# ON MODULES $M$ FOR WHICH ALL FINITELY GENERATED MODULES IN $\sigma[M]$ ARE LIFTING

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**Abstract.** *A module $L$ is called a lifting module if for any submodule $K$ of $L$ there is a direct summand $X$ of $L$ such that $X \subset K$ and $K/X$ is small in $L/X$. In this paper we show that for a finitely generated, self-projective module $M$, if all finitely generated modules in $\sigma[M]$ are lifting then $M$ has a decomposition $M = M_1 \oplus \cdots \oplus M_n$, where each $M_i$ is simple or is local with $Soc(M_i) = Rad(M_i)$.*

*In particular, a ring $R$ is semiprimary with $J(R)^2 = 0$ if all finitely generated right $R$-modules are lifting.*

## 1. Introduction and Preliminaries

Throughout, we consider associative rings with identity and all modules are unitary For a module $M$ over a ring $R$ we write $M_R$ to indicate that $M$ is a right $R$-module. Th Socle and the Jacobson radical of $M$ are denoted by $Soc(M)$ and $Rad(M)$, respectively The Jacobson radical of a ring $R$ is denoted by $J(R)$.

For a module $M_R$, we denote by $\sigma[M]$ the subcategory of Mod-$R$ whose objects ar submodules of $M$-generated modules (cf. Wisbauer [10]).

Let $M$ be a right $R$-module. We consider following conditions:

($alE$) All modules in $\sigma[M]$ are extending.

($fE$) All finitely generated modules in $\sigma[M]$ are extending.

($alL$) All modules in $\sigma[M]$ are lifting.

($fL$) All finitely generated modules in $\sigma[M]$ are lifting.

It is well known that, all modules in $\sigma[M]$ are extending if and only if every modul in $\sigma[M]$ is a direct sum of modules of length at most 2, if and only if all modules in $\sigma[M$ are lifting (cf. [2, 13.3], [3] and [9]). Recently, Dinh Van Huynh and others [4] showe that a finitely generated module $M$ is noetherian if all finitely generated modules in $\sigma[M$ are extending.

In this note we consider a finitely generated self-projective module $M$ for which al finitely generated modules in $\sigma[M]$ are lifting. We also discuss the following: When ar the conditions ($alL$) and ($fL$) equivalent?

Recall that a module $M$ is called an extending module or $CS$-module if ever submodule of $M$ is essential in a direct summand of $M$. A submodule $A$ of $M$ is calle all in $M$, written $A \ll M$, if $A + B \neq M$ for every proper submodule $B$ of $M$. Module is called a lifting module if every submodule $K$ of $M$ lies above a direct summand $M$, i.e. there is a direct summand $X$ of $M$ such that $X \subset K$ and $K/X$ is small in $/X$. It is clear that, $M$ is lifting if and only if for any submodule $K$ of $M$, there exists a composition $M = M_1 \oplus M_2$ where $M_1 \subset K$, $M_2 \cap K$ is small in $M_2$. For characterizations lifting property refer to [10, 41.11 and 41.12] or to [6]. We call a non-zero $R$-module hollow if every proper submodule is small in $M$. A module $M$ is called a local module $M$ has the largest proper submodule, i.e. a proper submodule which contains all other oper submodules. In this case, the largest submodule has to be equal to the Jacobson lical of $M$ and $Rad(M)$ is small in $M$ (cf. [10, 41.3; 41.4] or [6, Definition 4.1]).

## The results

**A. Condition** $(fL)$: All finitely generated modules in $\sigma[M]$ are lifting.

For convenience, we say that a module $M$ satisfies the condition $(fL)$ if all finitely nerated modules in $\sigma[M]$ are lifting.

**Remark 1.** Let $M_R$ be a finitely generated, self-projective module. Assume that is lifting. Then every submodule of $M$ has a supplement by [6, Proposition 4.8]. cording to Proposition 4.39 in [6], $M$ is discrete. By [6, Theorem 4.15] therefore $M$ has ecomposition $M = \oplus_{i=1}^{n} H_i$, where each $H_i$ is hollow. Since $M$ is finitely generated, -projective, each $H_i$ is self-projective local, hence the endomorphism ring $End_R(H_i)$ is al by [10, 41.19].

Now we consider modules with local endomorphism rings. We first prove the following result.

**Lemma 1.** *Let $L$ be a right $R$-module with the local endomorphism ring $End_R(L)$ l $S$ be a simple right $R$-module. Assume $L \oplus S$ is lifting. Then for any diagram*

$$\begin{array}{ccc} & & S \\ & & \downarrow h \\ L & \xrightarrow{p} & L/K, \end{array}$$

*which $K$ is a submodule of $L$, $p$ is natural epimorphism, $h$ is a non-zero homomorphism l it is not epimorphism, there exists a homomorphism $\tilde{h} : S \longrightarrow L$ such that $p\tilde{h} = h$.*

**Proof.** Assume that $h(S) = C/K$ with a submodule $C$ of $L$, $K \subset C$. Since $h$ is epimorphism, $C \neq L$. Denote $L \oplus S$ by $M$ and set

$$N := \{(x, y) \ / \ x \in C,\ y \in S,\ p(x) = h(y)\}.$$

It is clear that $N$ is a submodule of $L \oplus S$, $N \cap S = 0$ and $N \cap L = K$. Let $\pi_s$ ote the projection of $M = L \oplus S$ onto $S$ and $\pi_\ell$ denote the projection of $M$ onto $L$.

Then we easily see that $\pi_s(N) = S$; $\pi_\ell(N) = C$. By the hypothesis, $M$ is lifting, henc
$M$ has a decomposition $M = M_1 \oplus M_2$ with $M_1 \subset N$ and $M_2 \cap N \ll M_2$.

If $M_1 = 0$ then $N \ll M$. By [5, 51.3 Lemma], $\pi_s(N) \ll S$. On the other hand
$\pi_s(N) = S$, a contradiction. This argument shows that $M_1 \neq 0$.

Consider the decomposition $M = L \oplus S$. Note that $L$ and $S$ has local endomorphism
rings. By [1, 12.7 Corollary], the decomposition $M = L \oplus S$ complements direct summand
Then, for direct summand $M_1$, $M$ must have decomposition $M = M_1 \oplus L$ or $M = M_1 \oplus S$

Suppose that $M = M_1 \oplus S$. Consider the projection $\pi_\ell : M = L \oplus S \longrightarrow L$. W
have $\pi_\ell(M_1) \subset \pi_\ell(N) = C \neq L$. On the other hand, $L = \pi_\ell(M) = \pi_\ell(M_1 \oplus S) = \pi_\ell(M_1$
a contradiction. Thus $M$ has the decomposition $M = M_1 \oplus L$. Consider the projectio
$\pi_s : M = L \oplus S \longrightarrow S$. Since $M_1 \cap L = 0$, $g := \pi_s\big|_{M_1} : M_1 \longrightarrow S$ is an isomorphism. Se

$$\eta := \pi_\ell\big|_{M_1} : M_1 \longrightarrow L \quad \text{and} \quad \tilde{h} = \eta g^{-1} : S \longrightarrow L.$$

Let $y$ be any element in $S$. Assume $y = g(m)$ for an $m \in M_1 \subset N$. By the definition of
and $g$, we have $m = (x, y)$ with an $x \in C$ such that $h(y) = p(x)$. Since $\tilde{h}(y) = \eta(g^{-1}(y))$
$\eta(m) = \eta(x, y) = x$, $(p\tilde{h})(y) = p(x) = h(y)$. Thus $p\tilde{h} = h$. The proof of Lemma 2.1
complete.

Now, we consider a module $M_R$ satisfying the Condition $(fL)$. We have the firs
result.

**Proposition 2.** *Let $M$ be a right $R$-module satisfying the Condition $(fL)$. Assum that $L$ is a self-projective local module in $\sigma[M]$. Then $L$ is simple or $Soc(L) = Rad(L)$.*

**Proof.** By [10, 41.19], the endomorphism ring $End_R(L)$ is local. We proceed i
two steps.

*Step 1.* First we prove that $Soc(L)$ is essential in $L$. Since $L$ is local, $Soc(L) =$
if $Soc(L) \not\subset Rad(L)$. In this case, $L$ is simple.

Assume $Soc(L) \subset Rad(L)$. This implies that $Rad(L) \neq 0$. Let $x$ be any non-zer
element in $Rad(L)$ and set $L_1 := xR$. Then $L_1$ has a maximal submodule, say $K$, such tha
the factor module $L_1/K$ is simple. Consider the module $L \oplus (L_1/K)$, which is a finitel
generated module in $\sigma[M]$. By hypothesis for $M$, the module $L \oplus (L_1/K)$ is lifting.

Consider the following diagram

$$\begin{array}{ccc} & & L_1/K \\ & & \downarrow h \\ L & \xrightarrow{p} & L/K, \end{array}$$

in which $p$ is natural epimorphism, $h$ is inclusion homomorphism. Apply Lemma 1,
follows that there exists a homomorphism $\tilde{h} : L_1/K \longrightarrow L$ such that $p\tilde{h} = h$. Therefor
$L$ contains a simple submodule $S_1 = \tilde{h}(L_1/K)$ such that $(S_1 + K)/K = L_1/K$. We easil

see that $L_1 = S_1 \oplus K$. This argument shows that $Soc(L)$ is essential in $Rad(L)$, hence is essential in $L$.

*Step 2.* We will show next that $L$ is simple or $Soc(L) = Rad(L)$.

Suppose that $L$ is not simple. Then $Soc(L) \subset Rad(L)$. We show that $Soc(L) = Rad(L)$.

Note that $L$ is self-projective. Hence, by [10, 18.2(4) and 21.2], the factor module $L/SocL$ is also a self-projective module. By the same argument as in the Step 1, $Soc(L/Soc(L))$ is essential in $L/Soc(L)$.

If $L/Soc(L)$ is simple then $Soc(L) = Rad(L)$.

Suppose that $L/Soc(L)$ is not simple. Let $T$ be a simple submodule of $L/Soc(L)$. Then the inclusion homomorphism $q : T \longrightarrow L/Soc(L)$ is not epimorphism. By the Condition $(fL)$ for $M$, module $T \oplus L$ is lifting. Consider the following diagram

$$\begin{array}{ccc} & & T \\ & & \downarrow q \\ L & \xrightarrow{p} & L/\ Soc(L), \end{array}$$

where $p$ is natural projection. From Lemma 1 we conclude that there exists a homomorphism $g : T \longrightarrow L$ such that $pg = q$. Then $g(T)$ is a simple submodule of $L$. It implies $pg = 0$, a contradiction. Therefore $L/Soc(L)$ is simple, and thus $Soc(L) = Rad(L)$. □

**Therem 3.** Let *M be a finitely generated, self-projective right R-module. Assume that M satisfies the Condition* $(fL)$. *Then M has a decomposition*

$$M = M_1 \oplus \cdots \oplus M_n,$$

*where each* $M_i$ *is simple or local with* $Soc(M_i) = Rad(M_i)$, $i = 1, 2, ..., n$.

**Proof.** By assumption, $M$ is lifting. According to Remark 1, the module $M$ has a decomposition $M = M_1 \oplus \cdots \oplus M_n$, where every $M_i$ is a self-projective local module in $\sigma[M]$ for which the endomorphism ring $End_R(M_i)$ is local. By Proposition 2, each $M_i$ is simple or local with $Soc(M_i) = Rad(M_i)$, $i = 1, ..., n$. □

Now, putting $M_R = R_R$, then we need only to assume that all finitely generated right $R$-modules are lifting to obtain following result.

**Theorem 4.** *Let R be any ring for which all finitely generated right R-modules are lifting. Then R is a semiprimary ring with* $J(R)^2 = 0$.

**Proof.** We recall that a ring $R$ is called semiprimary if the factor ring $R/J(R)$ is semi-simple and the Jacobson radical $J(R)$ is nilpotent, i.e., $J(R)^k = 0$ with a positive integer $k$. □

Applying Theorem 3, the module $R_R$ has a decomposition $R_R = R_1 \oplus \cdots \oplus R_n$, where each $R_i$ is either a simple right $R$-module or a self-projective local right $R$-module with $Soc(R_i) = Rad(R_i)$. Then

$$J(R) = Rad(R_R) = \bigoplus_{i=1}^{n} Rad(R_i) \subset Soc(R_R).$$

It is clear that the right $R$-module $R_R/J(R)$ is semi-simple. From this, the factor ring $R/J(R)$ is semi-simple. Because $J(R) \subset Soc(R_R)$, $J(R)^2 = 0$ by [5, 9.3.5]. It follows that $R$ is semiprimary. □

**Remark 2.** For the module $M$ in Theorem 3, $Soc(M)$ is finitely generated if and only if $M_R$ has finite uniform dimension. In this case, $M_R$ is of finite length. For a ring $R$, we get the following Corollary from Theorem 4.

**Corollary 5.** *Let $R$ be any ring for which every finitely generated right $R$-module is lifting. Assume $R$ has finite right uniform dimension. Then $R$ is right artinian with $J(R)^2 = 0$.*

**B. When are Conditions $(fL)$ and $(alL)$ equivalent?**

Recall that for a ring $R$, every right $R$-module is extending if and only if $R$ is a generalized uniserial ring with $J(R)^2 = 0$, if and only if every right $R$-module is lifting (cf. [2, 13.5] and [9, 2.5]). On the other hand, K. Oshiro [8] obtained a characterization for a generalized unserial ring: A ring $R$ is a generalized uniserial ring if and only if every extending right $R$-module is lifting. From this and above results we have following result.

**Proposition 6.** *Let $R$ be a generalized uniserial ring. The following statements are equivalent:*

*(1) Every finitely generated right $R$-module is extending;*
*(2) Every finitely generated right $R$-module is lifting;*
*(3) Every right $R$-module is lifting;*
*(4) Every right $R$-module is extending.*

Proof. Straightforward. □

Now we generalization above result for a finitely generated self-projective right $R$-module to obtain the following

**Theorem 7.** *Let $M$ be a finitely generated self-projective right $R$-module. The following statements are equivalent:*

*(1) $M$ satisfies Conditions $(fL)$ and $(fE)$;*

*(2) $M$ satisfies Condition $(fL)$ and every finitely generated, indecomposable module in $\sigma[M]$ is extending;*

*(3) $M$ satisfies Condition $(alL)$;*

*(4) $M$ satisfies Condition $(alE)$.*

**Proof.** (1) $\Rightarrow$ (2). It is clear.

(2) ⇒ (3). By the assumption and Theorem 3, $M$ has a decomposition $M = M_1 \oplus \cdots \oplus M_n$, where each $M_i$ is simple or local with $Soc(M_i) = Rad(M_i)$. Since $M_i$ is extending, $M_i$ is uniform and hence $Soc(M_i)$ is simple. Then $M$ is of finite length.

Consider a finitely generated, indecomposable module $K$ in $\sigma[M]$. Then $K$ is lifting and extending. This implies that $Soc(K)$ is simple and $K$ is local, hence $K/Rad(K)$ is simple. By [10, 55.14], every module in $\sigma[M]$ is serial, in particular, every indecomposable module in $\sigma[M]$ is uniserial.

Let $N$ be any indecomposable module in $\sigma[M]$. Then the $M$-injective hull $\widehat{N}$ of $N$ is uniserial. Suppose that $N$ is not simple and $\widehat{N} \neq N$. This implies that in $\widehat{N}$, there exists a composition serial:

$$0 \subset Soc(N) \subset N_1 \subset N_2,$$

where $N_1$ and $N_2$ are finitely generated submodules of $\widehat{N}$. According to [10, 55.14], $N_2$ is self-injective hence the endomorphism ring $End(N_2)$ is local. By assumption, $N_2 \oplus N_1/Soc(N)$ is lifting.

Consider the following diagram

$$\begin{array}{ccc} & & N_1/\ Soc(N) \\ & & \downarrow h \\ N_2 & \xrightarrow{p} & N_2/\ Soc(N), \end{array}$$

where $h$ is the inclusion homomorphism, $p$ is natural epimorphism. Applying Lemma 1, there exists a homomorphism $\tilde{h} : N_1/Soc(N) \longrightarrow N_2$ such that $p\tilde{h} = h$. Since $N_2$ is uniserial, $p\tilde{h} = 0$, a contradiction.

Therefore, if $N$ is not simple then $N = \widehat{N}$ and $N$ is a module of length 2.

Now, from this, the module $M$ is a direct sum of simple modules and modules of length 2, which are $M$-injective and $M$-projective. Thus every module in $\sigma[M]$ is extending, also is lifting by [2, 13.3] and [9, 2.5].

(3) ⇒ (1). From [2, 13.3] and [9, 2.5], it is clear. □

Put $M_R = R_R$, we have the following.

**Corollary 8.** *The following statements are equivalent for a ring $R$:*

*(1) Every finitely generated right $R$-module is lifting and is extending;*

*(2) Every finitely generated right $R$-module is lifting and every finitely generated, indecomposable right $R$-module is extending;*

*(3) Every right $R$-module is lifting;*

*(4) Every right $R$-module is extending.*

In following theorem, the module $M$ is not necessary self-projective.

**Theorem 9.** *Let $M$ be a finitely generated module. The following assertions are equivalent:*

*(1) M satisfies Conditions $(fE)$ and $(fL)$;*

*(2) M satisfies Conditions $(fE)$ and every finitely generated, indecomposable mod ule in $\sigma[M]$ is lifting;*

*(3) M satisfies Conditions $(alE)$;*

*(4) M satisfies Conditions $(alL)$.*

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) By the assumption and [4, Theorem 5], $M$ is extending noetherian. Hence $M$ has a decomposition

$M = \oplus_{i=1}^{n} M_i$, where each $M_i$ is a finitely generated uniform module.

Let $N$ be any finitely generated uniform module in $\sigma[M]$. Then $N$ is lifting, hence is local. This implies that every uniform module in $\sigma[M]$ is uniserial by [10, 55.1]. Since $M$ is noetherian, $N$ is noetherian and every submodule of $N$ is too. Set

$$N_0 := N,\ N_1 = Rad(N),\ N_2 = Rad(N_1), \ldots.$$

We have a descending chain

$$N_0 = N \supset N_1 \supset N_2 \supset N_3 \supset \cdots,$$

where each $N_k/N_{k+1}$ is simple, $k = 0, 1, 2, \ldots$.

If $N_2 \neq 0$ then we have following composition series

$$(N/N_3) \supset (N_1/N_3) \supset (N_2/N_3) \supset 0.$$

Note that $N/N_3$ is uniserial. Now, according to [2, 7.4 Corollary], $(N/N_3) \oplus (N_1/N_2$ is not a extending module. This is contrary to the assumption (2). Thus $N_2 = 0$ and hence $N$ is a module of length at most 2.

From this, $M$ is of finite length. By [10, 55.14], every module in $\sigma[M]$ is a serial module. By above argument for the module $N$, we conclude that every uniserial module in $\sigma[M]$ is a module of length at most 2. Now, all modules in $\sigma[M]$ are extending by [2, 13.3].

(3) $\Rightarrow$ (1) follows from [2, 13.3] and [9, 2.5]. □

**Acknowledgements**

The author would like to thank Prof. Dinh Van Huynh for his useful suggestions during the preparation of this paper.

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TẠP CHÍ KHOA HỌC ĐHQGHN, KHTN, t.XVI, $n^0 2$ - 2000

## VỀ CÁC MÔĐUN M CÓ MỌI MÔĐUN HỮU HẠN SINH TRONG $\delta[M]$ LÀ LIFTING

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Một môđun L được gọi là lifting nếu với môđun con tùy ý $K$ của $L$ có một hạng tử rực tiếp $X$ của $L$ sao cho $X \subset K$ và $K/X$ là môđun con nhỏ trong $L/X$. Trong bài này, hứng tôi chứng minh rằng với một môđun hữu hạn sinh tự xạ ảnh $M$, nếu mọi môđun ữu hạn sinh trong $\delta[M]$ là lifting thì $M$ có một sự phân tích $M = M_1 \oplus ... \oplus M_n$, trong ló mỗi một $M_i$ là đơn hoặc là địa phương với $Soc(M_i) = Rad(M_i)$.

Nói riêng, một vành $R$ là nửa nguyên so với $J(R)^2 = 0$ nếu mọi $R$-môđun phải ữu hạn sinh là lifting