq (z_1, ..., z_{d-1}, y_d)A \subseteq \cdots \subseteq (z_1, y_2, ..., y_d)A \subseteq (y_1, ..., y_d)A.\end{aligned}$$

By [4; 4.2], we have $J_M(\underline{y}) \leq J_{M,\underline{x}}(\underline{r}) \leq K(\underline{x})$ where $\underline{r} = (r_1, ..., r_d)$ as required. □

In Conjecture 1 of [8], Hochster has given the monomial conjecture for the case $M = A$ such that, for every system of parameters $\underline{x} = (x_1, ..., x_d)$ of A, it holds

$$x_1^t \cdots x_d^t \notin (x_1^{t+1}, ..., x_d^{t+1})A$$

for all $t > 0$. Hochster proved in [8] that this monomial conjecture is true for hight powers of systems of parameters. He also gave an example to show that the monomial conjecture is not true for modules. However, the authors in [4] said that a system of parameters $\underline{x} = (x_1, ..., x_d)$ on M satisfies the *condition of the monomial conjecture* (MC) if

$$x_1^t \cdots x_d^t M \nsubseteq (x_1^{t+1}, ..., x_d^{t+1})M$$

for all $t > 0$. Clearly, $\underline{x}$ satisfies the condition of the monomial conjecture if and only if $Q_M(\underline{x}) \neq M$ i.e. $\ell\Big(M/Q_M(\underline{x})\Big) > 0$. By the counterexample of Hochster mentioned above, we can not show in general that every system of parameter of M satisfies the condition (MC). By [4; 3.3], we have a uniform bound for high powers of all systems of parameter of M satisfying the condition (MC). The following result is an immediate consequence of Theorem 2.3.

Corollary 2.4. *With the notations as above. Assume that $pf(M) \leq 0$. Then there exist a constant N such that for every s.o.p $\underline{x} = (x_1, ..., x_d)$ on M, $\underline{x}(\underline{n})$ satisfies the condition of the monomial conjecture for all $n_1, ..., n_d \geq N$.*

Proof. Since $pf(M) \leq 0$, by Theorem 2.3 we get a constant K such that $J_{M,\underline{x}}(\underline{n}) \leq K$ for every s.o.p $\underline{x}$ on M and for all $\underline{n}$. Set $N = K + 1$. Suppose that there exist positve integers $n_1, ..., n_d \geq N$ such that $\ell_A(M/Q_M(\underline{x}(\underline{n}))) = 0$. It follows that $n_1 \cdots n_d e(\underline{x}; M) \leq K$. We have the contradiction. The proof is complete. □

III. Cohen-Macaulay filtered modules

First of all we need some notations which have been introduced in [15] as follows.

Let M be a finitely generated A-module with $\dim_A M = d \geq 1$. For an integer $0 \leq i \leq d$, let M_i denote the largest submodule of M such that $\dim_A M_i \leq i$. Because M is a Noetherian A-module, the submodules M_i of M are well-defined. Moreover, it follows that $M_{i-1} \subseteq M_i$ for all $1 \leq i \leq d$. The increasing filtration $\mathcal{M} = \{M_i\}_{0 \leq i \leq d}$ of submodules of M is called the *dimension filtration* of M (see [15; 2.1]). A finitely generated A-module M is called a *Cohen-Macaulay filtered module* (CMF module), whenever $\mathcal{M}_i = M_i/M_{i-1}$ is either zero or an i-dimensional Cohen-Macaulay module for all $1 \leq i \leq d$. An s.o.p $\underline{x} = (x_1, ..., x_d)$ on M is called a distinguished system of parameters of M provided $(x_{i+1}, ..., x_d)M_i = 0$ for all $i = 0, ..., d-1$ (see [15; 2.5]). Note that any module M always admits a distinguished system of parameters (see [15; 2.6]).

Next, we show a relation between $pf(M_i)$ and $pf(\mathcal{M}_i)$ when $\mathcal{M}_i \neq 0$.

Proposition 3.1. *With the above notations. Let $\mathcal{M}_i = M_i/M_{i-1}$ for all $i = 1, ..., d$. Then if $\mathcal{M}_i \neq 0$, we have $pf(\mathcal{M}_i) = pf(M_i)$.*

Proof. Let $\underline{x} = (x_1, ..., x_d)$ be a distinguished system of parameters of M. Whenever $M_i \neq 0$ then $\underline{y} = (x_1, ..., x_i)$ is a system of parameters of $\mathcal{M}_i$. Let $\underline{n} = (n_1, ..., n_i)$ an i-tuple of positive integers and set $\underline{y}(\underline{n}) = (x_1^{n_1}, ..., x_i^{n_i})$.

One can easily check that the map

$$\varphi : \mathcal{M}_i/Q_{\mathcal{M}_i}(\underline{y}(\underline{n})) \longrightarrow M_i/Q_{M_i}(\underline{y}(\underline{n}))$$

defined by $\varphi(u + Q_{\mathcal{M}_i}(\underline{y}(\underline{n}))) = u + Q_{M_i}(\underline{y}(\underline{n}))$ for any $u \in M_i$, is a well defined and surjective homomorphism. Because of $x_i^{n_i} M_{i-1} = 0$, we can check that φ is an isomorphism. By $\dim M_{i-1} < i$, we get $e(\underline{y}(\underline{n}); \mathcal{M}_i) = e(\underline{y}(\underline{n}); M_i)$. It follows that $J_{M_i,\underline{y}}(\underline{n}) = J_{\mathcal{M}_i,\underline{y}}(\underline{n})$ for all $\underline{n} = (n_1, ..., n_i)$. Therefore $pf(\mathcal{M}_i) = pf(M_i)$ as required. □

Corollary 3.2. *Assume that M is a Cohen-Macaulay filtered module. Then the following statements are true:*

(i) If $\mathcal{M}_i \neq 0$ then $pf(M_i) = -\infty$.
(ii) $pf(M) = -\infty$.

Proof. It is trivial.

Corollary 3.3. *Assume that* M *is a CMF-module. Then* $pf(M_{\mathfrak{p}}) = -\infty$ *for all prime ideal* $\mathfrak{p} \in \mathrm{Supp}(M)$.

Proof. By [15; 4.8], we have $M_{\mathfrak{p}}$ which is a CMF $A_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Supp}(M)$. It follows that $pf(M_{\mathfrak{p}}) = -\infty$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$ as required. □

Let M be a finitely generated A-module such that $pf(M) = -\infty$. From the above proposition, a natural question is whether M is a Cohen-Macaulay filtered module? Unfornately, this question is not true in general because of the following counterexample.

Example 3.4. There exist a finitely generated A-module with $pf(M) = -\infty$, but M is not a CMF module.

Let $(A, \mathfrak{m})$ denote a local Noetherian ring and M a finitely generated A-module. Consider the idealization $A \propto M$ of M over A. That is, the additive group $A \propto M$ coinsides with the direct sum of abelian groups A and M. The multiplication is given by

$$(a, m)\ (b, n) = (ab, an + bm).$$

Then $B = A \propto M$ is a d-dimensional local ring (see [2; 3.3.22] or [9; 1.1]) for more details.

Now suppose that $(A, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay ring. Let M be not a CMF A-module with $\dim_A M = t < d$. Let $\mathcal{M} = \{M_i\}_{0 \leq i \leq d}$ denote the dimension filtration of M. Now put

$$B_i = \begin{cases} A \propto M & \text{for } i = d, \\ 0 \propto M & \text{for } i = t+1, ..., d-1 \text{ and} \\ 0 \propto M_i & \text{for } i = 0, ..., t. \end{cases}$$

Then $\{B_i\}_{0 \leq t \leq d}$ is a dimension filtration of $B = A \propto M$ such that $B_d = A \propto M$.

Note that

$$B_i/B_{i-1} \cong \begin{cases} A & \text{for } i = d, \\ 0 & \text{for } i = t+1, ..., d-1 \text{ and} \\ M_i/M_{i-1} & \text{for } i = 1, ..., t. \end{cases}$$

Then, B is not a CMF B-module. On the other hand, we have an exact sequence of the B-homomorphisms as follows

$$0 \longrightarrow 0 \propto M \longrightarrow B \xrightarrow{p} A \longrightarrow 0$$

where p is defined by $p(a, m) = a$ for any $(a, m) \in B$. It follows that $B/0 \propto M \cong A$ as B-module. Thus, we get $pf(B/0 \propto M) = pf_B(A)$. Finally, we have to prove $pf_B(B) = pf_B(B/0 \propto M)$. Indeed, set $N = 0 \propto M$ and $\overline{B} = B/N$. Let $I = \mathrm{Ann}(N)$. Since $\dim_B N = t < d$, we have $I \not\subset \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(B)$ with $\dim B/\mathfrak{p} = d$. So we can choose an element $x_1 \in I$ such that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(B)$ with $\dim B/\mathfrak{p} = d$. Then, x_1 is a parameter element of B and $x_1 N = 0$. Let $\underline{x} = (x_1, ..., x_d)$ be an s.o.p on B. We can show with the same method as in the proof of Proposition 3.1 that $J_{B,\underline{x}}(\underline{n}) = J_{\overline{B},\underline{x}}(\underline{n})$ for all $\underline{n} = (n_1, ..., n_d)$. Because A is a Cohen-Macaulay B-module, we have $pf_B(B) = -\infty$ as required. □

IV. Modules having $pf(M) = -\infty$

First of all we need some notations which has been introduced in [3] and [5] a follows.

The difference between lengths and multiplicities

$$I_M(\underline{n}, \underline{x}) = \ell_A\Big(M/(x_1^{n_1}, ..., x_d^{n_d})M\Big) - n_1 \cdots n_d\, e(\underline{x}; M)$$

considered as a function of $\underline{n}$, where $\underline{x} = (x_1, ..., x_d)$ is an s.o.p on M and $\underline{n} = (n_1, ..., n_d)$ d-tuple positive integers. This function in general is not a polynomial of $\underline{n}$ for all $n_1, ..., n_d$ large enough ($\underline{n} \gg 0$ for short). It is bounded above by the polynomial $n_1 \cdots n_d\, I_M(\underline{x})$.

Theorem 4.1. (see [3; 2.3]) *The least degree of all polynomial of $\underline{n}$ bounding abo $I_M(\underline{n}, \underline{x})$ is independent of the choice of $\underline{x}$.*

Definition 4.2. (see [3; 2.4]) The numberial invariant of M is called *the polynomial ty* of M and we denote by $p(M)$.

In [3], for convenience the author stipulate that the degree of the zero-polynomi equal to $-\infty$.

Let $H^i_{\mathfrak{m}}(M)$ be the i-th local cohomology module of M with respect to the maxim ideal $\mathfrak{m}$. Denote by $E(A/\mathfrak{m})$ the injective hull $A/\mathfrak{m}$ and $\widehat{A}$ the $\mathfrak{m}$-adic completion of A. We denote

$$K^i(M) = \operatorname{Hom}_A\ (H^i_{\mathfrak{m}}(M),\ E(A/\mathfrak{m})),$$

the Matlis' dual of $H^i_{\mathfrak{m}}(M)$, considering as finitely generated modules over $\widehat{A}$. We w recall the modules M which satisfy the following condition:

$$(*)\quad \begin{cases} \text{either } K^i(M) \text{ is a Cohen-Macaulay module of dimension } i \\ \text{or } K^i(M) = 0 \qquad \text{for } i = 1, ..., d-1\ (d = \dim M). \end{cases}$$

The class of modules M satisfying the condition $(*)$ was given first by N.T. Cuong an V.T. Khoi in [5].

Remark 4.3.

(i) In case A has a dualizing complex, P. Schenzel has proved in [15; Theorem 5.5] th M is a CMF-module if and only if for all $0 \leq i < d$ the module of deficiency $K^i(M$ is either zero or an i-dimensional Cohen-Macaulay module. Hence, each module satifies the condition $(*)$ is, under the assumption that $A = \widehat{A}$, a CMF-module.

(ii) Each A-module M holds the condition $(*)$, we have $pf(M) = -\infty$ (see [3; 1.1]).

Let $\underline{x} = (x_1, ..., x_d)$ be an s.o.p on M. We denote

$$M_i = M/(x_1, ..., x_i)M$$

for all $i = 1, ..., d$.

Following we have the results.

Theorem 4.4. *With the notations as above and* $\dim M = d \geq 1$. *Then the followi statements are equivalent:*

i) *For all* $i = 1,...,d-1$, *either* $K^i(M)$ *is Cohen-Macaulay module of dimension* i *or* $K^i(M) = 0$,

i) *There exist an s.o.p* $\underline{x} = (x_1,...,x_d)$ *on* M *such that* x_i *is a regular element of* $M_{i-1})$ *for all* $j = 1,...,d-i$, $i = 1,...,d-1$ *and* $\underline{x}$ *is a filter regular sequence.*

oof. (i) $\Rightarrow$ (ii). We will prove by induction on d. For $d = 1$ there is nothing to prove. $d \geq 2$ and suppose that the statement is true for all modules with dimension $< d$. By Prime Avoidance Theorem, we can choose an element $x_1 \in \mathfrak{m}$ such that

$$x_1 \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M) \bigcup_{i=1}^{d-1} \mathrm{Ass}(K^i(M)) \setminus \{\mathfrak{m}\}} \mathfrak{p}$$

refore, we can take x_1 as a first element of the system of parameters on M. $M_1 = M/x_1M$. Since for all $j = 1,...,d-1$, either $K^j(M)$ is a Cohen-Macaulay lule of dimension j or $K^j(M) = 0$, then x_1 is a regular element of $K^j(M)$ for all $1,...,d-1$. By [5; 3.2, (iv)], M_1 holds the condition (i). Applying the inductive othesis for M_1, there exists an s. p. o. $\underline{x}' = (x_2,...,x_d)$ on M_1 and it satisfies the lition (ii). It follows that an s.o.p $\underline{x} = (x_1,...,x_d)$ holds the condition (ii).

(ii) $\Rightarrow$ (i). We will prove by induction d. For $d = 1$, there is nothing to prove. For $d \geq 2$ and suppose that the statement is true for all modules with dimension . Since x_1 is a filter regular element of M, we have $H^i_{\mathfrak{m}}(M/(0:x_1)) \simeq H^i_{\mathfrak{m}}(M)$ for all 1 . The exact sequence

$$0 \longrightarrow M/(0:x_1) \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

ls the following exact sequence of local cohomology modules

$$0 \longrightarrow H^j_{\mathfrak{m}}(M)/x_1H^j_{\mathfrak{m}}(M) \longrightarrow H^j_{\mathfrak{m}}(M/x_1M) \longrightarrow (0:x_1)_{H^{j+1}_{\mathfrak{m}}}(M) \longrightarrow 0$$

ll $j = 1,...,d-2$.

ng the Matlis duality of this exact sequence, we get exact sequence

$$0 \longrightarrow K^{j+1}(M)/x_1K^{j+1}(M) \longrightarrow K^j(M/x_1M) \longrightarrow (0:x_1)_{K^j(M)} \longrightarrow 0$$

ll $j = 1,...,d-2$. By the hypothesis of x_1, we obtain $(0:x_1)_{K^j(M)} = 0$ for all $1,...,d-2$. It follows that

$$K^{j+1}(M)/x_1K^{j+1}(M) \cong K^j(M/x_1M)$$

ll $j = 1,...,d-2$. Applying the inductive hypothesis for M/x_1M, we get for all $j = d-3$, either $K^j(M/x_1M)$ is a Cohen-Macaulay of dimension j or $K^j(M/x_1M) = 0$. each $j = 1,...,d-3$, if $K^j(M/x_1M) = 0$, there is $K^{j+1}(M) = x_1 K^{j+1}(M)$. It ws that $K^{j+1}(M) = 0$ by Lemma Nakayama. Because of x_1 is a regular element of ${}^1(M)$ and if $K^j(M/x_1M)$ is a Cohen-Macaulay of dimension j, we can easily check $K^{j+1}(M)$ is a Cohen-Macaulay module of dimension $j+1$. Finally, we can easily that either $K^1(M)$ is a Cohen-Macaulay of dimension 1 or $K^1(M) = 0$ by the choice as required. □

Proposition 4.5. *With the notations as above and* $\dim M = d \geq 2$. *Assume that there exist an s.o.p* $\underline{x} = (x_1, ..., x_d)$ *on* M *such that for all* $i = 1, ..., d-2$, $x_i \notin \mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Ass}(M_{i-1})$ *with* $\dim A/\mathfrak{p} \geq d-i-1$ *and* x_i *is a regular element of* $K^{d-i-1}(M_{i-1})$ *and* $\mathfrak{m} \notin \mathrm{Ass}\, K^1(M_{d-2})$. *Then for all* $i = 1, ..., d-1$, *either* $K^{d-i}(M_{i-1})$ *is a Cohen-Macaulay module of dimension* $d-i$ *or* $K^{d-i}(M_{i-1}) = 0$.

Proof. Note that $H^i_{\mathfrak{m}}(M/(0:x_1)) \cong H^i_{\mathfrak{m}}(M)$ for $i \geq d-2$ by the hypothesis of x_1. Therefore, from the derived local cohomology sequence of the exact sequence

$$0 \to M/(0:x_1) \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0,$$

we have an exact sequence of local cohomology modules

$$0 \to H^{d-2}_{\mathfrak{m}}(M)/x_1H^{d-2}_{\mathfrak{m}}(M) \longrightarrow H^{d-2}_{\mathfrak{m}}(M/x_1M) \to (0:x_1)_{H^{d-1}_{\mathfrak{m}}(M)} \to 0.$$

Taking the Matlish duality of this exact sequence, we get an exact sequence

$$0 \to K^{d-1}(M)/x_1K^{d-1}(M) \to K^{d-2}(M/x_1M) \to (0:x_1)_{K^{d-2}(M)} \to 0.$$

Since x_1 is a regular element of $K^{d-2}(M)$, it follows that

$$K^{d-1}(M)/x_1K^{d-1}(M) \cong K^{d-2}(M_1).$$

With the same method as in the above proof, we can check that

$$K^{d-i}(M_{i-1})/x_iK^{d-i}(M_{i-1}) \cong K^{d-i-1}(M_i)$$

for all $i = 2, ..., d-2$. If $K^1(M_{d-2}) \neq 0$, there is $\mathrm{depth}\, K^1(M_{d-2}) = \dim K^1(M_{d-2})$. Thus $K^1(M_{d-2})$ is a Cohen-Macaulay module of dimension 1. We can show with the same method as in the conversly proof of Theorem 4.3 that for all $i = 2, ..., d-2$, either $K^{d-i}(M_{i-1})$ is a Cohen-Macaulay module of dimension $d-i$ or $K^{d-i}(M_{i-1}) = 0$ as required. □

Proposition 4.6. *Suppose that* $\dim M = d > 2$ *and* $p(M) \leq 2$. *Then if* $pf(M) \leq 0$, *we have* $pf(M_p) = -\infty$ *for all* $\mathfrak{p} \in \mathrm{Supp}(M) \setminus \{\mathfrak{m}\}$.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}\, A$ and $\mathfrak{q}$ be an element of $\mathrm{Ass}\left(\widehat{A}/\mathfrak{p}\widehat{A}\right)$ such that $\dim \widehat{A}/\mathfrak{q} = \dim A/\mathfrak{p}$. Let $f : A_{\mathfrak{p}} \longrightarrow \widehat{A}_{\mathfrak{q}}$ be a natural homomorphism. Since f is faithfull flat and $\dim(M_{\mathfrak{p}}) = \dim(\widehat{M}_{\mathfrak{q}})$, we can check that $pf(M_{\mathfrak{p}}) = pf(\widehat{M}_{\mathfrak{q}})$. Therefore, without loss of any generality we may assume that A is complete with respect to the $\mathfrak{m}$-adic topology. It follows by [14; 2.2.3] that

$$K^j(M_{\mathfrak{p}}) \cong \left(K^{j+\dim A/\mathfrak{p}}(M)\right)_{\mathfrak{p}}$$

for all $\mathfrak{p} \in \mathrm{Supp}(M)$ and all $j = 1, ..., \dim M_{\mathfrak{p}}$. If $p(M) \leq 1$, by [6; 4.2] we have $\ell\left(H^i_{\mathfrak{m}}(M)\right) < \infty$ for $i = 2, ..., d-1$. So it follows by [7; 2.5] that for all $\mathfrak{p} \in \mathrm{Supp}(M) \setminus \{\mathfrak{m}\}$ and all $j = 1, ..., \dim M_{\mathfrak{p}}$, either $K^j(M_{\mathfrak{p}})$ is a Cohen-Macaulay module of dimension j or $K^j(M_{\mathfrak{p}}) = 0$. By [5; 1.1] we get $pf(M_{\mathfrak{p}}) = -\infty$.

If $p(M) = 2$, by [6; 5.4] we have $\ell(H^i_{\mathfrak{m}}(M)) < \infty$ for $i = 3, ..., d-1$ and $K^2(M)$ is a generalized Cohen-Macaulay module of dimension 2. By [7; 2.5] and [5; 1.1], we obtained $pf(M_{\mathfrak{p}}) = -\infty$ as required. □

Acknowlegment: I am grateful to Prof. Nguyen Tu Cuong for drawing my attention to the subject and for hearty guidance during this research.

REFERENCES

[1] M. Auslander and D. A. Buchsbaum, Codimension and multiplicity, *Ann. of Math.* **68**(1958), 625-657.

[2] W. Bruns and J. Herzog. *"Cohen-Macaulay ring"*, Cambr. Univ. Press, 1993.

[3] N. T. Cuong, *On the least degree of polynomials bounding above the differences between lengths and multiplications of certain systems of parameters in local rings*, Nagoya Math. J. **125**(1992), 105-114.

[4] N. T. Cuong, N. T. Hoa and N. T. H. Loan, *On certain lengths functions associated to a system of parameters in local rings,* Vietnam J. Math. **27(3)** (1999), 259-272.

[5] N. T. Cuong and V. T. Khoi, *Module whose local cohomology modules have Cohen-Macaulay Matlis duals*, In: Proc. of Hanoi Conference on Algebra Geometry, Commutative Algebra and Computation Methods, D. Eisenbud (Ed.), Springer-Verlag, 1999, 223-231.

[6] N. T. Cuong and N. D. Minh, *Lengths of generalized fractions of modules have small polynomial,* Math. Proc. Cambr. Phil. Soc. **128**(2000), 269-282.

[7] N. T. Cuong, N. V. Trung and P. Schenzel, *Verallgemeinerte Cohen - Macaulay Moduln,* Math. Nachr. **85**(1978), 57-75.

[8] M. Hochster, *Contraced ideals from integral extensions of regular rings,* Nagoya Math. J. **51**(1973), 25-43.

[9] M. Nagata, "Local rings", Interscience, 1962.

[10] H. Matsumura, "Theory of commutative rings", Cambr. Univ. Press, 1986.

[11] N. D. Minh, *On the least degree of polynomials bounding above the differences between multiplicities and length of generalized fractions,* Acta Math. Vietnam. **20(1)** (1995), 115-128.

[12] R. Y. Sharp and M. A. Hamieh, *Lengths of certain generalized fractions,* J. Pure Appl. Algebra, **38** (1985), 323-336.

[13] R.Y. Sharp and H. Zakeri, *Modules of generalized fractions*, Mathematika **29**(1982), 32-41.

[14] P. Schenzel, " Dualisierende Komplexe in der lokalen Algebra und Buchsbaum Ringe", Lecture Notes in Mathematics **907**, Springer-Verlag, Berlin- Heidelberg-New York, 1982.

[15] P. Schenzel, *On the dimension filtration and Cohen-Macaulay filtered modules,* In: Proc. of the Ferrara meeting in honour of Mario Fiorentini, University of Antwerp Wilrijk, Belgium, 1998, 245-264.

[16] J.R. Strooker, *Homological questions in local algebra,* London Math. Soc. Lecture Notes Series, Vol. 145, Cambr. Univ. Press, 1990.

TẠP CHÍ KHOA HỌC ĐHQGHN, KHTN, t.XVII, n^{0}4 - 2001

MÔĐUN VỚI $pf(M) \leq 0$

Nguyễn Thái Hòa

Khoa Toán, Đại học sư phạm Quy Nhơn

Trong bài này, chúng tôi đưa ra một đặc trưng của những môđun M với $pf(M) \leq 0$. Tiếp theo, chúng tôi tìm một số mối liên hệ giữa bất biến $pf(M)$ với những môđun lọc hóa Cauhen-Macaulay được định nghĩa bởi P. Schenzel. Cuối cùng, đưa ra một số tính chất của những môđun có $pf(M) = -\infty$.