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ON SOME CONDITIONS FOR A PARAMETER DEPENDENCE OF THE SOLUTIONS TO A LINEAR DIFFERENTIAL EQUATION IN THE BANACH SPACE

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1. Introduction

In the Banach Space E, we consider the following differential equations:

$$\frac{dx_\lambda}{dt} = A(t)x_\lambda + R(t,\lambda)x_\lambda \tag{1}$$

$$\frac{dx}{dt} = A(t)x \tag{2}$$

$$\frac{dy_\lambda}{dt} = R(t,\lambda)y_\lambda\ , \tag{3}$$

where $A(t), R(t,x)$ are linear bounded operators for $t \in [0,T]$, $\lambda \in [0,\alpha]$ and strongly continuous on $[0,T]$; $R(t,o) = 0$.

Let us denote the Cauchy operators to the equations (1)-(3) with the initial conditions

$$X(0,\lambda) = X(0) = Y(0,\lambda) = I,$$

$\lambda \in [0,\alpha]$ by $X(t,\lambda)$, $X(t)$, $Y(t,\lambda)$ respectively (here I is the unit operator).

Levin [1] proved the following theorem

Theorem 1. *The pair of relations*

$$X(t,\lambda) \to X(t)\ ,\ \ X^{-1}(t,\lambda) \to X^{-1}(t)\ \ in\ \mathbf{L}_p \tag{4}$$

is equivalent to the pair of relations

$$Y(t,\lambda) \to I\ ,\ \ Y^{-1}(t,\lambda) \to I\ \ in\ \mathbf{L}_p$$

for any $p \geq 1$ *as* $\lambda \to 0$. *Here* $\mathbf{L}_p = \mathbf{L}_p[0,T]$.

For $p = \infty$ (the case in which one actually deal with convergence of solutions in $C[0,T]$. Levin [2] presented the folowing result

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Theorem 2. *Let one of the conditions*

1) $\|R(\lambda)\|_{\mathbf{L}_1} \leq c$;

2) $\|\tilde{R}(\lambda)R(\lambda)\|_{\mathbf{L}_1} \to 0$;

3) $\|R(\lambda)\tilde{R}(\lambda)\|_{\mathbf{L}_1} \to 0$;

4) $\|\tilde{R}(\lambda)R(\lambda) - R(\lambda)\tilde{R}(\lambda)\|_{\mathbf{L}_1} \to 0$.

be satisfied as $\lambda \to 0$. *Then the condition*

$$\tilde{R}(t,\lambda) \to 0 \text{ in } C[0,T]$$

is necessary and sufficient for the convergence

$$X(t,\lambda) \to X(t) \text{ in } C[0,T],$$

where

$$\tilde{R}(t,\lambda) = \int_0^t R(s,\lambda)ds.$$

In the case of convergence in $C[0,T]$, Theorem 1 was extended by Strauss and Yourke [3] for the nonlinear equation.

Using Theorem 1, the authors of papers [4], [5] have extended and generalised the result of Theorem 2.

In this paper we present necessary conditions on the coefficients of equation (1) for the validity of (4). We give an example, which is satisfied our condition but is not satisfied Levin's condition. The main result of this paper are theorem 3 and 4.

. Auxiliary lemmas

From now we denote:

$$\tilde{F}(t) = \int_0^t F(s)ds$$

Let us consider the sequence of operators

$$M_0(t,\lambda)\,,\ M_1(t,\lambda)\,,\dots M_p(t,\lambda)\,,\dots,$$

$$N_0(t,\lambda)\,,\ N_1(t,\lambda)\,,\dots,N_p(t,\lambda)\,,\dots$$

determined by the following induction relation:

$$M_0(t,\lambda) = R(t,\lambda)\,;\ \ M_p(t,\lambda) = \sum_{k=1}^{p} \tilde{R}^k(t,\lambda)R(t,\lambda) + \sum_{k=2}^{p} \frac{d}{dt}\tilde{R}^k(t,\lambda)$$

$$N_0(t,\lambda) = R(t,\lambda)\,;\ \ N_p(t,\lambda) = \sum_{k=1}^{p} R(t,\lambda)\tilde{R}^k(t,\lambda) - \sum_{k=2}^{p} \frac{d}{dt}\tilde{R}^k(t,\lambda)$$

Lemma 1 *The solution of the problem*

$$\frac{dY}{dt} = R(t)Y\,;\ \ Y(0) = I \tag{5}$$

can be represented in the form

$$Y(t) = I + \sum_{k=1}^{p} \tilde{R}^k(t)Y(t) - \int_0^t M_p(\tau)Y(\tau)d\tau. \quad (6)$$

and the solution of the problem

$$\frac{dZ}{dt} = -ZR(t)\ , \quad Z(0) = I \quad (7)$$

can be represented in the form

$$Z(t) = I + \sum_{k=1}^{p} Z(t)\tilde{R}^k(t) + \int_0^t Z(\tau)N_p(\tau)d\tau. \quad (8)$$

Proof. First let us prove formula (6).
For $p = 1$ we have

$$\begin{aligned} I + \tilde{R}(t)Y(t) - \int_0^t \tilde{R}(\tau)R(\tau)Y(\tau)d\tau =& I + \tilde{R}(t)Y(t) - \int_0^t \tilde{R}(\tau)\Big(\frac{d}{d\tau}Y(\tau)\Big)d\tau \\ =& I + \tilde{R}(t)Y(t) - \tilde{R}(t)T(t) + \int_0^t R(\tau)Y(\tau)d\tau \\ =& I + \int_0^t R(\tau)Y(\tau)d\tau \\ =& Y(t). \end{aligned}$$

i.e., formula (6) is valid for $p = 1$

Let us now assume that (6) is valid for $p > 1$. Then

$$\begin{aligned} &\sum_{k=1}^{p+1} \tilde{R}^k(t)Y(t) - \int_0^t M_{p+1}(\tau)Y(\tau)d\tau \\ =& \sum_{k=1}^{p} \tilde{R}^k(t)Y(t) + \tilde{R}^{p+1}(t)Y(t) \\ &- \int_0^t \sum_{k=1}^{p} \tilde{R}^k(\tau)R(\tau)Y(\tau)d\tau - \int_0^t \tilde{R}^{p+1}(\tau)R(\tau)Y(\tau)d\tau \\ &- \int_0^t \Big(\sum_{k=2}^{p} \frac{d}{d\tau}\tilde{R}^k(\tau)\Big)Y(\tau)d\tau - \int_0^t \Big(\frac{d}{d\tau}\tilde{R}^{p+1}(\tau)\Big)Y(\tau)d\tau \\ =& \sum_{k=1}^{p} \tilde{R}^k(t)Y(t) - \int_0^t M_p(\tau)Y(\tau)d\tau + \tilde{R}^{p+1}(t)Y(t) \\ &- \int_0^t \tilde{R}^{p+1}(\tau)[\frac{d}{d\tau}Y(\tau)]d\tau - \int_0^t \Big(\frac{d}{d\tau}\tilde{R}^{p+1'}(\tau)\Big)Y(\tau)d\tau \end{aligned}$$

'sing the intergration in part we have then

$$\sum_{k=1}^{p+1} \tilde{R}^k(t)Y(t) - \int_0^t M_{p+1}(\tau)Y(\tau)d\tau = \sum_{k=1}^{p} \tilde{R}(t)Y(t) - \int_0^t M_p(\tau)Y(\tau)d\tau$$
$$= Y(t) - I,$$

'herefore,

$$Y(t) = I + \sum_{k=1}^{p} \tilde{R}^k(t)Y(t) - \int_0^t M_p(\tau)Y(\tau)d\tau$$

e., formula (6) is valid for $p+1$ □

Formula (5) is proved similarly.

Let us now consider the following Volterra equation

$$x(t) = \int_0^t h(\varepsilon)x(\tau)d\tau + f(t) \tag{9}$$

'here $f(x) \in \mathbf{L}_p[0,T]$ and $h(t) \in C[0,T]$.

emma 2. *The solution of (9) satisfies the estimate*

$$\|x\|_{\mathbf{L}_p} \le (1 + T^{\frac{1}{p}})\|h\|_{\mathbf{L}_q} \exp \|h\|_{\mathbf{L}_1})\|f\|_{\mathbf{L}_p}$$

'here $\frac{1}{p} + \frac{1}{q} = 1$

The proof of this lemma was given in [5] □

. The case of $\mathbf{L}_p[0,T]$, $p \ge 1$

Let us construct operators $M_k(t,\lambda)$, $N_k(t,\lambda)$, $\tilde{M}_k(t,\lambda)$ and $\tilde{N}_k(t,\lambda)$ using the same ·chnique as in Section 2, but with

$$M_0(t,\lambda) = N_0(t,\lambda) = R(t,\lambda)$$

'e set

$$H_i(t,\lambda) = \sum_{k=1}^{i} R^k(t,\lambda) \ ; \ \ P_i(t,\lambda) = \sum_{k=1}^{i} R^k(t,\lambda) - M_i(t,\lambda)$$

heorem 3. *Let the conditions*

$$\|\tilde{H}_i(\lambda)\|_C \le \delta < 1 \ , \ \ \|M_i(\lambda)\|_{\mathbf{L}_q} \le \sigma < \infty$$

satisfies as $\lambda \to 0$ for some nonnegative integer i. Then the condition

$$\tilde{P}_i(t,\lambda) \to 0 \ in \ \mathbf{L}_p[0,T] \ , \ \ \lambda \to 0$$

necessary and sufficient for

$$Y(t,\lambda) \to I \ in \ \mathbf{L}_p[0,T] \ , \ \ \lambda \to 0.$$

Proof: By virtue of Lemma 1 the Cauchy operator $Y(t,\lambda)$ for equation (3) can b represented in the form

$$Y(t,\lambda) = I + \sum_{k=1}^{i} \tilde{R}^k(t,\lambda)Y(t,\lambda) - \int_0^t M_i(\tau,\lambda)Y(\tau,\lambda)d\tau. \quad (10$$

From this, by our construction, we obtain

$$\begin{aligned} Y(t,\lambda) - I &= \sum_{k=1}^{i} \tilde{R}^k(t,\lambda)[Y(t,\lambda) - I] + \tilde{P}_i(t,\lambda) \\ &\quad - \int_0^t M_i(\tau,\lambda)[Y(\tau,\lambda) - I]d\tau \\ &= \tilde{H}_i(t,\lambda)[Y(t,\lambda) - I] + \tilde{P}_i(t,\lambda) - \int_0^t M_i(\tau,\lambda)[Y(\tau,\lambda) - I]d\tau \end{aligned}$$

or

$$[I - \tilde{H}_i(t,\lambda)][Y(t,\lambda) - I] = \tilde{P}_i(\tau,\lambda) - \int_0^t M_i(\tau,\lambda)[Y(\tau,\lambda) - I]d\tau. \quad (11$$

By virtue of the condition of the theorem, the operator $[I - \tilde{H}_i(t,\lambda)]^{-1}$ exists for small and satisfies the estimate

$$\|[I - \tilde{H}_i(t,\lambda)]^{-1}\| \le \sum_{k=0}^{\infty} \delta^k = K < \infty.$$

The last inequality in conjunction with (11) yields

$$\|Y(t,\lambda) - I\| \le K\|\tilde{P}_i(t,\lambda)\| + K\int_0^t \|M_i(\tau,\lambda)\|\|Y(\tau,\lambda) - I\|d\tau.$$

Applying the theorem on integral inequalities (see [6], p.154) we obtain

$$\|Y(t,\lambda) - I\| \le \phi(t,\lambda),$$

where $\phi(t,\lambda)$ is the solution of the Volterra equation

$$x(t) - K\int_0^t \|M_i(\tau,\lambda)\|x(\tau)d\tau = K\|\tilde{P}_i(t,\lambda)\|.$$

By Lemma 2, the solution of the last equation satisfies the estimate

$$\|\phi(t,\lambda)\|_{\mathbf{L}_p} \le (1 + KT^{\frac{1}{p}}\|M_i(\lambda)\|_{\mathbf{L}_q}\exp K\|M_i(\lambda)\|_{\mathbf{L}_1})K\|\tilde{P}_i(\lambda)\|_{\mathbf{L}_p}.$$

Therefore,

$$\|Y(\lambda) - I\|_{\mathbf{L}_p} \le (1 + KT^{\frac{1}{p}}\sigma\exp K\sigma)K\|\tilde{P}_i(\lambda)\|_{\mathbf{L}_p}.$$

The sufficiency is thereby proved.

Furthemore, from (11) we have

$$\tilde{P}_i(t,\lambda) = |I - \tilde{H}_i(t,\lambda)||Y(t,\lambda) - I| + \int_0^t M_i(\tau,\lambda)|Y(\tau,\lambda) - I|d\tau$$

Ve apply the Holder inequalyty to obtain

$$\|\tilde{P}_i(\lambda)\|_{\mathbf{L}_p} \le (1+\delta)\|Y(\lambda) - I\|_{\mathbf{L}_p} + T^{\frac{1}{p}}\|M_i(\lambda)\|_{\mathbf{L}_q}\|Y(\lambda) - I\|_{\mathbf{L}_p}.$$

'onsequently, $\tilde{P}_i(t,\lambda) \to 0$ in $\mathbf{L}_p[0,T]$ if $Y(t,\lambda) \to I$ in $\mathbf{L}_p[0,T]$ $(\lambda \to 0)$. This completes he proof □

We set now

$$Q_j(t,\lambda) = N_j(t,\lambda) + \sum_{k=1}^{j} R^k(t,\lambda).$$

'heorem 4. *Let the conditions*

$$\|\tilde{H}_j(\lambda)\|_C \le \delta < 1\,, \quad \|N_j(\lambda)\|_{\mathbf{L}_q} \le \sigma < \infty$$

e satisfied as $\lambda \to 0$ for some integer $j \ge 0$. Then the condition

$$\tilde{Q}_j(t,\lambda) \to 0 \ \textit{in}\ \mathbf{L}_p[0,T]\,, \quad \lambda \to 0$$

necessary and sufficient for

$$Y^{-1}(t,\lambda) \to I \ \textit{in}\ \mathbf{L}_p[0,T]\,, \quad \lambda \to 0.$$

Proof: By Lemma 1, the solution of the problem

$$\frac{dZ}{dt} = -ZR(t,\lambda)\,, \quad Z(0,\lambda) = I$$

in be represented in the form

$$Z(t,\lambda) = I + \sum_{k=1}^{j} Z(t,\lambda)\tilde{R}^k(t,\lambda) + \int_0^t Z(\tau,\lambda)N_j(\tau,\lambda)d\tau$$

$$|Z(t,\lambda) - I||I - \tilde{H}_j(t,\lambda)| = \tilde{Q}_j(t,\lambda) + \int_0^t |Z(\tau,\lambda) - I|N_j(\tau,\lambda)d\tau \qquad (12).$$

pplying the same reasoning as in the proof of Theorem 3, we obtain the estimate

$$\|Z(t,\lambda) - I\|_{\mathbf{L}_p} \le K(1 + KT^{\frac{1}{p}}\|N_j(\lambda)\|_{\mathbf{L}_q} \exp K\|N_j(\lambda)\|_{\mathbf{L}_1}\|\tilde{Q}_j(\lambda)\|_{\mathbf{L}_p}.$$

remains to pay attention that $Z(t,\lambda) = Y^{-1}(t,\lambda)$.

The proof of necessity follows from (12) and the same observation.

It is now clear that the combinations of Theorem 1,3 and 4 provides different sufficient conditions for (4). In particular, by setting $H_0(t,\lambda) = 0$ we obtain the following easy to verify condition.

Theorem 5. *Let* $\|R(\lambda)\|_{\mathbf{L}_q} \leq \sigma < \infty$ *as* $\lambda \to 0$. *Then the condition* $\tilde{R}(t,\lambda) \to 0$ *in* $\mathbf{L}_p[0,T]$, $\lambda \to 0$ *is necessary and sufficient for the validity (4).*

4. The case of $C[0,T]$

Since the relations

$$Y(t,\lambda) \to I\,,\ \ Y^{-1}(t,\lambda) \to I\,,\ \ \lambda \to 0$$

are equivalent in $C[0,T]$, we set $p = \infty$ (and hence $q = 1$) and obtain the following results from theorems 1,3 and 4.

Theorem 6. *Let the conditions*

$$\|\tilde{H}_i(\lambda)\|_C \leq \delta < 1\,,\ \ \|M_i(\lambda)\|_{\mathbf{L}_1} \leq \sigma < \infty$$

be satisfied as $\lambda \to 0$ *for some nonnegative integer* i. *Then* $\tilde{P}_i(t,\lambda) \to 0$ *in* $C[0,T]$ *as* $\lambda \to 0$ *if and only if* $X(t,\lambda) \to X(t)$ *in* $C[0,T]$ *as* $\lambda \to 0$.

Theorem 7. *Let*

$$\|\tilde{H}_j(\lambda)\|_C \leq \delta < 1\,,\ \ \|N_j(\lambda)\|_{\mathbf{L}_1} \leq \sigma < \infty$$

as $\lambda \to 0$ *for some integer* $j \geq 0$. *Then* $\tilde{Q}_j(t,\lambda) \to 0$ *in* $C[0,T]$ *as* $\lambda \to 0$ *if and only if* $X(t,\lambda) \to X(t)$ *in* $C[0,T]$ *as* $\lambda \to 0$.

The straightfoward proof of Theorems 6 and 7 can be obtained from (11), (12) respectively, which the help of the Gronwall-Belman Lemma.

In closing, let us consider the following example:

$$\frac{dx_\lambda}{dt} = A(t)x_\lambda + R(t,\lambda)x_\lambda,$$

where

$$R(t,\lambda) = \begin{pmatrix} 0 & 0 & \frac{1}{\lambda}\cos\frac{t}{\lambda^2} \\ 0 & \frac{1}{\lambda}\cos\frac{t}{\lambda^2} & 0 \\ \frac{1}{\lambda}\sin\frac{2t}{\lambda^2} & 0 & 0 \end{pmatrix}.$$

It is easy to check that $\tilde{R}(t,\lambda) \to 0$ in $C[0,T]$ as $\lambda \to 0$ and that none of Levin's Theorem is satislied. However, it is not difficult to verify that the condition of Theorem 6 is satisfied with $i = 1$

Remark: Some extension for strong convegence of operators have been given in [7].

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REFERENCES

1. A. Yu. Levin, *Dokl. Akad. Nauk SSSR*, **225**, No. 3(1975), p 503-506 (in Russian).

2. A. Yu. Levin, *Dokl. Akad. Nauk SSSR*, **176**, No. 4(1967), p 774-777 (in Russian).

3. A. Strauss, J. A. Yorke, *Proc. Am. Math. Soc.*, Vol. **26**(1970), p. 255

4. P. P. Zabreiko, C.I. Constadinov, Nguyen Hong Thai, *Differential'nye Uravneniya*, **24**, No. 12(1988), p 2056-2063, (in Russian).

5. Nguyen The Hoan, *Differential'nye Uravneniya*, **29**, No 6(1973), p 970-975, (in Russian).

6. Yu. L. Daletskii and M.G. Krein, *Stability of Solutions for Differential Equations in Banach Space*, Moscow (1970) (in Russian).

7. Nguyen The Hoan and Pham Ngoc Boi. On the continuous dependence upon parameter of solutions of linear differential equations in Banach space, *Ucrainian Mathematical Journal*, **51**, No 2(1999), p 275- 280.

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MỘT VÀI ĐIỀU KIỆN VỀ SỰ PHỤ THUỘC THAM SỐ CỦA NGHIỆM
ỦA PHƯƠNG TRÌNH VI PHÂN TUYẾN TÍNH TRONG KHÔNG GIAN BANACH

Phạm Ngọc Bội
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Trong không gian Banach E, ta xét sự hội tụ của nghiệm của phương trình vi phân ến tính có nhiễu:

$$\frac{dx}{dt} = A(t)x + R(t,\lambda)x$$

nghiệm của phương trình vi phân tuyến tính thuần nhất:

$$\frac{dx}{dt} = A(t)x$$

ng đó t thuộc đoạn hữu hạn $[0,T]$, λ thuộc đoạn hữu hạn $[0,\alpha]$, $A(t)$, $R(t,\lambda)$ là các toán tuyến tính bị chặn của E và liên tục mạnh.

Bài báo này là sự tiếp nối các kết của của Lêvin [1] và [2], đưa ra một số điều kiện ıc của [2] và ví dụ chứng tỏ khi đó các điều kiện của Lêvin [2] không thoả mãn mà h lý 6 (của bài báo này) thoả mãn