GENERAL INTERPOLATION PROBLEMS INDUCED BY GENERALIZED RIGHT INVERTIBLE OPERATORS

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Abstract. *The general interpolation problem induced by a right invertible operator D with dim $\ker D = s$ $(0 < s < +\infty)$ was introduced and investigated by Przeworska-Rolewicz D., and then devoloped by Ng. V. Mau and Ph. Q. Hung (see [1], [2], [4]). Later, Ng. V. Mau and Ng. M. Tuan have constructed the generalized right invertible operators (see [3]).*
In this paper, we deal with a general interpolation problem induced by a generalized right invertible operator V with dim $\ker V = s$ $(0 < s < +\infty)$.

. Prelimilaries and notations.

Let X be a linear space over the field $\mathcal{K}$ of scalars, where $\mathcal{K} = \mathbb{R}$ or $\mathcal{K} = \mathbf{C}$. Denote y $L(X)$ the set of all linear operators with domains and ranges in X and by $R_1(X)$ the t of all generalized right invertible operators belong to $L(X)$. For every $V \in R_1(X)$, e denote by $\mathcal{R}_V^1$ the set of all right inverses of V and by $\mathcal{F}_V$ the set of all right initial perators for V, i.e,

$$\mathcal{R}_V^1 := \{W \in L(X): \ VWV = V, \ V^2W = V\},$$
$$\mathcal{F}_V := \{F \in L(X): \ FX = \ker V, \ F^2 = F \ \text{ and } \exists W \in \mathcal{R}_V^1 \ \text{ such that } FW = 0\}.$$

In the sequel, we shall assume that dim $\ker V = s$ $(0 < s < +\infty)$ and $\{e_1, \dots, e_s\}$ a basis of $\ker V$. Then $\ker V = \bigoplus_{m=1}^{s} Z_m$, where $Z_m = \text{lin}\{e_m\}$, $m = 1, \dots, s$. Write

$$P_{N_m}(W) = \text{lin}\{W^k e_m, \ k = 0, \dots, N-1\}, \ m = 1, \dots, s. \tag{1.1}$$

Definition 1.1. *Let $V \in R_1(X)$ and $W \in \mathcal{R}_V^1$. Every element*

$$u = \sum_{k=0}^{N-1} W^k z_k, \tag{1.2}$$

here $z_{N-1} \neq 0$, $z_0, \dots, z_{N-1} \in \ker V$ is said to be V - polynomial u of degree $N-1$.

. General interpolation problem (GIP)

Consider the following problem

Given n finite sets I_i of different non-negative integers with $\# I_i = r_i$, $r_1 + \cdots + r_n =$ $(i = 1, \dots, n)$. Find a V - polynomial u of degree $N-1$ satisfying N conditions

$$F_i V^k u = u_{ik}, \ k \in I_i, \ i = 1, \dots, n, \tag{2.1}$$

where $u_{ik} \in \ker V$ are given and $F_1, \dots, F_n$ are different right initial operators of V.

In the sequel, this problem will be called a general interpolation problem for gene alized right invertible operator (Shorthy : GIP).

Suppose that the elements of sets I_i are ordered r_i - tuples $(k_{i_1}, \dots, k_{ir_i})$, 0 $k_{i_1} < \dots < k_{ir_i}$, $i = 1, \dots, n$. Then (2.1) is of the form

$$F_i V^{k_{ij}} u = u_{ik_{ij}}, \ i = 1, \dots, n; \ j = 1, \dots, r_i. \tag{2.}$$

Write

$$u_{ik_{ij}} := u_{r_0 + \dots + r_{i-1} + j}, \ r_0 = 0, \tag{2.}$$

$$u_v := \sum_{\eta=1}^{s} u'_{(v-1)s+\eta} e_\eta, \ u'_{(v-1)s+\eta} \in \mathcal{K}, \ v = 1, \dots, N, \tag{2.}$$

$$z_k := \sum_{\mu=1}^{s} z'_{ks+\mu} e_\mu, \ z'_{ks+\mu} \in \mathcal{K}, \ k = 0, \dots, N-1. \tag{2.}$$

Now, we can rewrite (1.2) in the form

$$u = \sum_{k=0}^{N-1} \sum_{\mu=1}^{s} z'_{ks+\mu} W^k e_\mu. \tag{2.}$$

It follows

$$F_i V^{k_{ij}} u = \sum_{k=0}^{N-1} \sum_{\mu=1}^{s} z'_{ks+\mu} F_i V^{k_{ij}} W^k e_\mu = \sum_{k=k_{ij}}^{N-1} \sum_{\mu=1}^{s} z'_{ks+\mu} F_i V W^{k-k_{ij}+1} e_\mu.$$

Since $F_i \in \mathcal{F}_V$ and $F_i X = \ker V$, we get

$$F_i V W^{k-k_{ij}+1} e_\mu = \sum_{\eta=1}^{s} \beta_{i,(k-k_{ij}),\mu,\eta} e_\eta; \ \beta_{i,(k-k_{ij}),\mu,\eta} \in \mathcal{K}. \tag{2.}$$

These equalities imply that

$$\sum_{k=k_{ij}}^{N-1} \sum_{\mu=1}^{s} z'_{ks+\mu} \beta_{i,(k-k_{ij}),\mu,\eta} = u'_{(r_0+\dots+r_{i-1}+j-1)s+\eta}. \tag{2.}$$

Write

$$\beta_{i,(k-k_{ij}),\mu,\eta} = \begin{cases} \beta'_{(r_0+\dots+r_{i-1}+j-1)s+\eta, ks+\mu} & \text{if } k \geq k_{ij}, \\ 0 & \text{if } 0 \leq k < k_{ij} \end{cases} \tag{2.}$$

$$i = 1, \dots, n; \ k = 0, \dots, N-1; \ j = 1, \dots, r_i; \ \mu, \eta = 1, \dots, s.$$

Rewrite (2.8) in the form

$$\sum_{q=k_{ij}s+1}^{Ns} z'_q \beta'_{p,q} = u'_p, \qquad p = (r_0 + r_1 + \dots, + r_{i-1} + j - 1)s + \eta. \tag{2.1}$$

Write

$$d_p = G_i^{k_{ij}\eta} := (\underbrace{0,\dots,0}_{k_{ij}s\ zero}, \beta'_{p,k_{ij}s+1},\dots,\beta'_{p,Ns}), \tag{2.11}$$

$$G_i^{(k_{ij})} := (G_i^{k_{ij}1},\dots,G_i^{k_{ij}s})^T, \tag{2.12}$$

$$G_i := (G_i^{(k_{i1})},\dots,G_i^{(k_{ir_i})})^T, \tag{2.13}$$

$$G := (G_1,\dots,G_n)^T, \tag{2.14}$$

$$F_{ik_{ij}} := F_i V^{k_{ij}}, \tag{2.15}$$

$$\widehat{F}_i^{(k_{ij})} := (F_{ik_{ij}}, F_{ik_{ij}}W,\dots,F_{ik_{ij}}W^{N-1}), \tag{2.16}$$

where A^T is the transposed matrix of A.

Lemma 2.1.. *The system of operators* $\{F_{ik_{ij}};\ i=1,\dots,n;\ j=1,\dots,r_i\}$ *is linearly independent on every* $P_{N_m}(W)$ $(m=1,\dots,s)$ *if and only if the system of vector-operators* $\{\widehat{F}_i^{(k_{ij})};\ i=1\dots,n;\ j=1,\dots,r_i\}$ *is linearly independent on every* Z_m $(m=1,\dots,s)$, *where* $P_{N_m}(W), F_{ik_{ij}}, \widehat{F}_i^{(k_{ij})}$ *are defined by* (1.2), (2.15), (2.16), *respectively.*

Proof. The system $\{\widehat{F}_i^{(k_{ij})},\ i=1,\dots,n;\ j=1,\dots,r_i\}$ is linearly independent on Z_m $(m=1,\dots,s)$, i.e, the equality

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\alpha_{ij}\widehat{F}_i^{(k_{ij})}u_m=0,\ \ \forall u_m\in Z_m,\ \ \alpha_{ij}\in\mathcal{K},\ \ m=1,\dots,s,$$

implies $\alpha_{ij}=0$ for $i=1,\dots,n;\ j=1,\dots,r_i$. It means that

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\alpha_{ij}F_{ik_{ij}}W^k u_m=0;\ \ k=0,\dots,N-1.$$

i.e,

$$\sum_{k=0}^{N-1}\beta_k\sum_{i=1}^{n}\sum_{j=1}^{r_i}\alpha_{ij}F_{ik_{ij}}W^k e_m=0,\ \ \forall\beta_k\in\mathcal{K},\ \ Z_m=\text{lin}\{e_m\},$$

if and only if

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\alpha_{ij}F_{ik_{ij}}\left(\sum_{k=0}^{N-1}\beta_k W^k e_m\right)=0,\ \ \forall\beta_k\in\mathcal{K}.$$

Therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\alpha_{ij}F_{ik_{ij}}x_m=0,\ \ \forall x_m\in P_{N_m}(W).$$

The proof is complete. △

Lemma 2.2. *The system* $\{\widehat{F}_i^{(k_{ij})};\ i=1,\dots,n;\ j=1,\dots,r_i\}$ *is linearly independent on every* Z_m $(m=1,\dots,s)$ *if and only if the system* $\{d_p;\ p=1,\dots,Ns\}$ *is linearly independent, where* d_p *is defined by formula* (2.11).

Proof. Let the system $\{d_p;\ p=1,\dots,Ns\}$ be linearly independent and

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\alpha_{ijm}\widehat{F}_i^{(k_{ij})}u_m=0,\ \forall u_m\in Z_m,\ \alpha_{ijm}\in\mathcal{K},\ m=1,\dots,s.$$

Then

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\sum_{m=1}^{s}\alpha_{ijm}F_iV^{k_{ij}}W^ku_m=0,\ k=0,\dots,N-1.$$

Hence,

$$\sum_{k=0}^{N-1}\gamma_k\sum_{i=1}^{n}\sum_{j=1}^{r_i}\sum_{m=1}^{s}\alpha_{ijm}F_iV^{k_{ij}}W^ku_m=0,\ \forall\gamma_k\in\mathcal{K},$$

i.e.

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\sum_{m=1}^{s}\sum_{k=k_{ij}}^{N-1}\alpha_{ijm}\gamma_kF_iVW^{k-k_{ij}+1}e_m=0,\ \forall\gamma_k\in\mathcal{K}.$$

By the choosing $\gamma_k\in\mathcal{K}$ and by formula (2.7) we have

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\sum_{m=1}^{s}\alpha_{ijm}\sum_{\mu=1}^{s}\beta_{i,(k-k_{ij}),m,\mu}e_\mu=0.$$

Since $\{e_1,\dots,e_s\}$ is a basis of $\ker V$ then

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\sum_{m=1}^{s}\alpha_{ijm}\beta_{i,(k-k_{ij}),m,\mu}=0,\ k=0,\dots,N-1;\ \mu=1,\dots,s.$$

The formulas (2.8) and (2.9) together imply

$$\sum_{p=1}^{Ns}\alpha'_pd_p=0.$$

Now, our assumption implies that $\alpha'_p=0$, i.e. $\alpha_{ijm}=0$. So the system $\{\widehat{F}_i^{(k_{ij})};i=1,\dots,n,\ j=1,\dots,r_i\}$ is linearly independent on Z_m $(m=1,\dots,s)$.

Conversely, suppose that the system $\{\widehat{F}_i^{(k_{ij})}\}$ is linearly independent on every Z_m $(m=1,\dots,s)$ i.e, the equality

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}\sum_{m=1}^{s}\alpha_{ijm}\widehat{F}_i^{(k_{ij})}u_m=0,\ \forall u_m\in Z_m,\ \alpha_{ijm}\in\mathcal{K}.$$

implies $\alpha_{ijm}=0$ for $i=1,\dots,n;\ j=1,\dots,r_i$.

On the other hand, if $\sum_{p=1}^{Ns}\alpha'_pd_p=0,\ \alpha'_p\in\mathcal{K}$ then $\alpha_{ijm}=\alpha'_{(r_0+\dots+r_{i-1}+j)s+m}$. S $\alpha'_p=0$ i.e. the system $\{d_p;\ p=1,\dots,Ns\}$ is linearly independent.

The proof is complete. △

By combining Lemmas 2.1 and 2.2, it follows:

Theorem 2.1. *A necessary and sufficient condition for det $\widehat{G} \neq 0$ is that the system $\{F_{ik_{ij}};\ i = 1,\dots,n;\ j = 1,\dots,r_i\}$ is linearly independent on every $P_{N_m}(W)$ $(m = 1,\dots,s)$, where G and $P_{N_m}(W)$ are defined by (2.11) – (2.14) and (1.1).*

Theorem 2.2. *The GIP has a unique solution for every $u_{ik} \in \ker V$ $(i = 1,\dots,n;\ j = 1,\dots,r_i)$ if and only if the system $\{F_{ik_{ij}};\ i = 1,\dots,n;\ j = 1,\dots,r_i\}$ is linearly independent on every $P_{N_m}(W)$ $(m = 1,\dots,s)$, where $r_1 + \cdots, + r_n = N$. If this condition is satisfied then the unique solution of the GIP is of the form*

$$u = \sum_{k=0}^{N-1} \sum_{\mu=1}^{s} z'_{ks+\mu} W^k e_\mu, \tag{2.17}$$

where $(z'_1,\dots,z'_{Ns})$ is a unique solution of the system

$$G(z'_1,\dots,z'_{Ns})^T = (u'_1,\dots,u'_{Ns})^T, \tag{2.18}$$

in which G is defined by (2, 11) – (2.14) and the elements $u'_1,\dots,u'_{Ns}$ are defined by (2.3) – (2.4).

Proof. Note that every solution of the GIP is of the form

$$u = \sum_{k=0}^{N-1} W^k z_k = \sum_{k=0}^{N-1} \sum_{\mu=1}^{s} W^k z'_{ks+\mu} e_\mu = \sum_{k=0}^{N-1} \sum_{\mu=1}^{s} z'_{ks+\mu} W^k e_\mu,$$

where $z'_{ks+\mu}$ is defined by (2.8), (2.9), (2.10).

The assumption and Theorem 2.1 imply that det $G \neq 0$ and we have the proof of the theorem. △

As an application of theorems 2.1, 2.2 we shall give a solution of Hermite classical interpolation problems for generalized right invertible operators.

Let $I_i = \{0,\dots,r_i - 1\}$, $i = 1,\dots,n$, then we obtain the following Hermite interpolation problem: Find a V- polynomial u of degree $N - 1$ satisfying N conditions

$$F_i V^j u = u_{ij};\ \ i = 1,\dots,n;\ \ j = 0,\dots,r_i - 1,$$

where $r_1 + \cdots, + r_n = N$, $u_{ij} \in \ker V$ are given.

Theorem 2.3. *The Hermite interpolation problem has a unique solution for every $u_{ik} \in \ker V$ $(i = 1,\dots,n\ \ j = 0,\dots,r_i - 1)$ if and only if the system $\{F_i V^j; i = 1,\dots,n;\ j = 0,\dots,r_i - 1\}$ is linearly in dependent on every $P_{N_m}(W)$ $(m = 1,\dots,s)$. If this condition is satisfied, then the unique solution of this problem is of the form*

$$u = \sum_{k=0}^{N-1} \sum_{\mu=1}^{s} z'_{ks+\mu} W^k e_\mu,$$

where $(z'_1, \ldots, z'_{Ns})$ *is a unique solution of the system*

$$G(z'_1, \ldots, z'_{Ns})^T = (u'_1, \ldots, u'_{Ns})^T,$$

in which G *is defined by* (2.11) – (2.14) *with* $k_{ij} = j,\ j = 0, \ldots, r_i - 1$ *and the element* $u'_1, \ldots, u'_{Ns}$ *are defined as follows*

$$u_{ij} := u_{r_0 + \cdots + r_{i-1} + j + 1},$$

$$u_v := \sum_{\eta=1}^{s} u'_{(v-1)s+\eta} e_\eta, \quad v = 1, \ldots, N.$$

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BÀI TOÁN NỘI SUY TỔNG QUÁT ĐỐI VỚI TOÁN TỬ KHẢ NGHỊCH PHẢI SUY RỘNG

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Bài toán nội suy tổng quát trong lớp toán tử khả nghịch phải D với dim ker D = $(0 < s < +\infty)$ đã được Przeworska-Rolewicz, Nguyễn Văn Mậu và Phạm Quang Hưn nghiên cứu trong [1], [2], [4]. Sau đó Nguyễn Văn Mậu và Nguyễn Minh Tuấn đã xá dựng khái niệm toán tử khả nghịch phải suy rộng (xem [3]).

Trong bài này, chúng tôi giải quyết bài toán nội suy tổng quát trong lớp toán t khả nghịch phải suy rộng V với dim ker $V = s$ $(0 < s < +\infty)$.