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A CAUCHY TYPE FREE BOUNDARY PROBLEM FOR THE HEAT EQUATION

Nguyen Dinh Binh
Faculty of Mathematics
Hanoi University of Technology

.bstract. *In this paper we shall consider a free boundary problem for the heat equation. :xistence and uniquenees of the solution of the problem are proved by the method of semi-iscretization with respect to t.*

Introduction

We deal with the free boundary problem

Problem (P). Find a pair $(s(t), u(x,t))$ such that

1. $s(t) \in C^1[0,T]$, $s(t) > 0$ for $t \in [0,T]$, $s(0) = b > 0$,

2. $u(x,t) \in C^{2,1}(\overline{D}_T)$, for $D_T = \{(x,t) : 0 < x < s(t),\ 0 < t < T\}$, $\overline{D}_T$ is the osure of D_T.

3. The following equation and conditions are satisfied

$$u_{xx} - u_t = 0 \qquad \text{in } D_T, \tag{1.1}$$
$$u(x,0) = \varphi(x), \qquad 0 \le x \le s(0) = b, \tag{1.2}$$
$$u_x(0,t) = 1, \qquad 0 \le t \le T, \tag{1.3}$$
$$u(s(t),t) = f(t), \qquad 0 \le t \le T, \tag{1.4}$$
$$u_x(s(t),t) = g(t), \qquad 0 \le t \le T, \tag{1.5}$$

here b, T being given positive numbers, $f(t), g(t), \varphi(x)$ being given functions.

The existence of the solution of this problem is established by using the method of midiscretization with respect to t. This scheme has been used by Gary G. Sackett [2], guyen Dinh Tri [3] for the somewhat different problems. An other method for solving auchy type free boundary problem for nonlinear parabolic equations can be found in].

The following assumptions are made:

A.

$$\varphi(x) \in C^4[0,b],\ f(t) \in C^2[0,T],\ g(t) \in C^2[0,T],$$
$$\varphi''(x) < 0,\ \ \varphi''(x) - f'(0) < 0 \ \text{ for } \ 0 \le x \le b,$$
$$f'(t) < 0,\ \ g(t) > 0,\ \ g'(t) > 0 \ \text{ for } \ 0 \le t \le T.$$

B.

$$\varphi'(b) = g(b),\ \varphi'(0) = 1,\ \varphi(b) = f(0).$$

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II. Uniqueness of the solution

Theorem 1. *The following assumptions are made:* $f'(t),\ g(t),\ g'(t), \varphi"(x)$ *are continuous,* $f'(t) < 0,\ g(t) > 0,\ g'(t) > 0,\ \varphi"(x) < 0$ *for* $0 \le t \le T,\ 0 \le x \le b$ *, then the problem (P) cannot have more than one solution such that* $s'(t) > 0$.

Proof. The function $z = u_t$ satisfies

$$z_t - z_{xx} = 0 \qquad \text{in } D_T, \tag{2.1}$$
$$z_x(0,t) = 0, \qquad 0 \le t \le T, \tag{2.2}$$
$$z(x,0) = \varphi"(x) < 0, \qquad 0 \le x \le b, \tag{2.3}$$
$$z(s(t),t) = f'(t) - g(t)s'(t) < 0, \quad 0 \le t \le T. \tag{2.4}$$

By the maximum principle, $z(x,t)$ takes its maximum value at $x = s(t)$ or for t=0 because $z_x(0,t) = 0$. From (2.3) and (2.4) $z(x,t) = u_t(x,t) < 0$ in $\overline{D}_T$, hence $u_{xx}(x,t) < 0$ in $\overline{D}_T$ and $u_x(x,t)$ is a decreasing function with respect to x. Hence $u_x(x,t) > 0$ in $\overline{D}_T$ because $u_x(s(t),t) = g(t) > 0$.

Now assume that there exist two solutions of problem (P): $\{s_1(t), u_1(x,t)\}$ and $\{s_2(t), u_2(x,t)\}$. Then $v(x,t) = u_1(x,t) - u_2(x,t),\ s(t) = \min\limits_{0 \le t \le T}\{s_1(t), s_2(t)\}$, satisfying the following equation and conditions

$$\begin{aligned} v_t - v_{xx} &= 0 \quad \text{in } D_T = \{(x,t) : 0 < x < s(t),\ 0 < t < T\}, \\ v_x(0,t) &= 0, \quad 0 \le t \le T, \\ v(x,0) &= 0, \quad 0 \le x \le s(0). \end{aligned}$$

Hence $v(x,t)$ can reach positive maximum or negative minimum only on $x = s(t)$. If $s(t) = s_1(t)$, we have

$$\begin{aligned} v(s(t),t) &= u_1(s_1(t),t) - u_2(s_1(t),t) = f(t) - u_2(s_1(t),t) \\ &= u_2(s_2(t),t) - u_2(s_1(t),t) = |s_2(t) - s_1(t)| u_{2x}(\xi(t),t) \ge 0, \end{aligned} \tag{2.5}$$
$$\begin{aligned} v_x(s(t),t) &= u_{1x}(s_1(t),t) - u_{2x}(s_1(t),t) = g(t) - u_{2x}(s_1(t),t) \\ &= u_{2x}(s_2(t),t) - u_{2x}(s_1(t),t) = |s_2(t) - s_1(t)| u_{2xx}(\eta(t),t) \le 0, \end{aligned} \tag{2.6}$$

It's a contradiction . We get the same conclusion if $s(t) = s_2(t)$. □

III. Existence of the solution

Let $t_n = n\Delta t$, with $\Delta t > 0$

$$u_0(x) = \varphi(x), \quad u_n(x) = u(x,t_n), \quad s_n = s(t_n), \quad s_0 = b.$$

The problem (P) is approximated by the following problems (P_n).

Problem (P_n). Find the functions $u_n(x)$ and s_n such that

$$u_n'' = \frac{u_n(x) - u_{n-1}(x)}{\Delta t}, \quad 0 \le x \le s_n, \tag{3.1}$$

$$u_n'(0) = 1, \tag{3.2}$$

$$u_n(s_n) = f(t_n); \tag{3.3}$$

$$u_n'(s_n) = g(t_n). \tag{3.4}$$

We have to prove:

1/ The existence and uniqueness of the solution of problems (P_n),

2/ The uniform boundedness of some quantities, to be used for establishing the
ıvergence of the approximation scheme,

3/ The convergence of the scheme.

Proposition 1. *Under the assumptions A and B, the problem (P_n) has an unique ution $\{s_n, u_n(x)\}$ such that $s_n \ge s_{n-1}, \forall n$.*

Proof. We put $u_n(x) = f(t_n) + g(t_n)(x - s_n)$ for $x > s_n, \ n \ge 0$.

The general solution of the equation (3.1) is:

$$u_n(x) = A_n sh\frac{x}{\sqrt{\Delta}} + B_n ch\frac{x}{\sqrt{\Delta}} - \frac{1}{\sqrt{\Delta t}}\int_0^x sh\frac{x-\xi}{\sqrt{\Delta t}} u_{n-1}(\xi)d\xi. \tag{3.5}$$

quiring that u_n satisfy the conditions (3.2)-(3.4) we have

$$A_n = \sqrt{\Delta t},$$

$$f(t_n) = \sqrt{\Delta t} sh\frac{s_n}{\sqrt{\Delta t}} + B_n ch\frac{s_n}{\sqrt{\Delta t}} - \frac{1}{\sqrt{\Delta t}}\int_0^{s_n} sh\frac{s_n-\xi}{\sqrt{\Delta t}} u_{n-1}(\xi)d\xi, \tag{3.6}$$

$$g(t_n) = ch\frac{s_n}{\sqrt{\Delta t}} + \frac{1}{\sqrt{\Delta t}} B_n sh\frac{s_n}{\sqrt{\Delta t}} - \frac{1}{\sqrt{\Delta t}}\int_0^{s_n} ch\frac{s_n-\xi}{\sqrt{\Delta t}} u_{n-1}(\xi)d\xi \tag{3.7}$$

ninating A_n, B_n from (3.6),(3.7) , we get

$$f(t_n) sh\frac{s_n}{\sqrt{\Delta}} - \sqrt{\Delta t} g(t_n) ch\frac{s_n}{\sqrt{\Delta}} = -\sqrt{\Delta t} + \frac{1}{\sqrt{\Delta t}}\int_0^{s_n} ch\frac{\xi}{\sqrt{\Delta t}} u_{n-1}(\xi)d\xi.$$

have to prove the existence of the solution of the equations

$$_n(s) = f(t_n) sh\frac{s}{\sqrt{\Delta}} - \sqrt{\Delta t} g(t_n) ch\frac{s}{\sqrt{\Delta}} + \sqrt{\Delta t} - \frac{1}{\sqrt{\Delta t}}\int_0^{s} ch\frac{\xi}{\sqrt{\Delta t}} u_{n-1}(\xi)d\xi = 0. \tag{3.8}$$

s possible to check that $\forall n$

$$\Phi_n(s_{n-1}) > 0, \Phi_n'(s_{n-1}) < 0, \Phi_n''(s) < 0 \text{ for } s > s_{n-1}. \tag{3.9}$$

Because (3.9), the equation $\Phi_n(s) = 0$ has an unique solution $s_n > s_{n-1}, \forall n$.

The proof is completed by induction.

Proposition 2. (Estimation of $z_n(x) := \frac{u_n(x) - u_{n-1}(x)}{\Delta}$). *Under the assumptio A and B, we have* $\forall n$, *for* $x \in [0, s_n]$

$$|z_n(x)| \leq M_1, \tag{3.10}$$

in this paper we denote by M_i *the constants which don't depend on* Δt.

Proof. $z_n(x)$ satisfies the following equation and conditions

$$z_n''(x) = \begin{cases} \dfrac{z_n(x) - z_{n-1}(x)}{\Delta t} & \text{for} \quad 0 \leq x \leq s_{n-2}, \\ \dfrac{z_n(x)}{\Delta t} & \text{for} \quad s_{n-2} \leq x \leq s_{n-1}, \end{cases} \tag{3.11}$$

$$z_0(x) = \varphi''(x) < 0, \quad 0 \leq x \leq b,$$

$$z_n'(0) = 0,$$

$$z_n(s_{n-1}) = \frac{[u_{n-1}(s_{n-1}) - u_{n-2}(s_{n-1})]}{\Delta t} = \frac{\Delta f(t_{n-1})}{\Delta t} - g(t_{n-2})\frac{\Delta s_n}{\Delta t} < 0, \tag{3.12}$$

$$z_n'(s_{n-1}) = \frac{[u_{n-1}'(s_{n-1}) - u_{n-2}'(s_{n-1})]}{\Delta t} = \frac{g(t_{n-1}) - g(t_{n-2})}{\Delta t} = \frac{\Delta g(t_n)}{\Delta t} > 0, \tag{3.13}$$

$$z_n'(s_{n-2}) = \frac{[u_{n-1}'(s_{n-2}) - u_{n-2}'(s_{n-2})]}{\Delta t} = -u_{n-1}''(\eta)\frac{\Delta s_{n-1}}{\Delta t} + \frac{\Delta g(t_{n-2})}{\Delta t} > 0. \tag{3.14}$$

Therefore, we can deduce that $z_n < 0$ and takes its negative minimum value the endpoints or for t=0. But because $z_n'(s_{n-2}) > 0$, $z_n'(0) = 0$ it must in fact take i negative minimum for $t = 0$. We have $\forall n, 0 \leq x \leq S_n$

$$|z_n(x)| = |u''_n(x)| \leq \max_{0 \leq x \leq b} |\varphi''(x)| = M_1 \tag{3.15}$$

Corollary. *We have* $\forall n$, *for* $x \in [0, s_n]$

$$\left|\frac{\Delta s_n}{\Delta t}\right| \leq \left|\frac{s_n - s_{n-1}}{\Delta t}\right| \leq M_2, \tag{3.16}$$

$$|u_n'(x)| \leq M_3, \tag{3.17}$$

$$|u_n(x)| \leq M_4, \tag{3.18}$$

$$|s_n| \leq M_5. \tag{3.19}$$

Proposition 3. (Estimation of $z_n'(x) = u_n'''(x)$). *Under the assumptions A and we have* $\forall n$, for $x \in [0, s_{n-1}]$

$$|z_n'(x)| = |u_n'''(x)| \leq M_6 \tag{3.20}$$

Proof. $q_n = z_n'$ satisfies the following equation and conditions

$$q_n''(x) = \begin{cases} \dfrac{q_n(x) - q_{n-1}(x)}{\Delta t} & \text{for } 0 \le x \le s_{n-2}, \\ \dfrac{q_n(x)}{\Delta t} & \text{for } s_{n-2} \le x \le s_{n-1}, \end{cases} \tag{3.21}$$

$$|q_0(x)| = |\varphi'''(x)|, \ 0 \le x \le b, \tag{3.22}$$

$$|q_n(0)| = |z_n'(0)| = 0, \tag{3.23}$$

$$|q_n(s_{n-2})| = |z_n'(s_{n-2})| = \frac{|u_{n-1}'(s_{n-2}) - u_{n-2}'(s_{n-2})|}{\Delta t}$$
$$= \left| -u_{n-1}''(\eta)\frac{\Delta s_{n-1}}{\Delta t} + \frac{\Delta g(t_{n-2})}{\Delta t} \right| \le M_1 . M_2 + \max_{0 \le t \le T} |g'(t)| = M_6. \tag{3.24}$$

ombining (3.21)-(3.24) we get

$$|q_n| = |z_n'| = |u_n'''| \le \max\{\max_{0 \le x \le b} |\varphi'''(x)|, \ M_6, \ \max_{0 \le t \le T} |g'(t)|\} = M_7 \ \text{ for } 0 \le x \le s_{n-1}. \tag{3.25}$$

The last quantity to be dealt with is an estimate for $\dfrac{\Delta^2 u_n}{\Delta t^2}$. Put $\omega_n = \dfrac{\Delta z_n}{\Delta t}$ and ake an argument parallel to that for z_n to obtain a uniform bound for ω_n

$$|\omega_n| \le M_8. \tag{3.26}$$

Theorem 2 . *Under the assumptions A and B, there exists a solution of problem (P) uch that u, u_t, u_{xx} are continuous in D_T, s(t) is differentiable, and $s'(t) > 0$ for $0 \le$ $\le T$.*

Proof. From the remark following (3.16)-(3.19) we know there is a sequence $\Delta t_i \to 0$ nd a function s such that $s^{\Delta t_i} \to s$ uniformly on [0,T]. Now for each Δt_i we construct function of two variables $u^{\Delta t_i}(x,t)$ with domain $\{x \ge 0, t \ge 0\}$ by first requiring that coincide with each u_n^i on the lines $t = n\Delta t_i$,that is

$$u^{\Delta t_i}(x, n\Delta t_i) = u_n^i(x), \tag{3.27}$$

$^{\Delta t}$, $s^{\Delta t}$ is to be completed by linear interpolation

$$s^{\Delta t}(t) = \frac{t-(n-1)\Delta t}{\Delta t} s_n + \frac{n\Delta t - t}{\Delta t} s_{n-1}, \tag{3.28}$$

$$u^{\Delta t}(x,t) = \frac{t-(n-1)\Delta t}{\Delta t} u_n(x) + \frac{n\Delta t - t}{\Delta t} u_{n-1}(x). \tag{3.29}$$

Finally restrict $u^{\Delta t_i}$ to have domain $\overline{D}_T$.

By the uniform boundedness of u_n' and z_n, the u_n^i form an equicontinuous iiformly bounded family on $\overline{D}_T$, and hence , there exists a subsequence of Δt_i (which e shall still designate as Δt_i) such that $u_n^i \to u$ uniformly on $\overline{D}_T$.

We have to prove : u(x,t), s(t) are the solution of problem (P).

1. u(x,t) satisfies the differential equation $u_t = u_{xx}$ in D_T. By the constructio of $u^{\Delta t_i}(x,t)$ and the uniform boundedness of ω_n that u_t, and u_{xx} exist. Also, it is nc difficult to show as in [2] that

$$u_t(x,t) = \lim_{i>N} \frac{u_k^i(x) - u_{k-1}^i(x)}{\Delta t_i}, \tag{3.30}$$

where k are chosen so that $k\Delta t_i \to t$. At points of the form $(x, k\Delta t_i)$ we have

$$(u_k^i)'' = \frac{1}{\Delta t_i}[u_k^i - u_{k-1}^i], \tag{3.31}$$

and in view of (3.30) $u_t = u_{xx}$ for points of this form.

2. The conditions at the fixed boundary $u(x,0) = \varphi(x)$, $u_x(0,t) = 1$ can b dispatched by $u_n^i \to u$ uniformly.

3. For the condition $u(s(t),t) = f(t)$, we compute

$$\begin{aligned}|u(s(t),t) - f(t)| &\le |u(s(t),t) - u^{\Delta t_i}(s(t),t)| + |u^{\Delta t_i}(s(t),t) - u^{\Delta t_i}(s(t),k\Delta t_i)| \\ &+ |u^{\Delta t_i}(s(t)),k\Delta t_i) - u^{\Delta t_i}(s_k^i,k\Delta t_i)| \\ &+ |u^{\Delta t_i}(s_k^i,k\Delta t_i)) - f(t)| = C_1 + C_2 + C_3 + C_4.\end{aligned} \tag{3.32}$$

Since $u_n^i \to u$ uniformly, we may choose n large enough so that for arbitrary ϵ

$$C_1 < \frac{\epsilon}{3}. \tag{3.33}$$

Also recall that $u^{\Delta t_i}(x, n\Delta t_i) = u_n^i(x)$ so that

$$C_3 = u_k^i(s(t)) - u_k^i(s_k^i) \le |(u_k^i)'(\xi)|.|s(t) - s_k^i| \le M_5|s(t) - s_k^i|. \tag{3.34}$$

Moreover,

$$C_4 = u_k^i(s_k^i) - f(t) \le |f(k\Delta t_i) - f(t)| \le \max_{0\le t\le T} |f'(t)|.|k\Delta t_i - t| \tag{3.35}$$

Finally, by the interpolation contruction of u_n^i we have

$$\begin{aligned}C_2 &= \frac{|t - k\Delta t_i|}{\Delta t_i}|u^{\Delta t_i}(s(t),(k+1)\Delta t_i) - u^{\Delta t_i}(s(t),k\Delta t_i)| \\ &= \frac{|t - k\Delta t_i|}{\Delta t_i}|u_{k+1}^i(s(t) - u_k^i(s(t)| \le M_3|t - k\Delta t_i|.\end{aligned} \tag{3.36}$$

We now take n large enough such that

$$|s(t) - s_k^i| < \frac{\epsilon}{3M_5} \quad \text{and} \quad |t - k\Delta t_i| < \frac{\epsilon}{3(M_4 + \max_{0\le t\le T}|f'(t)|)}.$$

For such large n, we have, according to (3.32)-(3.33)

$$|u(s(t),t) - f(t)| < \epsilon.$$

4. We proceed $|u_x(s(t),t) - g(t)| < \epsilon$ as section 3.

We show that $s'(t)$ exists and $s'(t) > 0$. From the condition $u(s(t),t) = f(t)$, by omputing we get

$$\frac{1}{\delta}[s(t+\delta) - s(t)] = \Big[\frac{f(t+\delta) - f(t)}{\delta} - \frac{u(s(t), t+\delta) - u_t(s(t),t)}{\delta}\Big]$$
$$\Big[\frac{u(s(t+\delta), t+\delta) - u(s(t), t+\delta)}{\delta}\Big]^{-1}.$$

'aking the limit as $\delta \to 0^+$, $\delta \to 0^-$, all limits on the right-hand side exist (since s is ontinuous) giving

$$\lim_{\delta \to 0} \frac{(s(t+\delta) - s(t))}{\delta} = [f'(t) - u_t(s(t),t)][g(s(t),t)]^{-1}.$$

Hence exists $s'(t)$. Note that $s(t+\delta) > s(t)$ therefore $s'(t) > 0$.

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BÀI TOÁN BIÊN TỰ DO DẠNG CÔSI ĐỐI VỚI PHƯƠNG TRÌNH TRUYỀN NHIỆT

Nguyễn Đình Bình

Khoa Toán - Đại học Bách khoa Hà Nội

Bài báo nghiên cứu một bài toán biên tự do, với các điều kiện cho trên biên có ạng Côsi đối với phương trình truyền nhiệt. Tìm các hàm $s(t)$, $u(x,t)$ sao cho

$$\begin{aligned}
&u_{xx} - u_t = 0 && 0 < x < s(t), 0 < t < T,\\
&u(x,0) = \varphi(x), && 0 \le x \le s(0) = b,\\
&u_x(0,t) = 1, && 0 \le t \le T,\\
&u(s(t),t) = f(t), && 0 \le t \le T,\\
&u_x(s(t),t) = g(t), && 0 \le t \le T,
\end{aligned}$$

rong đó b, t là các hằng số dương và $f(t), g(t), \varphi(x)$ là các hàm số cho trước.

Sự tồn tại nghiệm của bài toán được giải quyết bằng phương pháp nửa rời rạc hóa heo biến t, tính duy nhất nghiệm được chứng minh bằng cách sử dụng nguyên lý cực ại đối với phương trình parabolic.