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MULTIPLIERS FOR GENERALIZED ENTIRE DIRICHLET SEQUENCE SPACES

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I. INTRODUCTION

Given a sequence (λ_k) with $\lambda_k \in \mathbf{C}$, $0 < |\lambda_k| \uparrow +\infty$ and $\rho > 0$, consider the generalized entire Dirichlet series

$$\sum_{k=1}^{\infty} c_k E_\rho(\lambda_k z), \quad z \in \mathbf{C}, \tag{1.1}$$

where coefficents c_k are complex numbers and $E_\rho(z)$ is the Mittag-Leffler function:

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\frac{n}{\rho}+1)} \quad (\Gamma \text{ being the Gamma function}).$$

In [5] we proved that if the series (1.1) converges absolutely for all $z \in \mathbf{C}$ then

$$\limsup_{k\to\infty} \frac{\log|c_k|}{|\lambda_k|^\rho} = -\infty, \tag{1.2}$$

and conversely, if the coefficients of the series (1.1) satisfy condition (1.2) and if

$$\limsup_{k\to\infty} \frac{\log k}{|\lambda_k|^\rho} < +\infty \tag{1.3}$$

then the series (1.1) converges absolutely for all $z \in \mathbf{C}$.

Next, in the case (λ_k) satisfies the condition (1.3), in the case of [2], we considered the following sequence space

$$\mathcal{A} = \{(c_k) : c_k \text{ satisfies } (1.2)\} = \{(c_k); \limsup_{k\to\infty} |c_k|^{1/|\lambda_k|^\rho} = 0\}.$$

Denoted by $\mathcal{A}^\alpha$ the Köthe dual of $\mathcal{A}$, i.e.,

$$\mathcal{A}^\alpha = \{(u_k) : \sum_{k=1}^{\infty} |c_k u_k| < +\infty \text{ for all } (c_k) \in \mathcal{A}\}.$$

we proved that $\mathcal{A}^\alpha = \mathcal{C}$, $\mathcal{A}^{\alpha\alpha} = \mathcal{A}$. Hence $\mathcal{C}^\alpha = \mathcal{A}$, where

$$\mathcal{C} = \left\{(u_k) : \limsup_{k\to\infty} |u_k|^{1/|\lambda_k|^\rho} < +\infty\right\}.$$

Furthermore, for each $c = (c_k) \in \mathcal{A}$, we defined

$$\|c\|_{\mathcal{A}} = \sup_{k\geq 1} |c_k|^{1/|\lambda_k|^\rho}$$

In [5], by using the same method as in [2], we proved that $\mathcal{A}$ is a complete separable, non-normable, metrizable space, where the metric is given by

$$d_{\mathcal{A}}(a, b) = \|a - b\|_{\mathcal{A}}; \quad a = (a_k) \in \mathcal{A}, \quad b = (b_k) \in \mathcal{A}.$$

In this note, we continue to study multipliers between these spaces and other sequence spaces on spaces $\mathcal{A}$ and $\mathcal{C}$.

We recall that for two sequence spaces X and Y, the symbol (X, Y) denotes the sequence space of multipliers from X to Y (see, e.g., [1]), i.e.,

$$(X, Y) = \{(u_k); \ (c_k u_k) \in Y \text{ for all } (c_k) \in X\}.$$

It is obvious that if

$$X_1 \subset X_2 \text{ and } Y_1 \subset Y_2, \text{ then } (X_2, Y_1) \subset (X_1, Y_2). \tag{1.4}$$

Also, it is clear that, the Köthe dual of a sequence space is, in fact, the sequence space of multipliers from this space to l_1 i.e., $(\mathcal{A}, l_1) = \mathcal{A}^\alpha$. A question arises: what about multipliers from $\mathcal{A}$ and $\mathcal{C}$ to l_p $(0 < p < +\infty)$ and vice-versa? This is the subject of the present note.

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II. MULTIPLIERS FOR GENERALIZED ENTIRE DIRICHLET SEQUENCE SPACES

First, we note the following result

Lemma 2.1. *We have*

$$\mathcal{A} \subset l_p \subset l_\infty \subset \mathcal{C}, 0 < p < +\infty.$$

We prove the following lemmas

Lemma 2.2. *We have*

a) $(\mathcal{A}, \mathcal{C}) \subset \mathcal{C}$,

b) $\mathcal{C} \subset (\mathcal{A}, \mathcal{A})$,

c) $\mathcal{C} \subset (\mathcal{C}, \mathcal{C})$.

Proof:

a) Let $(u_k) \in (\mathcal{A}, \mathcal{C})$. Suppose that $(u_k) \notin \mathcal{C}$. Then for arbitrary $M > 0$ and for a sequence (ε_n), $0 < \varepsilon_n \downarrow 0$, there exists an increasing sequence (k_n) of positive numbers such that

$$|u_{k_n}|^{1/|\lambda_{k_n}|^\rho} \geq M. - \varepsilon_n, \quad \forall n \geq 1.$$

We define the sequence (c_k) as follows

$$c_k = \begin{cases} |u_k|^{-1/2}, & \text{if } k = k_n, \ n = 1, 2, ..., \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{k\to\infty} |c_k|^{1/|\lambda_k|^\rho} = \limsup_{n\to\infty}\Big(|u_{k_n}|^{1/|\lambda_{k_n}|^\rho}\Big)^{-1/2} \le$$
$$\limsup_{n\to\infty}(M - \varepsilon_n)^{-1/2} = M^{-1/2} \to 0, \text{ as } M \to +\infty .$$

So $(c_k) \in \mathcal{A}$. However, we have

$$\limsup_{k\to\infty} |c_k u_k|^{1/|\lambda_k|^\rho} = \limsup_{n\to\infty}\Big(|u_{k_n}|^{1/|\lambda_{k_n}|^\rho}\Big)^{1/2} \ge$$
$$\limsup_{n\to\infty}(M - \varepsilon_n)^{1/2} = M^{1/2} \to \infty, \text{ as } M \to +\infty .$$

This iplies that $(c_k u_k) \notin \mathcal{C}$ which leads to a contradiction.

The implications b) and c) are obvious □

Now we can prove the following result

Theoem 2.1. *We have*

$$(\mathcal{A}, \mathcal{C}) = (l_p, \mathcal{C}) = (l_\infty, \mathcal{C}) = (\mathcal{C}, \mathcal{C}) = (\mathcal{A}, \mathcal{A}) = (\mathcal{A}, l_p) = (\mathcal{A}, l_\infty) = \mathcal{C}.$$

Proof: From Lemma 2.1, Lemma 2.2 and (1.4), it follows that

$$\mathcal{C} \subset (\mathcal{A}, \mathcal{A}) \subset (\mathcal{A}, l_p) \subset (\mathcal{A}, l_\infty) \subset (\mathcal{A}, \mathcal{C}) \subset \mathcal{C},$$

and

$$\mathcal{C} \subset (\mathcal{C}, \mathcal{C}) \subset (l_\infty, \mathcal{C}) \subset (l_p, \mathcal{C}) \subset (\mathcal{A}, \mathcal{C}) \subset \mathcal{C} .$$

The theorem is proved □

Next, we prove the following

Lemma 2.3. *We have*

a) $(l_p, \mathcal{A}) \subset \mathcal{A}$,

b) $(\mathcal{C}, l_\infty) \subset \mathcal{A}$,

c) $\mathcal{A} \subset (\mathcal{C}, \mathcal{A})$.

Proof:

a) First, we note that $(c_k) \in \mathcal{A}$ if and only if $(c_k^p) \in \mathcal{A}$ (with any appropriate choice of the power). Furthermore, we can check that the sequence (λ_k) satisfies Condition (1.3) if and only if there exists $\alpha > 0$ such that

$$\sum_{k=1}^{\infty} e^{-\alpha|\lambda_k|^\rho} < +\infty . \tag{2.1}$$

Now, let $(u_k) \in (l_p, \mathcal{A})$. Suppose that $(u_k) \notin \mathcal{A}$, which means that $(u_k^p) \notin \mathcal{A}$. Then there exists $M > 0$ such that for a sequence $(\varepsilon_n) \downarrow 0$, there exists an increasing sequence (k_n) of positive numbers such that

$$\frac{\log |u_{k_n}|^p}{|\lambda_{k_n}|^\rho} \geq M - \varepsilon_n\,, \quad \forall n \geq 1\,.$$

This implies that

$$|u_{k_n}|^{-p} \leq \exp\Big[(\varepsilon_n - M)|\lambda_{k_n}|^\rho\Big]\,, \quad \forall n \geq 1\,.$$

Define a sequence (c_k) as follows:

$$c_k = \begin{cases} \dfrac{\exp\big[p^{-1}(\gamma - \varepsilon_n)|\lambda_{k_n}|^\rho\big]}{|u_{k_n}|}\,, & \text{if } k = k_n,\ n = 1, 2, \dots, \\ 0\,, & \text{otherwise,} \end{cases}$$

where $\gamma < M - \alpha$ and $\alpha > 0$ is defined by (2.1). Then, we have

$$\sum_{k=1}^\infty |c_k|^p = \sum_{n=1}^\infty |c_{k_n}|^p = \sum_{n=1}^\infty \frac{\exp\big[(\gamma - \varepsilon_n)|\lambda_{k_n}|^\rho\big]}{|u_{k_n}|^p} \leq$$

$$\sum_{n=1}^\infty \exp\big[(\gamma - M)|\lambda_{k_n}|^\rho\big] \leq \sum_{n=1}^\infty \exp(-\alpha|\lambda_{k_n}|^\rho) < +\infty\,,$$

due to (2.1), which shows that $(c_k) \in l_p$. However,

$$\limsup_{k\to\infty} \frac{\log |c_k u_k|^p}{|\lambda_k|^\rho} = \limsup_{n\to\infty} \frac{\log |c_{k_n} u_{k_n}|^p}{|\lambda_{k_n}|^\rho} = \limsup_{n\to\infty} (\gamma - \varepsilon_n) = \gamma > -\infty\,,$$

which means that $((c_k u_k)^p) \notin \mathcal{A}$ or $(c_k u_k) \notin \mathcal{A}$. This is a contradiction. Hence $(l_p, \mathcal{A}4) \subset \mathcal{A}$.

b) Let $(u_k) \in (\mathcal{C}, l_\infty)$. Assume that $(u_k) \notin \mathcal{A}$, then there exists an increasing sequeence (k_n) of positive numbers such that

$$\lim_{n\to\infty} |u_{k_n}|^{1/|\lambda_{k_n}|^\rho} = +\infty\,. \tag{(2.2)}$$

Consider a sequence (c_k) as follows:

$$c_k = \begin{cases} k_n/|u_{k_n}|\,, & \text{if } k = k_n,\ n = 1, 2, , \dots, \\ 0\,, & \text{othewise}\,. \end{cases}$$

Then we have

$$\limsup_{k\to\infty} |c_k|^{1/|\lambda_k|^\rho} = \limsup_{k\to\infty} \big(k_n/|u_{k_n}|\big)^{1/|\lambda_{k_n}|^\rho} = 0 < +\infty\,,$$

due to (2.2) and (1.3). Hence $(c_k) \in \mathcal{C}$. However

$$\sup_{k\geq 1} |c_k u_k| = \sup_{n\geq 1} |c_{k_n} u_{k_n}| = \sup_{n\geq 1} k_n = +\infty\,.$$

Hence $(c_k u_k) \notin l_\infty$: a contradiction.

c) The implication $\mathcal{A} \subset (\mathcal{C}, \mathcal{A})$ is obvious. □

We can prove the following

Theorem 2.2. *We have*

$$(\mathcal{C}, l_\infty) = (\mathcal{C}, l_p) = (\mathcal{C}, \mathcal{A}) = (l_\infty, \mathcal{A}) = (l_p, \mathcal{A}) = \mathcal{A}\,.$$

Proof: From Lemma 2.1, Lemma 2.3 and (1.4), it follows that

$$\mathcal{A} \subset (\mathcal{C}, \mathcal{A}) \subset (l_\infty, \mathcal{A}) \subset (l_p, \mathcal{A}) \subset \mathcal{A}\,.$$

The theorem is proved □

Remark. Theorem 2.1 and 2.2 for the ordinary Dirichlet series of one and several complex variables were proved in [3] and [4].

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NHÂN TỬ CỦA KHÔNG GIAN DÃY DIRICHLET NGUYÊN SUY RỘNG

Trịnh Đào Chiến

Sở Giáo dục và Đào tạo Gia Lai

Với hai không gian dãy X và Y, không gian dãy của các nhân tử từ X vào Y, ký hiệu là (X, Y), được xác định như sau $(X, Y) := \{(u_k);\ (c_k u_k) \in Y,\ \forall (c_k) \in X\}$. Xét không gian dãy $\mathcal{A}$, các hệ số của chuỗi Dirichlet suy rộng dạng $\sum_{k=1}^{\infty} c_k E_\rho(\lambda_k z)$, trong đó $E_\rho(\cdot)$ là hàm Mittag - Leffler. Qua mô tả không gian $\mathcal{A}^\alpha$ đối ngẫu Köthe của $\mathcal{A}$, ta thấy rằng $(\mathcal{A}, l_1) = \mathcal{A}^\alpha$, trong đó $l_1 = \{(u_k);\ \sum_{k=1}^{\infty} |u_k| < \infty\}$. Một câu hỏi đặt ra: kết quả sẽ như thế nào đối với các không gian dãy của các nhân tử từ $\mathcal{A}, \mathcal{A}^\alpha$ vào các không gian quen thuộc khác, chẳng hạn $l_p (0 < p < \infty), l_\infty, \ldots$ và ngược lại? Bài báo này sẽ đề cập đến các nội dung đó.