CHARACTERIZATION AND GENERALIZED CHARACTERIZATION OF SINGULAR INTEGRAL EQUATIONS

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Abstract. *In this paper we give an algebraic method for reducing some classes of singular integral equations with a regular part to either the characteristic equation or a generalized characteristic equation.*

I. Introduction

Consider a singular integral equation of the form

$$(K\varphi)(t) := (K_0 + T)\varphi = f, \qquad (*)$$

where either

$$(K_0\varphi)(t) := a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau$$

or

$$(K_0\varphi)(t) := a(t)\varphi(t) + \frac{1}{\pi i}\int_\Gamma \frac{b(\tau)\varphi(\tau)}{\tau - t}d\tau$$

and

$$(T\varphi)(t) := \int_\Gamma T(t,\tau)\varphi(\tau)d\tau.$$

The Noether theory of the equation $(*)$ was considered by several authors under general assumptions about $a(t)$, $b(t)$ and $T(t,\tau)$ (see, e.g. [1], [2], [4], [7]). It is known that the characteristic equation and its associated characteristic equation admit effecive solutions. But in general, equations of the form $(*)$ do not admit effective solutions. However, there are some sufficient conditions which are given by Samko S. G., Ng. V. Mau,... (see, e.g. [4], [8]) so that the equations $(*)$ can be solved effectively. In order to get other sufficient conditions for the kernel $T(t,\tau)$, we consider a problem on characterization of singular integral equations, i.e. we find the operators T such that the equation $(*)$ can be reduced to either $K_0\varphi = g$ or a generalized characteristic equation $(K_0 + T_0)\varphi = g$, where T_0 is a compact operator with kernel $T_0(t,\tau)$ satisfying sufficient conditions which are given by the authors mentioned above.

This paper deals with either characterization or generalized characterization of some class equations of the form $(*)$.

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§1.

Let Γ be a simple regular closed arc and let X be the space $H^{\mu}(\Gamma)$ $(0 < \mu < 1)$. Let $L(X)$ be a space of all linear operators with domains and ranges in X.

Definition 1.1. (see [3]) *An operator $V \in L(X)$ is said to be generalizlly almost invertible if there is an operator $W \in L(X)$ (called a generalized almost inverse of V) such that $ImW \subset domV$, $ImV \subset domW$ and*

$$VWV = V \quad on \quad dom\, V.$$

The set of all generalized almost invertible operators in $L(X)$ will be denoted by $W(X)$. For a given $V \in W(X)$ we denote by $\mathcal{W}_V$ the set of all generalized almost inverse of V.

Consider equations of the form

$$(K\varphi)(t) := (V\varphi)(t) + \lambda \int_{\Gamma} T_n(t,\tau)\varphi(\tau)d\tau = f(t), \tag{1}$$

where $V \in W(X), T_n(t,\tau) = \sum_{k=1}^{n} a_k(t)b_k(\tau), \quad f(t), a_k(t), b_k(t) \in X \quad (k = 1, 2, \ldots, n)$, $\{a_k(t)\}_{k=\overline{1,n}}$ is a linearly independent system,

$$b_k(t) \neq 0 \quad (k = 1, 2, \ldots, n), \quad 0 \neq \lambda \in \mathbb{C}. \tag{2}$$

Let $W \in \mathcal{W}_V$. Assume that

$$\begin{aligned} &a_k(t) \in \text{dom}\, W, \quad k = 1, 2, \ldots, n; \\ &\big[(I - WV)\varphi\big](t) = -\sum_{k=0}^{\kappa_0} u_k(\varphi)\varphi_k(t) \quad \text{on} \quad \text{dom}\, V, \end{aligned} \tag{3}$$

where $\varphi_0(t) \equiv 0$; $\{\varphi_k(t)\}_{k=\overline{1,\kappa_0}}$ is a linearly independent system and $u_k \in X^*$ $(k = 0, \ldots, \kappa_0)$ are the given linear functionals (X^* is a conjugate space to X).

Let $\mathcal{A} = \left[K_{jk}\right]_{j,k=1}^{n+\kappa_0}$ be an $(n+\kappa_0) \times (n+\kappa_0)$ matrix that is defined by complex numbers K_{jk}, where

$$K_{jk} = \begin{cases} 1 + K'_{jk} & \text{if} \quad j = k, \\ K'_{jk} & \text{if} \quad j \neq k, \quad (j,k = 1, \ldots, n+\kappa_0) \end{cases} \tag{4}$$

and

$$K'_{jk} = \begin{cases} \lambda \int_{\Gamma} b_j(t)(Wa_k)(t)dt & \text{if} \ j,k = 1, \ldots, n; \\ \int_{\Gamma} b_j(t)\varphi_{k-n}(t)dt & \text{if} \ j = 1, \ldots, n; \ k = n+1, \ldots, n+\kappa_0; \\ \lambda u_{j-n}(Wa_k) & \text{if} \ j = n+1, \ldots, n+\kappa_0; \ k = 1, \ldots, n; \\ u_{j-n}(\varphi_{k-n}) & \text{if} \ j,k = n+1, \ldots, n+\kappa_0. \end{cases} \tag{5}$$

Let $\mathcal{A}^k(\varphi)$ be an $(n+\kappa_0) \times (n+\kappa_0)$ matrix, obtained from $\mathcal{A}$ by replacing the k^{th} column by the $\gamma(\varphi)$ - column, where

$$\gamma(\varphi) = \left[\gamma_1(\varphi), \gamma_2(\varphi), \ldots, \gamma_{n+\kappa_0}(\varphi)\right]^T,$$

$$\gamma_j(\varphi) = \begin{cases} \int\limits_\Gamma b_j(t)(W\varphi)(t)dt & \text{if} \quad j = 1, \dots, n; \\ u_{j-n}(W\varphi) & \text{if} \quad j = n+1, \dots, n+\kappa_0. \end{cases} \tag{6}$$

Let

$$\Delta = \det \mathcal{A} \text{ and } \Delta_k(\varphi) = \det \mathcal{A}^k(\varphi). \tag{7}$$

The set of all equations of the form

$$(V\varphi)(t) + \lambda \sum_{k=1}^{s} u_k(\varphi) d_k(t) = f(t)$$

will be denoted by H_V^s, where $\{d_k(t)\}_{k=\overline{1,s}}$ is a linearly independent system in X, $0 \neq u_k \in X^*$ $(k = 1, \dots, s)$ are linear functionals, $f(t) \in X$ is a given function, $0 \neq \lambda \in \mathbb{C}$.

Denote

$$H_V^\circ = \{(V\varphi)(t) = f(t) / f(t) \in X\},$$

$$\widetilde{H}_V^s = \bigcup_{l=0}^{s} H_V^l.$$

Evidently the equation (1) belongs to H_V^n.

Theorem 1.1. *For the equation (1), suppose that the condition (3) is satisfied. If $\Delta \neq 0$, then the equation $(\widetilde{K}K\varphi)(t) = (\widetilde{K}f)(t)$ belongs to H_V°, where*

$$\widetilde{K} = I - T_1,$$

$$(T_1\varphi)(t) = \lambda \sum_{k=1}^{n} \frac{\Delta_k(\varphi)}{\Delta} a_k(t).$$

Proof. It is easy to check that $\widetilde{K} \in L(X)$ and $\operatorname{dom}\widetilde{K} = \operatorname{dom}W \supset \operatorname{Im} K$.

We have

$$\begin{aligned}(\widetilde{K}K\varphi)(t) &= (I - T_1)(K\varphi)(t) \\ &= (V\varphi)(t) + \lambda \sum_{k=1}^{n} \alpha_k a_k(t) - \lambda \sum_{k=1}^{n} \frac{\Delta_k(K\varphi)}{\Delta} a_k(t),\end{aligned}$$

where

$$\alpha_k = \int\limits_\Gamma b_k(t)\varphi(t)dt, \quad k = 1, \dots, n.$$

Form (3), we have

$$(WV\varphi)(t) = \varphi(t) + \sum_{k=0}^{\kappa_0} u_k(\varphi)\varphi_k(t) \quad \text{on} \quad \operatorname{dom} V,$$

i.e.,

$$\Big[W(K\varphi - \lambda \sum_{j=1}^{n} \alpha_j a_j)\Big](t) = \varphi(t) + \sum_{k=0}^{\kappa_0} u_k(\varphi)\varphi_k(t) \quad \text{on} \quad \operatorname{dom} V.$$

This implies

$$(WK\varphi)(t) = \varphi(t) + \lambda \sum_{j=1}^{n} \alpha_j (Wa_j)(t) + \sum_{k=0}^{\kappa_0} u_k(\varphi)\varphi_k(t)$$
$$= \varphi(t) + \sum_{k=1}^{n+\kappa_0} \beta_k \psi_k(t) \quad \text{on} \quad \text{on } V, \tag{8}$$

where

$$\beta_k = \begin{cases} \alpha_k & \text{if } k = 1,\dots,n; \\ u_{k-n}(\varphi) & \text{if } k = n+1,\dots,n+\kappa_0; \end{cases}$$
$$\psi_k(t) = \begin{cases} \lambda(Wa_k)(t) & \text{if } k = 1,\dots,n; \\ \varphi_{k-n}(t) & \text{if } k = n+1,\dots,n+\kappa_0. \end{cases}$$

From (4), (5) and (6), we obtain

$$\gamma_j(K\varphi) = \begin{cases} \int\limits_{\Gamma} b_j(t)\big[\varphi(t) + \sum\limits_{k=1}^{n+\kappa_0} \beta_k \psi_k(t)\big]dt & \text{if } j = 1,\dots,n; \\ u_{j-n}\big[\varphi(t) + \sum\limits_{k=1}^{n+\kappa_0} \beta_k \psi_k(t)\big] & \text{if } j = n+1,\dots,n+\kappa_0 \end{cases}$$
$$= \beta_j + \sum_{k=1}^{n+\kappa_0} \beta_k K'_{jk} = \sum_{k=1}^{n+\kappa_0} \beta_k K_{jk}, \quad j = 1,\dots,n+\kappa_0.$$

Thus

$$\Delta_k(K\varphi) = \beta_k \Delta, \quad k = 1,\dots,n+\kappa_0.$$

Hence

$$\sum_{k=1}^{n} \frac{\Delta_k(K\varphi)}{\Delta} a_k(t) = \sum_{k=1}^{n} \beta_k a_k(t) = \sum_{k=1}^{n} \alpha_k a_k(t).$$

This implies

$$(\widetilde{K}K\varphi)(t) = (V\varphi)(t) = (\widetilde{K}f)(t).$$

This equation belongs to H_V°. The theorem is proved.

Corollary 1.1. *For the equation (1) suppose that the condition (3) is satisfied and* $u_k(\varphi) = e_k$ $(k = 1,\dots,\kappa_0)$, $e_k \in C$ *are given complex numbers. If* $\Delta \neq 0$, *then the equation* $(\widetilde{K}_1 K\varphi)(t) = (\widetilde{K}_1 f)(t)$ *belongs to* H_V°, *where*

$$\widetilde{K}_1 = I - T_2,$$
$$(T_2\varphi)(t) = \lambda \sum_{k=1}^{n} \frac{\Delta_k(\varphi)}{\Delta} a_k(t) + \sum_{j=n+1}^{n+\kappa_0} \frac{\Delta_j(\varphi)}{\Delta} \varphi_{j-n}(t).$$

Proof. By similar arguments as seen in the proof of theorem 1.1, we have

$$\Delta_k(K\varphi) = \beta_k \Delta, \quad k = 1,\dots,n+\kappa_0;$$

where

$$\beta_k = \begin{cases} \alpha_k & \text{if } k = 1,\dots,n; \\ u_{k-n}(\varphi) & \text{if } k = n+1,\dots,n+\kappa_0. \end{cases}$$

Hence,

$$(\widetilde{K}_1 K\varphi)(t) = (V\varphi)(t) - \sum_{j=n+1}^{n+\kappa_0} \beta_j \varphi_{j-n}(t) = (\widetilde{K}_1 f)(t).$$

i.e.,

$$(V\varphi)(t) = (\widetilde{K}_1 f)(t) + \sum_{j=1}^{\kappa_0} u_j(\varphi)\varphi_j(t).$$

From our assumption, we have

$$(V\varphi)(t) = (\widetilde{K}_1 f)(t) + \sum_{j=1}^{\kappa_0} e_j\varphi_j(t).$$

This equation belongs to H_V°. The corollary is proved.

Let $[L_{jk}]_{j,k=1}^n$ be an $n \times n$ matrix that is defined by complex numbers L_{jk}, where

$$L_{jk} = \begin{cases} 1 - L'_{jk} & \text{if } \ j = k, \\ -L'_{jk} & \text{if } \ j \neq k, \end{cases}$$

$$L'_{jk} = \lambda \frac{\Delta_j(a_k)}{\Delta}, \quad j, k = 1, \dots, n;$$

where Δ and $\Delta_j(\varphi)$ are defined by (7).

Proposition 1.1. *Suppose that all assumptions of theorem 1.1 are satisfied. If* $\det [L_{jk}]_{j,k=1}^n \neq 0$, *then* $\psi(t)$ *is a solution of (1) if and only if* $\psi(t)$ *is a solution of* $(\widetilde{K}K\varphi)(t) = (\widetilde{K}f)(t)$.

Proof. Suppose that $\psi(t)$ is a solution of (1), i.e. $(K\psi)(t) = f(t)$. Then $f(t) \in \mathrm{Im}K \subset \mathrm{dom}\widetilde{K}$ and $(\widetilde{K}K\psi)(t) = (\widetilde{K}f)(t)$, so $\psi(t)$ is a solution of $(\widetilde{K}K\varphi)(t) = (\widetilde{K}f)(t)$.

Conversely, if $\psi(t)$is a solution of $(\widetilde{K}K\varphi)(t) = (\widetilde{K}f)(t)$, then $(\widetilde{K}g)(t) = 0$, where $g(t) = (K\psi)(t) - f(t)$.
i.e.,

$$g(t) - \lambda \sum_{k=1}^n \frac{\Delta_k(g)}{\Delta} a_k(t) = 0.$$

Put $\dfrac{\Delta_k(g)}{\Delta} = \delta_k$ $(k = 1, \dots, n)$. Acting by linear functionals v_j to both sides of this equation, where $v_j(\varphi) = \dfrac{\Delta_j(\varphi)}{\Delta}$ $(j = 1, \dots, n)$, we get

$$\delta_j - \sum_{k=1}^n \delta_k L'_{jk} = 0, \quad j = 1, \dots, n;$$

i.e.,

$$\sum_{k=1}^n \delta_k L_{jk} = 0, \quad j = 1, \dots, n.$$

Since $\det \left[L_{jk}\right]_{j,k=1}^{n} \neq 0$, we obtain $\delta_j = 0$, $j = 1, \dots, n$. Hence, $g(t) = 0$, i.e., $(K\tilde{\psi})(t) = f(t)$. The proposition is proved.

Consider now the case $\Delta = 0$.

Suppose that r is the rank of matrix $\mathcal{A}$ and $\overline{\mathcal{A}} = \left[K_{\nu_j \mu_k}\right]_{j,k=1}^{r}$ is a submatrix of $\mathcal{A}$ such that

$$\Delta' = \det \overline{\mathcal{A}} \neq 0,$$

where

$$\begin{aligned}
&\nu_k < \nu_j, \ \ \mu_k < \mu_j \ \text{ if } \ k < j; \ j,k \in \{1, 2, \dots, r\},\\
&\nu_1, \nu_2, \dots, \nu_l, \ \ \mu_1, \mu_2, \dots, \mu_m \in \{1, \dots, n\},\\
&\nu_{l+1}, \nu_{l+2}, \dots, \nu_r, \ \ \mu_{m+1}, \dots, \mu_r \in \{n+1, \dots, n+\kappa_0\}.
\end{aligned}$$

Let $\overline{\mathcal{A}}^{\mu_k}(\varphi)$ be an $r \times r$ matrix, obtained from $\overline{\mathcal{A}}$ replacing the k^{th} column by the $\left[\gamma_{\nu_1}(\varphi), \dots, \gamma_{\nu_r}(\varphi)\right]^T$ column, where $\gamma_{\nu_j}(\varphi)$ $(j = 1, \dots, r)$ are defined by (5) and let $\Delta'_{\mu_k}(\varphi) = \det \overline{\mathcal{A}}^{\mu_k}(\varphi)$.

Theorem 1.2. *For the equation (1), suppose that the condition (3) is satisfied. If $\Delta' \neq 0$, then equation $(\tilde{\tilde{K}} K\varphi)(t) = (\tilde{\tilde{K}} f)(t)$ belongs to $\tilde{H}_V^{n+\kappa_0-r}$, where*

$$\tilde{\tilde{K}} = I - T_1',$$

$$(T_1'\varphi)(t) = \lambda \sum_{k=1}^{m} \frac{\Delta'_{\mu_k}(\varphi)}{\Delta'} a_{\mu_k}(t).$$

Proof. It is easy to see that $\tilde{\tilde{K}} \in L(X)$ and $\operatorname{dom}\tilde{\tilde{K}} = \operatorname{dom}W \supset \operatorname{Im} K$.
We have

$$\begin{aligned}
(\tilde{\tilde{K}} K\varphi)(t) &= (I - T_1')(K\varphi)(t)\\
&= (V\varphi)(t) + \lambda \sum_{k=1}^{n} \alpha_k a_k(t) - \lambda \sum_{k=1}^{m} \frac{\Delta'_{\mu_k}(K\varphi)}{\Delta'} a_{\mu_k}(t),
\end{aligned}$$

where $\alpha_k = \int\limits_{\Gamma} b_k(t)\varphi(t)dt$.

By similar arguments as seen in the proof of theorem 1.1, we have

$$\Delta'_{\mu_k}(K\varphi) = \beta_{\mu_k}\Delta' + \Delta''_{\mu_k} + \Delta'''_{\mu_k}, \quad k = 1, \dots, r;$$

where Δ''_{μ_k}, Δ'''_{μ_k} are determinants of the matrices which obtained from $\overline{\mathcal{A}}$ replacing the k^{th} column by the $[U_1, \dots, U_r]^T$ column and the $[U_1', \dots, U_r']^T$ column, respectively, where

$$U_j = \sum_{i \in \{\overline{1,n}\} \setminus \{\overline{\mu_1, \mu_m}\}} \beta_i K_{\nu_j i}, \quad j = 1, \dots, r, \tag{9}$$

$$U_j' = \sum_{i \in \{\overline{n+1, n+\kappa_0}\} \setminus \{\overline{\mu_{m+1}, \mu_r}\}} \beta_i K_{\nu_j i}, \quad j = 1, \dots, r,$$

$$\beta_i = \begin{cases} \alpha_i & \text{if } i = 1, \dots, n;\\ u_{i-n}(\varphi) & \text{if } i = n+1, \dots, n+\kappa_0. \end{cases}$$

Hence, we obtain

$$(\tilde{\tilde{K}} K\varphi)(t) = (V\varphi)(t) + \lambda \sum_{i\in\{\overline{1,n}\}\setminus\{\overline{\mu_1,\mu_m}\}} \alpha_i a_i(t) - \lambda\sum_{k=1}^{m} \frac{\Delta''_{\mu_k}}{\Delta'} a_{\mu_k}(t)$$
$$- \lambda\sum_{k=1}^{m} \frac{\Delta'''_{\mu_k}}{\Delta'} a_{\mu_k}(t) = (\tilde{\tilde{K}} f)(t).$$

This equation can be written in the form

$$(V\varphi)(t) + \lambda \sum_{i\in\{\overline{1,n}\}\setminus\{\overline{\mu_1,\mu_m}\}} \alpha_i c_i(t) + \lambda \sum_{j\in\{\overline{n+1,n+\kappa_0}\}\setminus\{\mu_{m+1},\mu_r\}} \beta_j d_j(t) = (\tilde{\tilde{K}} f)(t),$$

where $c_i(t)$, $d_j(t)$ are defined in terms of $a_k(t)$ $(k = 1,\dots,n)$. Thus, the last equation belongs to $\widetilde{H}_V^{n+\kappa_0-r}$. The theorem is proved.

Corollary 1.2. *Suppose that the condition (3) is satisfied and* $u_k(\varphi) = e_k$ $(k = 1,\dots,\kappa_0)$, $e_k \in C$ *are given complex numbers. If* $\Delta' \neq 0$ *then the equation* $(\tilde{\tilde{K}}_1 K\varphi)(t) = (\tilde{\tilde{K}}_1 f)(t)$ *belongs to* $\widetilde{H}_V^{n-m}$, *where*

$$\tilde{\tilde{K}}_1 = I - T'_2,$$
$$(T'_2\varphi)(t) = \lambda\sum_{k=1}^{m} \frac{\Delta'_{\mu_k}(\varphi)}{\Delta'} a_{\mu_k}(t) + \sum_{j=m+1}^{r} \frac{\Delta'_{\mu_j}(\varphi)}{\Delta'} \varphi_{\mu_j-n}(t).$$

Proof. By similar arguments as seen in the proof of theorem 1.2, we obtain

$$(\tilde{\tilde{K}}_1 K\varphi)(t) = (V\varphi)(t) + \lambda \sum_{i\in\{\overline{1,n}\}\setminus\{\overline{\mu_1,\mu_m}\}} \alpha_i a_i(t) - \lambda\sum_{k=1}^{m} \frac{\Delta''_{\mu_k}}{\Delta'} a_{\mu_k}(t)$$
$$- \lambda\sum_{k=1}^{m} \frac{\Delta'''_{\mu_k}}{\Delta'} a_{\mu_k}(t) - \sum_{j=m+1}^{r} \beta_{\mu_j}\varphi_{\mu_j-n}(t) - \sum_{j=m+1}^{r} \frac{\Delta''_{\mu_j}}{\Delta'} \varphi_{\mu_j-n}(t)$$
$$- \sum_{j=m+1}^{r} \frac{\Delta'''_{\mu_j}}{\Delta'} \varphi_{\mu_j-n}(t) = (\tilde{\tilde{K}}_1 f)(t),$$

where Δ''_{μ_j}, Δ'''_{μ_j} are defined by (9).
This equation can be written in the form

$$(V\varphi)(t) + \lambda \sum_{i\in\{\overline{1,n}\}\setminus\{\overline{\mu_1,\mu_m}\}} \alpha_i c_i(t) - \sum_{j=1}^{\kappa_0} u_j(\varphi) d_j(t) = (\tilde{\tilde{K}}_1 f)(t),$$

where $c_i(t)$, $d_j(t)$ are defined in terms of $a_k(t)$ $(k = 1,\dots,n)$, $\varphi_{\mu_l-n}(t)$ $(l = m+1,\dots,r)$.
From our assumption, we have

$$(V\varphi)(t) + \lambda \sum_{i\in\{\overline{1,n}\}\setminus\overline{\mu_1,\mu_m}\}} \alpha_i c_i(t) = (\tilde{\tilde{K}}_1 f)(t) + \sum_{j=1}^{\kappa_0} e_j d_j(t).$$

This equation belongs to $\widetilde{H}_V^{n-m}$. The corollary is proved.

§2. SOME EXAMPLES OF APPLICATION

Let Γ and X are defined as in §1. Denote by D^+ the domain bounded by Γ and D^- - its complement including the point at infinity. Assume that $0 \in D^+$. Let

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau,$$

$$(S_M\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{M(t,\tau)}{\tau - t} \varphi(\tau) d\tau.$$

1. Characterization of singular integral equations

Consider singular integral equations of the form

$$(K\varphi)(t) := a(t)\varphi(t) + b(t)(S\varphi)(t) + \lambda \int_\Gamma T_n(t,\tau)\varphi(\tau)d\tau = f(t), \tag{10}$$

where $f(t)$, $a(t)$, $b(t) \in X$, $a(t) \pm b(t) \neq 0$ for all $t \in \Gamma$; $T_n(t,\tau)$ is defined by (2).

Denote

$$(K_0\varphi)(t) := a(t)\varphi(t) + b(t)(S\varphi)(t),$$

$$(R_0\varphi)(t) := \frac{1}{a^2(t) - b^2(t)} \left[a(t)\varphi(t) - \frac{b(t)Z(t)}{\pi i} \int_\Gamma \frac{\varphi(\tau)d\tau}{Z(\tau)(\tau - t)} \right],$$

where

$$Z(t) = e^{\Gamma(t)} \sqrt{\frac{a^2(t) - b^2(t)}{t^\kappa}}, \quad \Gamma(t) = \frac{1}{2\pi i} \int_\Gamma \frac{\ln\left(\tau^{-\kappa} \frac{a(\tau) - b(\tau)}{a(\tau) + b(\tau)}\right)}{\tau - t} d\tau, \quad \kappa = \operatorname{Ind} K_0.$$

Denote

$$\kappa_0 = \begin{cases} \kappa & \text{if } \kappa > 0, \\ 0 & \text{if } \kappa \leq 0, \end{cases}$$

$$F_0 = I - R_0 K_0.$$

It is easy to verify the following lemma

Lemma 2.1. *The following equality is holds*

$$(F_0\varphi)(t) = -\sum_{k=0}^{\kappa_0} u_k(\varphi)\varphi_k(t) \quad \textit{on } X,$$

where $\varphi_0(t) = 0$, $\varphi_j(t) = [a^2(t) - b^2(t)]^{-1} b(t) Z(t) t^{j-1}$ $(j = 1, \dots, \kappa_0)$ *and* $u_k(\varphi)$ $(k = 0, \dots, \kappa_0)$ *are linear functionals which are defined by*

$$u_k(\varphi) = \begin{cases} 0 & \textit{if } k = 0, \\ \frac{1}{2\pi i} \int_\Gamma \frac{\tau^{\kappa_0 - k}}{e^{\Gamma^-(\tau)}} \left[-\varphi(\tau) + \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau_1)}{\tau_1 - \tau} d\tau_1 \right] d\tau & \textit{if } k = 1, \dots, \kappa_0, \end{cases}$$

where $\Gamma^-(t)$ *is a boundary value of the function* $\Gamma(z)$ *in* D^-.

It is known that (see [1])

$$\operatorname{Ker} K_0 = \operatorname{lin}(\varphi_0(t), \varphi_1(t), \dots, \varphi_{\kappa_0}(t)). \tag{11}$$

From (11) and lemma 2.1, we obtain

$$K_0 \in W(X), \quad R_0 \in \mathcal{W}_{K_0}.$$

Let $\boldsymbol{\Delta}$ and $\widetilde{\boldsymbol{K}}$ are contructed in the same way as Δ and $\widetilde{K}$ in §1 (where replacing the operators W, V by R_0, K_0)

According to the theorem 1.1, we obtain

Theorem 2.1.. *If $\boldsymbol{\Delta} \neq 0$, then the equation $(\widetilde{\boldsymbol{K}} K\varphi)(t) = (\widetilde{\boldsymbol{K}} f)(t)$ is the characteristic equation.*

2. Generalized characterization of singular integral equations

a) Consider singular integral equations of the form

$$(K\varphi)(t) := \varphi(t) + b(t)(S_k\varphi)(t) + \lambda \int\limits_{|\tau|=1} T_n(t,\tau)\varphi(\tau)d\tau = f(t), \tag{12}$$

where $b(t), f(t) \in X$; $T_n(t,\tau)$ is defined by (2) and

$$(S_k\varphi)(t) = \frac{1}{\pi i} \int\limits_{|\tau|=1} \frac{\tau^{m-1-k} t^k}{\tau^m - t^m} \varphi(\tau)d\tau, \quad 1 < m \in N,\ 0 \leq k \leq m-1.$$

Denote

$$\begin{aligned} &(K_1\varphi)(t) := \varphi(t) + b(t)(S_k\varphi)(t), \\ &(K_2\varphi)(t) := \varphi(t) + \widetilde{b}(t)(S\varphi)(t), \\ &(W_1\varphi)(t) := \varphi(\varepsilon_1 t),\ \varepsilon_1 = \exp(2\pi i/m),\ \varepsilon_j = \varepsilon_1^j \quad (j = 1, \dots, m), \end{aligned}$$

where $\widetilde{b}(t) = \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} b(\varepsilon_j t)$.

Assume that $1 \pm \widetilde{b}(t) \neq 0$ for all t such that $|t| = 1$.

Denote

$$\begin{aligned} &(R_1\varphi)(t) := \varphi(t) - b(t)(S_k R_2 P_k \varphi)(t), \\ &(R_2\varphi)(t) := \frac{1}{1 - \widetilde{b}^2(t)} \Big[\varphi(t) - \frac{\widetilde{b}(t)Z(t)}{\pi i} \int\limits_{|\tau|=1} \frac{\varphi(\tau)d\tau}{Z(\tau)(\tau - t)}\Big], \end{aligned}$$

where

$$P_k = \frac{1}{m}\sum_{j=1}^{m} \varepsilon_k^{m-j} W_1^j, \quad Z(t) = e^{\Gamma(t)} \sqrt{\frac{1 - \widetilde{b}^2(t)}{t^\kappa}},$$

$$\Gamma(t) = \frac{1}{2\pi i} \int\limits_{|\tau|=1} \frac{\ln\Big(\tau^{-\kappa} \frac{1 - \widetilde{b}(\tau)}{1 + \widetilde{b}(\tau)}\Big)}{\tau - t} d\tau, \quad \kappa = \operatorname{Ind} K_2.$$

Denote

$$\kappa_0 = \begin{cases} \kappa & \text{if } \kappa > 0, \\ 0 & \text{if } \kappa \le 0, \end{cases}$$
$$F_1 = I - R_1 K_1.$$

Lemma 2.2.. *The following equality holds*

$$(F_1\varphi)(t) = -\sum_{j=0}^{\kappa_0} u_j(\varphi)\varphi_j(t) \quad on\ X,$$

where $\varphi_0(t) = 0$, $\varphi_j(t) = b(t)(S_k\psi_j)(t)$, $\psi_j(t) = \dfrac{\widetilde{b}(t)Z(t)}{1-\widetilde{b}^2(t)}t^{j-1}$, $j = 1, \ldots, \kappa_0$ and

$$u_j(\varphi) = \begin{cases} 0 & if\ j = 0, \\ \frac{1}{2\pi i}\int\limits_{|\tau|=1} \frac{\tau^{\kappa_0 - j}}{e^{\Gamma^-(\tau)}}\Big[-(P_k\varphi)(\tau) + \frac{1}{\pi i}\int\limits_{|\tau|=1} \frac{(P_k\varphi)(\tau_1)}{\tau_1 - \tau}d\tau_1\Big]d\tau & if\ j = 1, \ldots, \kappa_0; \end{cases}$$

where $\Gamma^-(t)$ *is a boundary value of the function* $\Gamma(z)$ *outside the unit circle.*

Proof. We have

$$\begin{aligned}(F_1\varphi)(t) &= [(I - R_1K_1)\varphi](t) \\ &= \varphi(t) - (K_1\varphi)(t) + b(t)(S_kR_2P_kK_1)(t) \\ &= \varphi(t) - (K_1\varphi)(t) + b(t)(S_kR_2K_2P_k\varphi)(t) \\ &= \varphi(t) - (K_1\varphi)(t) + b(t)[S_k(I - F_2)P_k\varphi](t) \\ &= \varphi(t) - (K_1\varphi)(t) + b(t)(S_kP_k\varphi)(t) - b(t)(S_kF_2P_k\varphi)(t) \\ &\quad -b(t)(S_kF_2P_k\varphi)(t).\end{aligned}$$

On the other hand, according to the lemma 2.1, we obtain

$$(F_2\varphi)(t) = -\sum_{j=0}^{\kappa_0} v_j(\varphi)\psi_j(t),$$

where

$$\psi_j(t) = \begin{cases} 0 & \text{if } j = 0, \\ \frac{\widetilde{b}(t)Z(t)}{1-\widetilde{b}^2(t)}t^{j-1} & \text{if } j = 1, \ldots, \kappa_0, \end{cases}$$

$$v_j(\varphi) = \begin{cases} 0 & \text{if } j = 0, \\ \frac{1}{2\pi i}\int\limits_{|\tau|=1} \frac{\tau^{\kappa_0 - j}}{e^{\Gamma^-(\tau)}}\Big[-\varphi(\tau) + \frac{1}{\pi i}\int\limits_{|\tau|=1} \frac{\varphi(\tau_1)}{\tau_1 - \tau}d\tau_1\Big]d\tau & \text{if } j = 1, \ldots, \kappa_0. \end{cases}$$

Hence

$$(F_1\varphi)(t) = -\sum_{j=0}^{\kappa_0} u_j(\varphi)\varphi_j(t),$$

where

$$u_j(\varphi) = v_j(P_k\varphi), \quad \varphi_j(t) = b(t)(S_k\psi_j)(t).$$

The lemma is proved.

Without loss of generality, we assume that $\varphi_j(t)$, $j = 1, \dots, \kappa_0$ (if $\kappa_0 > 0$) are linearly independent.

It is easy to verify

$$\text{Ker } K_1 = \text{lin}(\varphi_0(t), \varphi_1(t), \dots, \varphi_{\kappa_0}(t)). \tag{13}$$

From (13) and lemma 2.2, we obtain

$$K_1 \in W(X), \quad R_1 \in W_{K_1}.$$

Let $\boldsymbol{\Delta}$ and $\widetilde{\boldsymbol{K}}$ are contructed in the same way as Δ and $\widetilde{K}$ in §1 (where replacing the operators W, V by the operators R_1, K_1).

Theorem 2.2.. *If $\boldsymbol{\Delta} \neq 0$, then the equation $(\widetilde{\boldsymbol{K}} K\varphi)(t) = (\widetilde{\boldsymbol{K}} f)(t)$ is a generalized characteristic equation.*

Proof. According to the Theorem 1.1, we have

$$(\widetilde{\boldsymbol{K}} K\varphi)(t) = (K_1\varphi)(t) = (\widetilde{\boldsymbol{K}} f)(t).$$

This equation has a solution in a closed form (see, e.g.[4]), i.e., it is a generalized characteristic equation.

b) Consider singular integral equations of the form

$$(K\varphi)(t) := a(t)\varphi(t) + b(t)(S\varphi)(t) + \frac{1}{\pi i}\int_\Gamma \frac{M(t,\tau) - M(t,t)}{\tau - t}\varphi(\tau)d\tau +$$

$$+ \lambda \int_\Gamma T_n(t,\tau)\varphi(\tau)d\tau = f(t), \tag{14}$$

where $a(t)$, $b(t)$, $f(t) \in X$, $a(t) \pm b(t) \neq 0$ for all $t \in \Gamma$; $M(t,\tau)$ is a function satisfying Hölder's condition in both variables $(t,\tau) \in \Gamma \times \Gamma$.

Let functions $N_j(t,\tau)$ $(j = 1, 2)$ satisfy the following conditions:

$$N_1(t,t) = N_2(t,t) = 0, \quad t \in \Gamma, \tag{15}$$

$$M(t,\tau) - M(t,t) = N_1(t,\tau) - N_2(t,\tau).$$

Denote

$$(K_3\varphi)(t) := a(t)\varphi(t) + b(t)(S\varphi)(t) + \frac{1}{\pi i}\int_\Gamma \frac{M(t,\tau) - M(t,t)}{\tau - t}\varphi(\tau)d\tau,$$

$$M_1(t,\tau) = \frac{N_1(t,\tau)}{a(t) + b(t)}, \quad M_2(t,\tau) = \frac{N_2(t,\tau)}{a(t) - b(t)},$$

$$M_{1,2}(t,\tau_1) = \frac{1}{\pi i}\int_\Gamma \frac{M_1(t,\tau)M_2(\tau,\tau_1)}{\tau - \tau_1}d\tau - \frac{1}{\pi i}\int_\Gamma \frac{M_1(t,\tau)M_2(\tau,\tau_1)}{\tau - t}d\tau,$$

$$M_{2,1}(t,\tau_1) = \frac{1}{\pi i}\int_\Gamma \frac{M_2(t,\tau)M_1(\tau,\tau_1)}{\tau - \tau_1}d\tau - \frac{1}{\pi i}\int_\Gamma \frac{M_2(t,\tau)M_1(\tau,\tau_1)}{\tau - t}d\tau,$$

$$(R_3\varphi)(t) := \frac{1}{2}\Big[(I + S_{M_2})\frac{1}{a(t) + b(t)} + (I - S_{M_1} - S_{M_{2,1}})\frac{1}{a(t) - b(t)}\Big]\varphi(t) +$$

$$+ \frac{1}{2\pi i}\Big[(I + S_{M_2})\frac{Z(t)}{a(t) + b(t)} - (I - S_{M_1} - S_{M_{2,1}})\frac{Z(t)}{a(t) - b(t)}\Big]\int_\Gamma \frac{\varphi(\tau)d\tau}{Z(\tau)(\tau - t)},$$

where

$$Z(t)=e^{\Gamma(t)}\sqrt{\frac{a^2(t)-b^2(t)}{t^\kappa}},\quad \Gamma(t)=\frac{1}{2\pi i}\int\limits_\Gamma \frac{\ln\left(\tau^{-\kappa}\frac{a(\tau)-b(\tau)}{a(\tau)+b(\tau)}\right)}{\tau-t}d\tau,\ \kappa=\operatorname{Ind}\frac{a(t)-b(t)}{a(t)+b(t)}.$$

Denote

$$\kappa_0=\begin{cases}\kappa & \text{if } \kappa>0,\\ 0 & \text{if } \kappa\le 0,\end{cases}$$
$$F_3=I-R_3K_3.$$

Lemma 2.3. *Let the functions $N_j(t,\tau)$ $(j=1,2)$ satisfy the condition (15) and let $M_1(t,\tau)$, $M_2(t,\tau)$ can be extended to the D^+, D^- in such a manner that they are analytic in both variables in D^+, D^- and continuous in $\overline{D}^+$, $\overline{D}^-$, respectively. If the function $M_{1,2}(t,\tau)$ admits analytic prolongation in both variables in D^+ and continue in $\overline{D}^+$, then the following equality holds*

$$(F_3\varphi)(t)=-\sum_{j=0}^{\kappa_0}u_j(\varphi)\varphi_j(t)\quad \text{on } X,$$

where

$$\varphi_j(t)=\begin{cases}0 & \text{if } j=0,\\ (I-S_{M_1}-S_{M_{2,1}})\frac{Z(t)t^{j-1}}{a(t)-b(t)}-(I+S_{M_2})\frac{Z(t)t^{j-1}}{a(t)+b(t)}, & \text{if } j=1,\dots,\kappa_0;\end{cases}$$
$$u_j(\varphi)=\begin{cases}0 & \text{if } j=0,\\ \frac{1}{2\pi i}\int\limits_\Gamma\frac{\tau^{\kappa_0-j}}{e^{\Gamma^-(\tau)}}\Big[\varphi^-(\tau)+(S_{M_2}\varphi^+)(\tau)\Big]d\tau, & \text{if } j=1,\dots\kappa_0;\end{cases}$$

where $\Gamma^-(t)$ is a boundary value of the function $\Gamma(z)$ in D^- and $\varphi^+(t)=\frac{1}{2}[(I+S)\varphi)](t)$, $\varphi^-(t)=\frac{1}{2}[(-I+S)\varphi](t)$.

Proof. We have

$$\begin{aligned}(F_3\varphi)(t)&=\varphi(t)-(R_3K_3\varphi)(t)\\
&=\varphi(t)-\frac{1}{2}(I+S_{M_2})\frac{1}{a(t)+b(t)}\Big[(K_3\varphi)(t)+\frac{Z(t)}{\pi i}\int\limits_\Gamma\frac{(K_3\varphi)(\tau)}{Z(\tau)(\tau-t)}d\tau\Big]+\\
&\quad+\frac{1}{2}(I-S_{M_1}-S_{M_{2,1}})\frac{1}{a(t)-b(t)}\Big[-(K_3\varphi)(t)+\frac{Z(t)}{\pi i}\int\limits_\Gamma\frac{(K_3\varphi)(\tau)}{Z(\tau)(\tau-t)}d\tau\Big]\\
&=\varphi(t)-\frac{1}{2}(I+S_{M_2})X^+(t)\Big[\frac{(K_3\varphi)(t)}{Z(t)}+\frac{1}{\pi i}\int\limits_\Gamma\frac{(K_3\varphi)(\tau)}{Z(\tau)(\tau-t)}d\tau\Big]+\\
&\quad+\frac{1}{2}(I-S_{M_1}-S_{M_{2,1}})X^-(t)\Big[-\frac{(K_3\varphi)(t)}{Z(t)}+\frac{1}{\pi i}\int\limits_\Gamma\frac{(K_3\varphi)(\tau)}{Z(\tau)(\tau-t)}d\tau\Big],\end{aligned}$$

where $X^+(t)=e^{\Gamma^+(t)}$, $X^-(t)=t^{-\kappa}e^{\Gamma^-(t)}$ ($\Gamma^+(t)$, $\Gamma^-(t)$ are boundary values of the function $\Gamma(z)$ in D^+, D^-, respectively).

On the other hand, it is easy to verify

$$\frac{1}{2}\Big[\frac{(K_3\varphi)(t)}{Z(t)} + \frac{1}{\pi i}\int\limits_{\Gamma}\frac{(K_3\varphi)(\tau)d\tau}{Z(\tau)(\tau-t)}\Big] =$$

$$= \begin{cases} \frac{\varphi^{+}(t)-\left(S_{M_1}\varphi^{-}\right)(t)}{X^{+}(t)} & \text{if } \kappa \leq 0, \\ \frac{\varphi^{+}(t)-\left(S_{M_1}\varphi^{-}\right)(t)}{X^{+}(t)} - \sum\limits_{j=1}^{\kappa}\frac{t^{j-1}}{2\pi i}\int\limits_{\Gamma}\frac{\tau^{\kappa-j}}{e^{\Gamma^{-}(\tau)}}\big[\varphi^{-}(\tau) + (S_{M_2}\varphi^{+})(\tau)\big]d\tau & \text{if } \kappa > 0, \end{cases}$$

$$\frac{1}{2}\Big[-\frac{(K_3\varphi)(t)}{Z(t)} + \frac{1}{\pi i}\int\limits_{\Gamma}\frac{(K_3\varphi)(\tau)d\tau}{Z(\tau)(\tau-t)}\Big] =$$

$$= \begin{cases} \frac{\varphi^{-}(t)+\left(S_{M_2}\varphi^{+}\right)(t)}{X^{-}(t)} & \text{if } \kappa \leq 0, \\ \frac{\varphi^{-}(t)+\left(S_{M_2}\varphi^{+}\right)(t)}{X^{-}(t)} - \sum\limits_{j=1}^{\kappa}\frac{t^{j-1}}{2\pi i}\int\limits_{\Gamma}\frac{\tau^{\kappa-j}}{e^{\Gamma^{-}(\tau)}}\big[\varphi^{-}(\tau) + (S_{M_2}\varphi^{+})(\tau)\big]d\tau & \text{if } \kappa > 0, \end{cases}$$

where $\varphi^{+}(t) = \frac{1}{2}\big[(I+S)\varphi\big](t),\ \varphi^{-}(t) = \frac{1}{2}\big[(-I+S)\varphi\big](t).$

Hence, from our assumptions, it is easy to check that

$$(F_3\varphi)(t) = \begin{cases} 0 & \text{if } \kappa \leq 0 \\ -\sum_{j=1}^{\kappa} u_j(\varphi)\varphi_j(t) & \text{if } \kappa > 0 \end{cases}$$

i.e.,

$$(F_3\varphi)(t) = -\sum_{j=0}^{\kappa_0} u_j(\varphi)\varphi_j(t).$$

The lemma is proved.

Without loss of generality, we assume that $\varphi_j(t)$, $j = 1, \ldots, \kappa_0$ (if $\kappa_0 > 0$) are linearly independent.

It is easy to verify.

$$\text{Ker } K_3 = \text{lin}(\varphi_0(t), \varphi_1(t), \ldots, \varphi_{\kappa_0}(t)). \tag{16}$$

From (16) and lemma 2.3, we obtain

$$K_3 \in W(X), \quad R_3 \in \mathcal{W}_{K_3}.$$

Let $\boldsymbol{\Delta}$ and $\widetilde{\boldsymbol{K}}$ are contructed in the same way as Δ and $\widetilde{K}$ in §1 (where replacing the operators W, V by R_3, K_3).

Theorem 2.3.. *Suppose that all assumptions of lemma 2.3 for $N_j(t,\tau)$, $M_j(t,\tau)$ ($j = 1,2$) and $M_{1,2}(t,\tau)$ are satisfied. If $\boldsymbol{\Delta} \neq 0$, then the equation $(\widetilde{\boldsymbol{K}}K\varphi)(t) = (\widetilde{\boldsymbol{K}}f)(t)$ is a generalized characteristic equation.*

Proof. According to the theorem 1.1, we have

$$(\widetilde{\boldsymbol{K}}K\varphi)(t) = (K_3\varphi)(t) = (\widetilde{\boldsymbol{K}}f)(t).$$

This equation has a solution in a closed form (see, e.g. [4]), i.e., it is a generalized characteristic equation.

Remark. If we apply the proposition 1.1 to the equations (10), (12), (14), we shall get sufficient conditions under which these equations have a solution in a closed form.

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ĐẶC TRƯNG HOÁ VÀ ĐẶC TRƯNG HOÁ SUY RỘNG CỦA PHƯƠNG TRÌNH TÍCH PHÂN KỲ DỊ

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