ON A CLASS OF EQUATIONS INDUCED BY A NILPOTENT OPERATOR AND ITS APPLICATIONS

Nguyen Tan Hoa
Gia Lai Education and Training Department

Abstract. *In this paper we deal with a class of equations of the form* $\sum_{k=0}^{n-1} A_k S^k x =$ *where* $A_k, S \in L_0(X)(k = \overline{0, n-1})$, *$S$ is a nilpotent operator. We show that under sor assumptions on operators* $A_k(k = \overline{0, n-1})$, *all solutions of these equations can be obtaine We shall give some applications to study some classes of singular integral and differenti equations.*

A particular case of these equations with $A_0, A_1, \cdots, A_{n-1}$ *being operators of mul plication by complex numbers* $a_0, a_1, \cdots, a_{n-1}(a_0 \neq 0)$ *respectively was solved by Przewor Rolewicz in [4].*

Let X be a linear space over the field of complex numbers. We denote by $L(X)$ tl set of all linear operatiors with the domains and the ranges contained in the space X aı

$$L_0(X) = \{A \in L(X) : \operatorname{dom} A = X\}.$$

An algebraic operator $S \in L_0(X)$ is said to be nilpotent operator of order $n(n \geq$ if its characteristic polynomial is of the form $P(t) = t^n$, i.e.,

$$S^n = 0, S^k \neq 0 (1 \leq k < n) \quad \text{on} \quad X. \qquad ($$

Theorem 1. *Let* $S, A_0, A_1, \cdots, A_{n-1} \in L_0(X)$, *and let S be a nilpotent operator of ord* $n(n \geq 2)$. *Suppost that A_0 is invertible and*

$$A_0 X_k = X_k, \qquad ($$

$$A_i X_k \subset X_k \quad (i, k = \overline{1, n-1}), \qquad ($$

where $X_k = \{x \in X : S^k x = 0\}$. *Then the equation*

$$\sum_{k=0}^{n-1} A_k S^k x = y \qquad ($$

(where $y \in X$) has a unique solution

$$x = A_0^{-1} y + \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \leq j_1 + j_2 + \cdots + j_{i-1} \leq n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y, \qquad ($$

where $B_k = A_k S^k \quad (k = \overline{1, n-1})$

Proof. We have

$$\sum_{k=0}^{n-1} A_k S^k x = A_0 x + \sum_{k=1}^{n-1} B_k x =$$

$$= y + \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y$$

$$+ B_1 A_0^{-1} y + \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_1 A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y$$

$$+ B_2 A_0^{-1} y + \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_2 A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y$$

$$\cdots$$

$$+ B_{n-1} A_0^{-1} y + \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_{n-1} A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y$$

$$= y + \sum_{i=3}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y$$

$$+ \sum_{k=1}^{n-1} \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_k A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y.$$

From (1), (2) and (3) we obtain $B_k A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} = 0$ if $k + j_1 + + \cdots + j_{i-1} \ge n (k, j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}, i = \overline{1, n})$. Hence

$$\sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_k A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y =$$

$$= \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_k A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y \quad (k = \overline{1, n-1}).$$

Thus

$$\sum_{k=0}^{n-1} A_k S^k x = y + \sum_{i=3}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y +$$

$$+ \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y = y.$$

Hence x defined by (5) is a solution of equation (4).

Suppose that $x_1, x_2 \in X$ are solution of (4). Then we have

$$\sum_{k=0}^{n-1} A_k S^k z = 0,$$

where $z = x_1 - x_2$.

Acting on both sides of equation (6) with the operators $S^{n-1}, S^{n-2}, \cdots, S^2, S$ a using (1) and (3) we obtain the system of equations

$$\begin{aligned} S^{n-1}A_0 z &= 0 \\ S^{n-2}A_0 z + S^{n-2}A_1 S z &= 0 \\ \cdots\cdots & \\ SA_0 z + SA_1 S z + \cdots + SA_{n-3}S^{n-3} z + SA_{n-2}S^{n-2} z &= 0 \\ A_0 z + A_1 S z + A_2 S^2 z + \cdots + A_{n-2}S^{n-2} z + A_{n-1}S^{n-1} z &= 0 \end{aligned}$$

By (2) and the first equation of system (7) we have $z \in X_{n-1}$. This result and imply that $S^{n-1-i}A_i S^i z = 0, \quad i = \overline{1, n-1}$.

Thus, the system (7) is equivalent to the system

$$\begin{aligned} S^{n-2}A_0 z &= 0 \\ S^{n-3}A_0 z + S^{n-3}A_1 S z &= 0 \\ \cdots\cdots & \\ SA_0 z + SA_1 S z + \cdots + SA_{n-4}S^{n-4} z + SA_{n-3}S^{n-3} z &= 0 \\ A_0 z + A_1 S z + A_2 S^2 z + \cdots + A_{n-3}S^{n-3} z + A_{n-2}S^{n-2} z &= 0 \end{aligned}$$

After $n-1$ steps, we obtain $A_0 z = 0$, hence $x_1 = x_2$. Thus equation (4) ha unique solution. Theorem is proved. □

Similarly, we can prove the following result.

Theorem 2. *If the assumptions of theorem 1 are satisfied, then equation*

$$\sum_{k=0}^{n-1} S^k A_k x = y,$$

(where $y \in X$) has a unique solution

$$x = A_0^{-1} y + \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} A_0^{-1} C_{j_1} A_0^{-1} C_{j_2} \cdots A_0^{-1} C_{j_{i-1}} A_0^{-1} y,$$

where $C_k = S^k A_k \quad (k = \overline{1, n-1})$.

Theorem 3. *Let $S, A_0, A_1, \cdots, A_{n-1} \in L_0(X)$, $X_k = \{x \in X : S^k x = 0\} (k = 0, 1, 2, ...)$ and $\tilde{X} = \bigcup_{k=1}^{\infty} X_k$. Suppose that A_0 is invertible, $A_0 X_k = X_k$ and $A_i X_k \subset X_k, i = \overline{1, n-1}, k = 1, 2, ...$ Then the equation*

$$\sum_{k=0}^{n-1} A_k S^k x = y, \tag{8}$$

where $y \in \tilde{X}$) has a unique solution in $\tilde{X}$. More precisely: if $y \in X_m$, then there is a unique $x \in X_N$ satisfying equation (8) and such that:

$$x = A_0^{-1} y + \sum_{i=2}^{N} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le N-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y,$$

where $N = \max(n, m)$ and $B_k = A_k S^k \quad (k = \overline{1, n-1})$

Proof. Let $y \in X_m$ and suppose that $m > n$. Putting $A_n = A_{n+1} = \cdots = A_{m-1} = 0$, we can write $\sum_{k=0}^{n-1} A_k S^k x = \sum_{k=0}^{m-1} A_k S^k x$. This and Theorem 1 imply that

$$x = A_0^{-1} y + \sum_{i=2}^{m} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le m-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, m-1}}} A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y,$$

where $B_k = A_k S^k, k = \overline{1, m-1}$.

By $B_k = 0 \quad (k = \overline{n, m-1})$, we have

$$x = A_0^{-1} y + \sum_{i=2}^{m} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le m-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y \in X_m,$$

In the second case we have $m \le n$ and $X_m \subset X_n$. Indeed, if $x \in X_m$, then $S^n x = S^{n-m}(S^m x) = 0$, which implies $x \in X_n$. We assume that $y \in X_n$. By theorem 1, with $X = X_n$, we obtain

$$x = A_0^{-1} y + \sum_{i=2}^{n} (-1)^{i-1} \sum_{\substack{1 \le j_1 + j_2 + \cdots + j_{i-1} \le n-1 \\ j_1, j_2, \cdots, j_{i-1} = \overline{1, n-1}}} A_0^{-1} B_{j_1} A_0^{-1} B_{j_2} \cdots A_0^{-1} B_{j_{i-1}} A_0^{-1} y \in X_n.$$

Suppose that $x_1, x_2 \in \tilde{X} (x_1 \neq x_2)$ are solution of (8), suppose that $x_1 \in X_{m_1}, x_2 \in X_{m_2}$, we have

$$\sum_{k=0}^{n-1} A_k S^k (x_1 - x_2) = 0.$$

If $m_1, m_2 \leq n$ then $x_1 - x_2 \in X_n$. Consider the equation (9) in X_n by simil arguments as in the proof of Theorem 1, we obtain $x_1 - x_2 = 0$ which contracdicts to ou assumption.

If $\max(m_1, m_2) = N_1 > n$ then $x_1 - x_2 \in X_{N_1}$. Consider the equation (9) in X_N We can write (9) as follows

$$\sum_{k=0}^{N_1-1} A_k S^k (x_1 - x_2) = 0,$$

where $A_n = A_{n+1} = \cdots = A_{N_1-1} = 0$. By similar arguments as in the proof of Theorem we obtain $x_1 - x_2 = 0$ which contracdicts to our assumption.

Thus equation (8) has a unique solution. Theorem is proved. □

Applications

Let Γ be a simple regular closed arc and let X be the space $H^\mu(\Gamma)(0 < \mu < 1$ Denote by D^+ the domain bounded by Γ and by D^- its complement including the poi at infinity.

Let

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau,$$

$$(N\varphi)(t) = \int_\Gamma N(t, \tau)\varphi(\tau) d\tau,$$

$$(T\varphi)(t) = \int_\Gamma \sum_{j=1}^{n} a_j(t) b_j(\tau) \varphi(\tau) d\tau,$$

where $a_j(t), b_j(t), \varphi(t) \in X (j = \overline{1, n}); \{a_j(t)\}_{j=\overline{1;n}}$ is a linear independent system.

Write $P = \frac{1}{2}(I + S), Q = \frac{1}{2}(I - S), X^+ = PX, X^- = QX$.

Corollary 1. *Suppose that $N(t, \tau)$ admits an analytic prolongation in every variable o D^+ and is continuous for $t, \tau \in \Gamma$. Then the equation*

$$c_0\varphi(t) + c_1(N\varphi)(t) = f(t),$$

where $c_0, c_1 \in C, c_0 \neq 0, f(t) \in X$ has a unique solution

$$\varphi(t) = c_0^{-1} f(t) - c_0^{-2} c_1 (Nf)(t).$$

Proof. By our assumption for $N(t, \tau)$ from the Cauchy integral Theorem, we get $N^2 =$ on X (see [1]). We have

$$c_0 I X_1 = X_1, \quad c_1 I X_1 \subset X_1 \text{ where } X_1\{g(t) \in X : (Ng)(t) = 0\}.$$

Applying theorem 1 we obtain the required result. □

Corollary 2. *Suppose that $N(t, \tau)$ admits an analytic prolongation in every variable o D^+ and is cotinuous for $t, \tau \in \Gamma$. Then the equation*

$$c_0\varphi(t) + c_1(S\varphi)(t) + c_2(NS\varphi)(t) = f(t), \qquad ($$

here $c_0, c_1, c_2 \in C, c_0 \pm c_1 \neq 0, f(t) \in X$ *has a unique solution*

$$\varphi(t) = \frac{1}{c_0^2 - c_1^2}[c_0 f(t) - c_1(Sf)(t) + c_2(Nf)(t)].$$

roof. By our assumption for $N(t,\tau)$ from the Cauchy integral Theorem, we get $N^2 =$ $, SN = N, NS = -N$ (see [2]). Rewrite to equation (10) as follows

$$[(c_0 I + c_1 S)\varphi](t) - c_2(N\varphi)(t) = f(t).$$

is easy to see that $c_0 I + c_1 S$ is invertible $(c_0 I + c_1 S)X_1 = X_1, -c_2 I X_1 \subset X_1$, where $X_1 = \{g(t) \in X : (Ng)(t) = 0\}$.

Applying Theorem 1, we obtain

$$\varphi(t) = [(c_0 I + c_1 S)^{-1} f](t) + c_2[(c_0 I + c_1 S)^{-1} N (c_0 I + c_1 S)^{-1} f](t) =$$
$$= \frac{1}{c_0^2 - c_1^2}[c_0 f(t) - c_1(Sf)(t) + c_2(Nf)(t)].$$

orollary 3. *Suppose that* $N(t,\tau)$ *admits an analytic prolongation in every variable onto* D^+ *and is cotinuous for* $t, \tau \in \Gamma$. *Let* $a_j(t) \in X^+ (j = \overline{1,n})$, *Then the equation*

$$c_0\varphi(t) + c_1(S\varphi)(t) + (TN\varphi)(t) = f(t), \tag{11}$$

here $c_0, c_1 \in C, c_0 \pm c_1 \neq 0, f(t) \in X$ *has a unique solution*

$$\varphi(t) = \frac{1}{c_0^2 - c_1^2}[c_0 f(t) - c_1(Sf)(t)] - \frac{1}{(c_0^2 - c_1^2)^2}[(c_0 I - c_1 S)TN(c_0 I - c_1 S)f](t).$$

roof. Rewrite the equation (11) as follows

$$[(c_0 I - c_1 S)\varphi](t) + (TN\varphi)(t) = f(t).$$

By our assumption we have $N^2 = 0, c_0 I + c_1 S$ is invertible, $(c_0 I + c_1 S)X_1 =$ $X_1, TX_1 \subset TX \subset X_1$, where $X_1 = \{g(t) \in X : (Ng)(t) = 0$.

Applying Theorem 1 we obtain the required result.

Let $X = C^\infty_{[a,b]}$ be the space of all real valued defined on a closed interval $[a,b]$ and aving continuous derivatives of an arbitrary order in (a,b). Let

$$(Dx)(t) = \frac{d}{dt}x(t),$$
$$(A_k x)(t) = \int_a^b \sum_{j=1}^q t^j a_{kj}(\tau)(D^j x)(\tau)d\tau \quad (k = \overline{1, n-1}, n \geq 2, q \geq 1),$$
$$A_0 = c_0 I,$$

here $c_0 \in R, c_0 \neq 0; a_{kj}(t), x(t) \in X, k = \overline{1, n-1}, j = \overline{1,q}$. Denote

$$X_\nu = \{x(t) \in X : (D^\nu x)(t) = 0\}, \quad \nu = 1, 2, \cdots$$
$$\tilde{X} = \bigcup_{\nu=1}^\infty X_\nu$$

Corollary 4. *The equation*

$$\sum_{k=1}^{n-1}(A_k D^k x)(t) = y(t), \tag{12}$$

where $y(t) \in \tilde{X}$ has a unique solution in $\tilde{X}$. More precisely if $y(t) \in X_m$, then there is unique $x(t) \in X_N$ satysfying equation (8) and such that

$$x = c_0^{-1}y(t) + \sum_{\mu=2}^{N}(-1)^{\mu-1} \sum_{\substack{1 \le j_1+j_2+\cdots+j_{\mu-1} \le N-1 \\ j_1, j_2, \cdots, j_{\mu-1} = \overline{1, n-1}}} c_0^{-\mu}(B_{j_1}B_{j_2}\cdots B_{j_{\mu-1}}y)(t),$$

where $N = \max(n, m)$, $B_k = A_k D^k (k = \overline{1, n-1})$.

Proof. By our assumption we have $A_0, A_1, \cdots, A_{n-1} \in L_0(X)$, A_0 is invertible an $A_0 X_\nu = X_\nu, \nu = 1, 2, \cdots$

It is easy to check that

$$A_k X_\nu \subset X_\nu, \quad k = \overline{1, n-1}, \nu = 1, 2, \cdots$$

Applying Theorem 3 we obtain the required result. □

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REFERENCE

1. F.D. Gakhov. *Boundary value problems.* Oxford, 1996 (3rd Russian complemente and corrected edition, Moscow, 1977.)
2. Ng. V. Mau. Generalized algebraic elements and linear singular integral equation with transformed arguments, *Wydawnictwa Politechniki Warszawskiej, Warszaw* 1989.
3. D. Przeworska - Rolewicz. *Algebraic analysis.* Amsterdam, Warszawa, 1988.
4. D. Przeworska - Rolewicz. *Equations with transformed arguments, An algebra approach.* Amsterdam, Warszawa, 1973.
5. D. Przeworska - Rolewicz. *Equations in linear space.* Amsterdam, Warszawa, 196

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VỀ MỘT LỚP PHƯƠNG TRÌNH SINH BỞI TOÁN TỬ LŨY LINH VÀ ÁP DỤNG

Nguyễn Tấn Hoà
Trường Cao đẳng Sư phạm Gia Lai

Bài báo này đề cập đến một lớp phương trình dạng $\sum_{k=0}^{n-1} A_k S^k x = y$, trong đ $A_k, S \in L_0(X)(k = \overline{0, n-1})$, S là toán tử lũy linh. Chúng ta chỉ ra rằng với một vài gi thiết của $A_k(k = \overline{0, 1-n})$ tất cả các nghiệm của phương trình này thu được dưới dạn đóng. Sau đó ta đưa ra một vài áp dụng đối với các phương trình vi phân và tích phâ