THE EXTENSION PROBLEM FOR A LINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS WITH FUNCTION COEFFICIENTS

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1. Introduction

It is well known that there are some functions on single variable complex, which are anilytic in the $C \setminus \{0\}$, but can not be analytically extended to the isolated singularity $z = 0$. The function $\omega(z) = \dfrac{1}{z}$ is an example of such ones.

According to Hartogs'extension theorem, this phenamenon is no longer valid for analytic functons on several complex veriables (see [1]).

Considering analytic fuctions on several complex variables as solutions of the Cauchy-Riemann system, the Hartogs'theorem deals actually with the extension of the solutions of linear system of first order partial differential equations with constant coefficients.

We are interested in the following problem: Does the Hartogs' theorem still hold for the linear system of partial differential equations of the form:

$$L^{(\ell)}(u) := \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^{(\ell)} \frac{\partial u_i}{\partial x_j} = f^{(\ell)}, \quad l = \overline{1, L}, \tag{1.1}$$

where $A_{ij}^{(\ell)}$ are fuctions for all $i = \overline{1,m}, j = \overline{1,n}, \ell = \overline{1,L}, u_i = u_i(x_1, ..., x_n)$ are real analytic on $x_1, ..., x_n$ and $u = (u_1, ..., u_m)$ is the unknown function.

In [3, 4], Le Hung Son showed some matrix criteritions for extension of solutions the above system with constant coefficiens. In this paper we prove similar result for the case, where the coefficients $A_{ij}^{(\ell)}$ are real analytic funtion on variables $x_1, ..., x_n$.

2. Preliminaries

Let G be a domain in R^n, $\sum$ be an open neighbourhood of ∂G,

Definition 2.1. *Suppose that $u = (u_1(x), ..., u_m(x))$ is a solution of the system (1.1) in G and $\tilde{u} = (\tilde{u}_1(x), ..., \tilde{u}_m(x))$ is a solution of the same system in another domain $\tilde{G}$, with $G \subseteq \tilde{G} \subsetneq R^n$, then $\tilde{u}$ is called a continuous extension of u if $\tilde{u} = u$ in G.*

The unique theorem on extention claims that if there exists an extension $\tilde{u}$ of u, then it is unique (see [1, 2]).

Noting:

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$$A^{(\ell)} = (A_{ij}^{(\ell)})_{m\times n}, \ l = \overline{1,L}, \ \vec{\lambda}_i = (\lambda_1^{(i)}, ..., \lambda_L^{(i)}) \qquad (2.1$$

$$D^{(i)} = \sum_{l=1}^{L} \lambda_l^{(i)} A^{(\ell)}, \quad D^{(i)} = (D_{kj}^{(i)})_{m\times n}, \ i = \overline{1,m} \qquad (2.2$$

$$B = [D^{(1)}, \ldots, D^{(m)}], \quad C = \begin{bmatrix} D^{(1)} \\ \vdots \\ D^{(m)} \end{bmatrix}, \qquad (2.3$$

then B is a $m \times m.n$ - matrix and C is a $m^2 \times n$ -matrix.

If $A_{ij}^{(\ell)}$ are constant, we get the following results (see [3, 4]):

Theorem 2.1. *Suppose that $m \leq n$ and there exist m vectors $\vec{\lambda}_1, ..., \vec{\lambda}_m$ such that:*

1. rank $D^{(i)} = 1, \quad i = \overline{1,m}$

2. rank $B = m$, rank $C = m$.

Then every real analytic solution of (2.1) in Σ can continuously be extended to solution of the same system in the whole domain G.

When m, n are arbitrary, we have:

Theorem 2.2. *Suppose that there exist m vectors $\vec{\lambda}_1, ..., \vec{\lambda}_m$ such that:*

1. rank $D^{(i)} = 1, \quad i = \overline{1,m}$

2. rank $B = m$, rank $C = 1$.

Then every real analytic solution of (2.1) in Σ can continuously be extended to solution of the same system in the whole domain G.

3. The extension theorem in the case $A_{ij}^{(\ell)}$ are functions

Assume that all coefficients $A_{ij}^{(l)}$ are (real) analytic functions in $G \cup \Sigma$, where G $G_1 \times \cdots \times G_n$, $G_j = \{(a_j, b_j)\} \subset R(X_j)$. We get:

Theorem 3.1. Suppose that $m \leq n$ and there exist m vectors $\vec{\lambda}_i$ satisfying

1. *For each i ($i = \overline{1,m}$), there exists an element $D_{kj}^{(i)} \neq 0$ defined in the whole $\Sigma \cup G$ and*

$$D_{k\ell}^{(i)} = D_{kj}^{(i)}.\alpha_{\ell i}, \quad \alpha_{\ell i} = \text{const}, \quad \ell = \overline{1,n}$$

2. *Rank $D^{(i)} = 1$, $i = \overline{1,m}$,*
3. *Rank $B = m$*
4. *Rank $C = m$*
5. $\sum\limits_{\ell=1}^{L} \lambda_\ell^{(i)} \Big(\sum\limits_{j=1}^{n} \dfrac{\partial A_{kj}^{(\ell)}}{\partial x_j} \Big) = 0, \ i, k = \overline{1,m}.$

Then every real analytic solution of (1.1) in Σ can continuously be extended to a olution of the same system in the whole domain G.

roof. First, because of the assumption, there exist the injective linear mappings

$$x = T(\xi) \quad (T: R^n(\xi_1, \dots, \xi_n) \to R^n(x_1, \dots, x_n)), \tag{3.1}$$

nd

$$u' = T_1(u) \quad (T_1: R^m(u_1, \dots, u_m) \to R^m(u'_1, \dots, u'_m)), \tag{3.2}$$

here $\xi = (\xi_1, \dots, \xi_n)$, $x = (x_1, \dots, x_n)$, $u = (u_1, \dots, u_m)$, $u' = (u'_1, \dots, u'_m)$.

We denote

$$G' = T(G) \tag{3.3}$$

$$\Sigma' = T(\Sigma). \tag{3.4}$$

Then it is easy to see that G' is a domain in $R^n(\xi_1, \dots, \xi_n)$ and Σ' is an open ighborhood of $\partial G'$. Since the functions $u_1(x), \dots, u_m(x)$ are given in Σ and real analytic $x_1, \dots, x_n$ then it follows that $u'_1(\xi), \dots u'_m(\xi)$ are given in Σ' and real analytic on $\dots, \xi_n$. Since

$$\frac{\partial u'_1}{\partial \xi_1} = \varphi_1(\xi_1, \dots, \xi_n), \dots, \frac{\partial u'_m}{\partial \xi_m} = \varphi_m(\xi_1, \dots, \xi_n), \tag{3.5}$$

here $\varphi_k(\xi_1, \dots, \xi_n)$ are defined in the whole of $\Sigma' \cup G'$ and real analytic on $\xi_1, \dots, \xi_n$, it llows that:

$$u'_1 = \int \varphi_1 d\xi_1 + \psi_1(\xi_2, \dots, \xi_n),$$

here $\int \varphi_1 d\xi_1$ is a primitive of φ_1 in respect to ξ_1, then:

$$\psi_1(\xi_2, \dots, \xi_n) = u'_1 - \int \varphi_1 d\xi_1. \tag{3.6}$$

Since u'_1 is given only in Σ', then the right hand side of (3.6) is defined only for ch ξ_1. But ψ_1 is independent of ξ_1, therefore this function can be continuously extended r all $\xi_1 \in G'$. This means that u'_1 can be continuously extended in the whole of G'. It easy to show that u'_1 is real analytic in G'. By an similar method we can show that all $, i = 2, \dots, n$ can be continuously extended in the whole of G', and these functions are al analytic on $\xi_1, \dots, \xi_n$ in G'. We denote these extension of u'_i by $\tilde{u}'_i(\xi)$ and consider

$$\tilde{u} = T_1^{-1}(\tilde{u}'), \quad \text{where } \tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_m),\ \tilde{u}' = (\tilde{u}'_1, \dots, \tilde{u}'_m). \tag{3.7}$$

It is easy to show that $\tilde{u}_i$ are defined in the whole domain G and real analytic on $, \dots, x_n$. On the other hand

$$\tilde{u}_i = u_i \text{ in } \Sigma. \tag{3.8}$$

d $u = (u_1, \dots, u_m)$ is a solution of (1.1) in Σ, it follows:

$$L^{(\ell)}(\tilde{u}) = L^{(\ell)}(u) = f^{(\ell)} \text{ in } \Sigma,\ \ell = \overline{1, L}. \tag{3.9}$$

Further it is obvious that $L^{(\ell)}(\tilde{u})$ is real analytic on $x_1, \dots, x_n$. Because of the uniquene theorem and of (3.9) we get:

$$L^{(\ell)}(\tilde{u}) = f^{(\ell)} \quad \text{in the whole of } G \text{ for } \ell = \overline{1, L}. \qquad (3.1$$

The condition (3.10) means that $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_m)$ is a solution of (1.1) in G. Hence, $\tilde{u}$ the extension of u in G.

The problem is now reduced to prove the existence of two injective linear mappin T_1 and T.

Let us consider the matrix $D^{(1)}$.

According to (1) there exists an element $D^{(1)}_{kj} \neq 0$ in $\Sigma \cup G$. Without loss generality we may assume that $D^{(1)}_{11} \neq 0$. Let $D^{(1)}_{11} = a_{11}(x) \cdot \alpha_{11}$, $(\alpha_{11} = 1)$. Th the first row of $D^{(1)}$ has the form: $|a^{(1)}_{11}\alpha_{11}, a^{(1)}_{11}\alpha_{21}, \dots, a^{(1)}_{11}\alpha_{n1}|$, where $\alpha_{11}, \dots, \alpha_{n1}$ a constant.

Under the above assumption of $D^{(1)}_{11} \neq 0$, the function $\gamma^{(1)}_k(x) = \dfrac{D^{(1)}_{k1}}{D^{(1)}_{11}}$ is the re analytic. Furthermore, since rank $D^{(1)} = 1$, the k- row of $D^{(1)}$ is of the form

$$|D^{(1)}_{k1}, D^{(1)}_{k2}, \dots, D^{(1)}_{kn}| = \gamma^{(1)}_k |a^{(1)}_{11}\alpha_{11}, a^{(1)}_{11}\alpha_{21}, \dots, a^{(1)}_{11}\alpha_{n1}|.$$

We denote $\gamma^{(1)}_k a^{(1)}_{11} = a_{1k}$ then $D^{(1)}_{kj} = a_{1k} \cdot \alpha_{j1}$, where $a_{1k}(x)$ are real analytic fu tions on $(x_1, \dots, x_n)$, $k = \overline{2, m}$, $j = \overline{1, n}$. Thus there exist the tuple $(a_{11}(x), \dots, a_{1m}($ of real analytic functions and the tuple of constants $(\alpha_{11}, \alpha_{21}, \dots, \alpha_{n1})$ such that: $D^{(1)}_{kj}$ $a_{1k} \cdot \alpha_{j1}$, $k = \overline{1, m}$, $j = \overline{1, n}$. Similarly, by using the assumptions (1) and (2), it follows th there exist the tuples $(a_{i1}(x), \dots, a_{im}(x))$ of real analytic functions on $x_1, \dots, x_n$ and t tuples of constants $(\alpha_{1i}, \dots, \alpha_{2i}, \dots, \alpha_{ni})$ such that: $D^{(i)}_{kj} = a_{ik}\alpha_{ji}$, $i = \overline{2, m}$. Since ra $B = m$ and by (1), we can extract in each matrix $D^{(i)}$ a column to obtain a sub-matri

$$B^* = \begin{pmatrix} a_{11} \cdot \alpha_{j_1 1} & a_{21}\alpha_{j_2 2} & \cdots & a_{m1}\alpha_{j_m m} \\ a_{12} \cdot \alpha_{j_1 1} & a_{22}\alpha_{j_2 2} & \cdots & a_{m2}\alpha_{j_m m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} \cdot \alpha_{j_1 1} & a_{2m}\alpha_{j_2 2} & \cdots & a_{mm}\alpha_{j_m m} \end{pmatrix} \qquad (3.1$$

such that det $B^* \neq 0$.

From (3.11) it follows that:

$$\det \begin{bmatrix} a_{11}(x) & a_{21}(x) & \dots & a_{m1}(x) \\ a_{12}(x) & a_{22}(x) & \dots & a_{m2}(x) \\ \vdots & \ddots & \vdots & \\ a_{1m}(x) & a_{2m}(x) & \dots & a_{mm}(x) \end{bmatrix} \neq 0 \quad \text{in } \Sigma \cup G. \qquad (3.$$

Since rank $C = m$ and by (1) we can exact in each matrix $D^{(i)}$ a row to obtai sub-matrix:

$$C^* = \begin{pmatrix} a_{1k_1}\alpha_{11} & a_{1k_1}\alpha_{21} & \cdots & a_{1k_1}\alpha_{n1} \\ a_{2k_2}\alpha_{12} & a_{2k_2}\alpha_{22} & \cdots & a_{2k_2}\alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mk_m}\alpha_{1m} & a_{mk_m}\alpha_{2m} & \cdots & a_{mk_m}\alpha_{nm} \end{pmatrix} \qquad (3.$$

such that rank $C^* = m$. Hence

$$\det\begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1m} & \alpha_{2m} & \dots & \alpha_{nm} \end{bmatrix} \neq 0. \tag{3.14}$$

From (3.14) we get m vectors:

$$\begin{aligned} \vec{\alpha}_1 &= (\alpha_{11}, \alpha_{21}, \dots, \alpha_{n1}) \\ \vec{\alpha}_2 &= (\alpha_{12}, \alpha_{22}, \dots, \alpha_{n2}) \\ \dots & \quad \dots \quad \dots \\ \vec{\alpha}_m &= (\alpha_{1m}, \alpha_{2m}, \dots, \alpha_{nm}), \end{aligned}$$

which are linearly independent. Because of $m \leq n$, we may add to this system $(n - m)$ vectors $\vec{\alpha}_{m+1}, \dots, \vec{\alpha}_n$ such that the resulting system $\{\vec{\alpha}_1, \dots, \vec{\alpha}_m, \dots, \vec{\alpha}_n\}$ is linearly independent. Thus exist the numbers α_{ij}, $i, j = \overline{1, n}$, such that

$$\det\begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{bmatrix} \neq 0. \tag{3.15}$$

Now we consider two injective mappings:

$$\begin{aligned} T &: R^n(\xi_1, \dots, \xi_n) \to R^n(x_1, \dots, x_n) \\ x_\ell &= \sum_{k=1}^{n} \alpha_{\ell k}\xi_k \\ T_1 &: R^m(u_1, \dots, u_m) \to R^m(u'_1, \dots, u'_m) \\ u'_i(x) &= \sum_{j=1}^{m} a_{ij}(x)u_j(x), \end{aligned}$$

and then we get

$$\begin{aligned} \frac{\partial u'_i}{\partial \xi_k} &= \sum_{j=1}^{m}\sum_{\ell=1}^{n} \frac{\partial a_{ij} \cdot u_j}{\partial x_\ell} \cdot \frac{\partial x_\ell}{\partial \xi_k} \\ &= \sum_{j=1}^{m}\sum_{\ell=1}^{n} \left(a_{ij}\frac{\partial u_j}{\partial x_\ell} + u_j\frac{\partial a_{ij}}{\partial x_\ell}\right)\alpha_{\ell k} \\ &= \sum_{j=1}^{m}\sum_{\ell=1}^{n} a_{ij}\alpha_{\ell k}\frac{\partial u_j}{\partial x_\ell} + \sum_{j=1}^{m}\sum_{\ell=1}^{n} u_j \cdot \alpha_{\ell k} \cdot \frac{\partial a_{ij}}{\partial x_\ell}. \end{aligned}$$

Let $i = k$, replace j by k and ℓ by j, we get:

$$\frac{\partial u'_i}{\partial \xi_i} = \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ik} \cdot \alpha_{ji}\frac{\partial u_k}{\partial x_j} + \sum_{k=1}^{m}\sum_{j=1}^{n} u_k \cdot \frac{\partial \alpha_{ji} \cdot a_{ik}}{\partial x_j}.$$

Putting

$$M = \sum_{k=1}^{m}\sum_{j=1}^{n} \alpha_{ji} \cdot a_{ik} \cdot \frac{\partial u_k}{\partial x_j}, \quad N = \sum_{k=1}^{m}\sum_{j=1}^{n} u_k \cdot \frac{\partial \alpha_{ji} a_{ik}}{\partial x_j},$$

we get:

$$\begin{aligned} M &= \sum_{k=1}^{m}\sum_{j=1}^{n} D_{kj}^{(i)} \frac{\partial u_k}{\partial x_j} \\ &= \sum_{k=1}^{m}\sum_{j=1}^{n} (\lambda_1^{(i)} A_{kj}^{(1)} + \cdots + \lambda_L^{(i)} A_{kj}^{(L)}) \frac{\partial u_k}{\partial x_j} \\ &= \sum_{\ell=1}^{L} \lambda_\ell^{(i)} f^{(l)} = \varphi_i(\xi_1, \ldots, \xi_n), \end{aligned}$$

where φ_i are real analytic on $\xi_1, \ldots, \xi_n$, and we have

$$N = \sum_{k=1}^{m}\sum_{j=1}^{n} u_k \cdot \frac{\partial D_{kj}^{(i)}}{\partial x_j}.$$

Since

$$\begin{aligned} \sum_{j=1}^{n} \frac{\partial D_{kj}^{(i)}}{\partial x_j} &= \sum_{j=1}^{n} \frac{\partial (\lambda_1^{(i)} A_{kj}^{(1)} + \cdots + \lambda_L^{(i)} \cdot A_{kj}^{(L)})}{\partial x_j} \\ &= \sum_{\ell=1}^{L} \lambda_\ell^{(i)} \sum_{j=1}^{n} \frac{\partial A_{kj}^{(\ell)}}{\partial x_j} = 0 \end{aligned}$$

(because of (5)), it follows $N = 0$, hence

$$\frac{\partial u_i'}{\partial \xi_i} = M + N = \varphi_i(\xi_1, \ldots, \xi_n).$$

Thus we have shown that there exist two injective linear mappings satisfying (3.1), (3. and (3.5). The proof of Theorem (3.3) is complete.

Remark. When m, n are arbitrary, by an argument analogous to that used for the pro of Theorem (3.1), we get:

Theorem 3.2 *Suppose that there exist m vectors $\vec{\lambda}_1, \ldots, \vec{\lambda}_m$ such that:*

1) *For each i ($i = \overline{1,m}$) there exists an element $D_{kj}^{(i)} \neq 0$ in $\Sigma \cup G$, and $D_{k\ell}^{(i)}$ $D_{kj}^{(i)} . \alpha_{\ell i}$, $\alpha_{\ell i} =$ const, $\ell = \overline{1,n}$*

2) *Rank $D^{(i)} = 1$, $i = \overline{1,m}$*

3) *Rank $B = m$*

4) *Rank $C = 1$*

5) $\sum_{\ell=1}^{L} \lambda_i^{(\ell)} \sum_{j=1}^{n} \frac{\partial A_{kj}^{(\ell)}}{\partial x_j} = 0,\ i = \overline{1,m},\ k = \overline{1,m}.$

Then every real analytic solution of (1.1) in Σ can continuously be extended to a *olution of the same system in the whole domain G.*

. Application

xample 4.1 We consider the system

$$\begin{cases} A_{11}^{(1)} \dfrac{\partial u_1}{\partial x_1} + A_{22}^{(1)} \cdot \dfrac{\partial u_2}{\partial x_2} = f^{(1)} \\ A_{12}^{(2)} \dfrac{\partial u_1}{\partial x_2} + A_{21}^{(2)} \dfrac{\partial u_2}{\partial x_1} = f^{(2)} \\ (x_1^2 + x_2^2 + 3)\dfrac{\partial u_1}{\partial x_3} + (x_1^2 + 5x_2^2 + 1) \cdot \dfrac{\partial u_2}{\partial x_4} = f^{(3)} \\ (x_1^2 + x_2^2 + 3)\dfrac{\partial u_1}{\partial x_4} + (x_1^2 + 5x_2^2 + 1) \cdot \dfrac{\partial u_2}{\partial x_3} = f^{(4)} \end{cases} \tag{4.1}$$

here $A_{11}^{(1)}, A_{22}^{(1)}, A_{12}^{(2)}, A_{21}^{(2)}, f^{(1)}, \dots, f^{(4)}$ are the given real analytic functions on $x_1, \dots, x_4$ the whole $\Sigma \cup G$.

The system (4.1) has the form of (2.1) with $m = 2$, $n = 4$, $L = 4$.

We choose: $\vec{\lambda}_1 = (0, 0, 1, 1)$, $\vec{\lambda}_2 = (0, 0, -1, 1)$

Then we get:

$$D^{(1)} = \begin{pmatrix} 0 & 0 & x_1^2 + x_2^2 + 3 & x_1^2 + x_2^2 + 3 \\ 0 & 0 & x_1^2 + 5x_2^2 + 1 & x_1^2 + 5x_2^2 + 1 \end{pmatrix},$$

$$D^{(2)} = \begin{pmatrix} 0 & 0 & -(x_1^2 + x_2^2 + 3) & x_1^2 + x_2^2 + 3 \\ 0 & 0 & x_1^2 + 5x_2^2 + 1 & -(x_1^2 + 5x_2^2 + 1) \end{pmatrix}$$

is clear that

* $D_{13}^{(i)} \neq 0$ and $D_{1j}^{(i)} = D_{13}^{(i)} \cdot \alpha_{ji}$, $\alpha_{ji} = \text{const}$, $i = 1, 2$, $j = 1, 2, 3$.

* rank $D^{(1)} =$ rank $D^{(2)} = 1$

We choose

$$B^* = \begin{pmatrix} (x_1^2 + x_2^2 + 3) & -(x_1^2 + x_2^2 + 3) \\ (x_1^2 + 5x_2^2 + 1) & (x_1^2 + 5x_2^2 + 1) \end{pmatrix} \quad C^* = \begin{pmatrix} (x_1^2 + x_2^2 + 3) & (x_1^2 + x_2^2 + 3) \\ (x_1^2 + 5x_2^2 + 1) & -(x_1^2 + 5x_2^2 + 1) \end{pmatrix}$$

get $\det B^* \neq 0$, $\det C^* \neq 0$. Then rank $B =$ rank $C = 2 = m$. It is clear that:

$$\frac{\partial A_{11}^{(3)}}{\partial x_1} + \frac{\partial A_{12}^{(3)}}{\partial x_2} + \frac{\partial A_{13}^{(3)}}{\partial x_3} + \frac{\partial A_{14}^{(3)}}{\partial x_4} = 0,$$

d

$$\frac{\partial A_{21}^{(3)}}{\partial x_1} + \frac{\partial A_{22}^{(3)}}{\partial x_2} + \frac{\partial A_{23}^{(3)}}{\partial x_3} + \frac{\partial A_{24}^{(3)}}{\partial x_4} = 0.$$

Thus

$$\sum_{j=1}^{4} \frac{\partial A_{kj}^{(3)}}{\partial x_j} = 0.$$

Similarly

$$\sum_{j=1}^{4} \frac{\partial A_{kj}^{(4)}}{\partial x_j} = 0,$$

hence

$$\lambda_1^{(i)} \sum_{j=1}^{4} \frac{\partial A_{kj}^{(1)}}{\partial x_j} + \cdots + \lambda_4^{(i)} \sum_{j=1}^{4} \frac{\partial A_{kj}^{(4)}}{\partial x_j} = 0, \quad i = 1,2;\ k = 1,2.$$

Thus the Theorem (3.1) can be applied.

Example 4.2 We consider the system:

$$\begin{cases} (2x_1^2 + 2x_2^2)\dfrac{\partial u_1}{\partial x_1} - e^{x_1 - x_2}\dfrac{\partial u_2}{\partial x_2} = f^{(1)} \\ (4x_1x_2 + 2)\dfrac{\partial u_1}{\partial x_2} + e^{x_1 - x_2}\dfrac{\partial u_2}{\partial x_1} = f^{(2)} \\ -((x_1 + x_2)^2 + 1)\dfrac{\partial u_1}{\partial x_1} + ((x_1 - x_2)^2 - 1)\dfrac{\partial u_1}{\partial x_2} - e^{x_1 - x_2} \cdot \dfrac{\partial u_2}{\partial x_1} = f^{(3)}, \end{cases} \qquad (4.2)$$

The system (4.2) has the form of (2.1) with $m = 2$, $n = 2$, $L = 3$,

Choose: $\vec{\lambda}_1 = (0,1,1)$, $\vec{\lambda}_2 = (1,0,1)$, then

$$D^{(1)} = \begin{pmatrix} -(x_1 + x_2)^2 - 1, & (x_1 + x_2)^2 + 1 \\ 0, & 0 \end{pmatrix}, \quad D^{(2)} = \begin{pmatrix} (x_1 - x_2)^2 - 1, & (x_1 - x_2)^2 - \\ -e^{x_1 - x_2}, & -e^{x_1 - x_2} \end{pmatrix}$$

It is clear that rank $D^{(1)}$ = rank $D^{(2)} = 1$.

Choose

$$B^* = \begin{pmatrix} (x_1 + x_2)^2 + 1, & (x_1 - x_2)^2 - 1 \\ 0 & -e^{x_1 - x_2} \end{pmatrix}, \quad C^* = \begin{pmatrix} -(x_1 + x_2)^2 - 1, & (x_1 + x_2)^2 + 1 \\ -e^{x_1 - x_2}, & -e^{x_1 - x_2} \end{pmatrix}$$

Then we get rank B^* = rank C^* = rank B = rank $C = 2 = m$.

On the other side

$$* \quad \frac{\partial A_{11}^{(1)}}{\partial x_1} + \frac{\partial A_{12}^{(1)}}{\partial x_2} + \frac{\partial A_{11}^{(3)}}{\partial x_1} + \frac{\partial A_{12}^{(3)}}{\partial x_2} = 0 \quad * \quad \frac{\partial A_{21}^{(1)}}{\partial x_1} + \frac{\partial A_{22}^{(1)}}{\partial x_2} + \frac{\partial A_{21}^{(3)}}{\partial x_1} + \frac{\partial A_{22}^{(3)}}{\partial x_2} = 0$$

$$* \quad \frac{\partial A_{11}^{(2)}}{\partial x_1} + \frac{\partial A_{12}^{(2)}}{\partial x_2} + \frac{\partial A_{11}^{(3)}}{\partial x_1} + \frac{\partial A_{12}^{(3)}}{\partial x_2} = 0 \quad * \quad \frac{\partial A_{21}^{(2)}}{\partial x_1} + \frac{\partial A_{22}^{(2)}}{\partial x_2} + \frac{\partial A_{21}^{(3)}}{\partial x_1} + \frac{\partial A_{22}^{(3)}}{\partial x_2} = 0$$

Thus:

$$\lambda_1^{(i)} \sum_{j=1}^{2} \frac{\partial A_{kj}^{(1)}}{\partial x_j} + \lambda_2^{(i)} \sum_{j=1}^{2} \frac{\partial A_{kj}^{(2)}}{\partial x_j} + \lambda_3^{(i)} \sum_{j=1}^{2} \frac{\partial A_{kj}^{(3)}}{\partial x_j} = 0, \quad i = 1,2;\ k = 1,2.$$

Hence, we can apply the Theorem (3.1) for the system (4.2).

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Trong bài này, một số tiêu chuẩn ma trận cho việc thác triển nghiệm của một hệ hương trình đạo hàm riêng tuyến tính tổng quát được chứng minh. Điều đó chỉ ra rằng ện tượng thác triển nói trên có quan hệ mật thiết với hạng của ma trận các hệ số của ệ phương trình đó.