A FINITE ALGORITHM FOR A CLASS OF NONLINEAR OPTIMIZATION PROBLEMS

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Abstract. *In this paper a finite algorithm is presented for solving a class of nonlinear optimization problems with special structure. It is based on numbering techniques to improve feasible solutions commonly used in solving problems of transportation type.*

I. PROBLEM STATEMENT

Given an $m \times n$ matrix $A = (a_{ij})_{m\times n}$, where $a_{ij} \in \{0, 1\}$, and given positive numbers p_i $(0 < p_i \leq n)$, $i = 1, 2, \ldots, m$. Consider the following optimization problem:

$$(P) \qquad \max_{1\leq j\leq n} \sum_{i=1}^{m} x_{ij} \longrightarrow \min, \tag{1}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = p_i, \; i = 1, 2, \ldots, m, \tag{2}$$

$$0 \leq x_{ij} \leq a_{ij}, \text{ integers}, \; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n. \tag{3}$$

Since the objective function (1) is convex and the constraints (2), (3) are linear and integer, problem (P) is a nonlinear integer programming problem. However, as shown below, (P) may be reduced to a linear integer problem with special structure.

The constraint (2) may also be replaced with inequality constraint (2') without changing the solution of (P):

$$\sum_{j=1}^{n} x_{ij} \geq p_i, \; i = 1, 2, \ldots, m. \tag{2'}$$

Problem (P) may be explained as follows: there are m students and n subjects for them. The number of subjects required for the i^{th}- student is p_i. Coefficients a_{ij} represent the agreeableness of student i to subject j ($a_{ij} = 1$ if student i is agreeable to subject j and $a_{ij} = 0$ if not). The question is how to arrange the students to learn the subjects so that each of students learn completely the number of subjects required for him and so that the number of students for each subject is as similar as possible.

It is easily seen that (P) is equivalent to the following 0 - 1 integer programming problem

$$\min\{t|\sum_{j=1}^{n} x_{ij} \geq p_i,\ \forall i;\ \sum_{i=1}^{m} x_{ij} \leq t,\ \forall j;\ x_{ij} \in \{0,1\},\ x_{ij} \leq a_{ij},\ \forall i,j\}.$$

The model of problem (P) was studied in [1] and [2]. In [2] the authors suggested a polynomial time algorithm for problem (P) by solving a finite number of maximum flow problems.

Exploiting the special structure of the problem, in the sequel we shall develop an improved algorithm for solving (P) which has the following features: (i) it is finite; (ii) it is based on numbering techniques to improve feasible solutions commonly used in solving problems of transportation type.

II. FOUNDATION OF THE SOLUTION METHOD

As usually for the convenience we agree that a matrix $x = \{x_{ij}\}$ whose entries satisfy (2) and (3) is called a *feasible solution* of (P), a feasible solution achieving the minimum of (1) is called an *optimal solution* of (P).

Let us denote

$$b_j = \sum_{i=1}^{m} a_{ij} \leq m,\ j = 1, 2, \ldots, n, \quad p = \sum_{i=1}^{m} p_i > 0,$$

(b_j represents the number of students agreeable to subject j, and p is the total number of subjects required for all students).

Letting

$$a_i = \sum_{j=1}^{n} a_{ij},\ i = 1, 2, \ldots, m,$$

it is proved in [2] that in order to (P) has an optimal solution. A necessary and sufficient condition for the existence of an optimal solution of (P) is

$$a_i \geq p_i \text{ for all } i = 1, 2, \ldots, m. \tag{4}$$

Condition (4) is very simple and easily to be checked. So, we assume that (P) satisfies this condition. Furthermore, without loss of generality we may assume that the students and subjects are numbered so that

$$a_1 - p_1 \geq a_2 - p_2 \geq \ldots \geq a_m - p_m \text{ and } m \geq b_1 \geq b_2 \geq \ldots \geq b_n > 0.$$

It is natural to suppose that $b_j > 0$ for all $j = 1, 2, \ldots, n$, because if $b_j = 0$ for some j then the subject j must be deleted (there is no student who wants to learn the subject)).

For the sake of convenience, for each feasible solution x we create a table consisting of m rows and n columns, in which each of rows corresponds to a student and each of columns corresponds to a subject. The cell lying at the intersection of row i and column j is denoted by (i, j). A feasible solution $x = \{x_{ij}\}$ of (P) will correspond to a table consisting of zeros and ones in its cells. A cell (i, j) is called *black* if $a_{ij} = 0$ (black cells will be forbidden to use, because student i is not agreeable to subject j, so that $x_{ij} = 0$)). The remaining cells will be divided into two classes: *white cells* if $x_{ij} = 0$ (student i is agreeable to subject j, but he is not allocated for this subject) and *blue cells* if $x_{ij} = 1$ (student i is allocated for subject j).

Denote

$$t_j^x = \sum_{i=1}^{m} x_{ij},\ \forall j = 1, 2, \ldots, n. \quad t^x = \max_{1 \le j \le n} t_j^x, \tag{5}$$

(t_j^x is the number of students allocated for subject j and t^x is the objective function value of x).

For any feasible solution $x = \{x_{ij}\}$ of (P), according to (5) we have

$$\sum_{j=1}^{n} t_j^x = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} p_i = p. \tag{6}$$

Column j is called *full* if $t_j^x = t^x$, and deficient if $t_j^x \le t^x - 2$. It should be noted that the notions of blue cell, white cell, full column and deficient column are concerned with a given feasible solution.

The following proposition gives a simple criterion for an optimal solution of (P).

Proposition 1. *If a feasible solution x has no deficient column, i.e.,*

$$t^x = t_{j_0}^x,\ t_j^x \ge t^x - 1,\ \forall j \ne j_0, \tag{7}$$

then x is an optimal solution of (P).

Proof: From (6) and (7) it follows

$$p = \sum_{j=1}^{n} t_j^x > n(t^x - 1). \tag{8}$$

Suppose the contrary that there exists a feasible solution y that is better than x, i.e.

$$t_j^y \le t^x - 1,\ \forall j = 1, 2, \ldots, n. \tag{9}$$

Combining (6) and (9) yields

$$p = \sum_{j=1}^{n} t_j^y \le n(t^x - 1),$$

which is contrary to (8). Thus, x is an optimal solution □

Consider now a feasible solution $x = \{x_{ij}\}$ of (P). Let $\mathcal{C}$ be a sequence of alternating white and blue cells with respect to x joining column j_0 and column j_k :

$$\mathcal{C} = \{(i_0, j_0), (i_0, j_1), \ldots, (i_{k-1}, j_{k-1}), (i_{k-1}, j_k), \ (k \geq 1)\}, \tag{10}$$

where (i_t, j_t), $t = 0, 1, \ldots, k-1$, are white cells ($x_{i_t j_t} = 0$), while (i_t, j_{t+1}), $t = 0, 1, \ldots, k-1$, blue cells ($x_{i_t j_{t+1}} = 1$). We introduce the following transformation of x.

Transformation A. On the sequence $\mathcal{C}$, replace all the former white cells by blue ones and all the former blue cells by white ones. This means that we set

$$x'_{i_t j_t} = 1, \ x'_{i_t j_{t+1}} = 0, \ t = 0, 1, \ldots, k-1, \ x'_{ij} = x_{ij}, \ \forall (i,j) \notin \mathcal{C}.$$

The following lemma shows that this transformation does not change the objective function value of x.

Lemma 1. *Assume that x' is obtained from x by Transformation A on some sequence of alternating white and blue cells joining two columns which are not full, then $t^{x'} = t^x$.*

Proof. Suppose that there exists a sequence of form (10) joining two columns j_0 and j_k which are not full ($t^x_{j_0} < t^x, t^x_{j_k} < t^x$). Since in each of rows i_t ($t = 0, 1, \ldots, k-1$) there are just two white and blue cells of $\mathcal{C}$, $x' = \{x'_{ij}\}$ satisfies (2), (3), i.e. x' is also a feasible solution of (P). Similarly, since in each of columns j_t ($t = 1, 2, \ldots, k-1$) there are just two white and blue cells of $\mathcal{C}$, we have

$$t^{x'}_j = t^x_j, \ \forall \, j \neq j_0, j_k. \tag{11}$$

On the other hand, as column j_0 has only one cell of $\mathcal{C}$ (white cell (i_0, j_0)), we have

$$t^{x'}_{j_0} = t^x_{j_0} + 1, \tag{12}$$

and as column j_k has only one cell of $\mathcal{C}$ (blue cell (i_{k-1}, j_k)), we get

$$t^{x'}_{j_k} = t^x_{j_k} - 1. \tag{13}$$

Finally, as columns j_0 and j_k are not full, from (11) - (13) it follows that $t^{x'} = t^x$. □

Similarly, suppose that $\mathcal{C}$ is a cycle of alternating white and blue cells:

$$(i_0, j_0), (i_0, j_1), \ldots, (i_k, j_k), (i_k, j_0), (i_0, j_0) \ \ (k \geq 1),$$

where (i_t, j_t), $t = 0, 1, \ldots, k$, are white cells ($x_{i_t j_t} = 0$), while (i_t, j_{t+1}), $t = 0, 1, \ldots, k-1$, and (i_k, j_0) are blue cells ($x_{i_t j_{t+1}} = x_{i_k j_0} = 1$). Consider the following transformation:

Transformation B. On the cycle $\mathcal{C}$ replace all the former white cells by blue ones and all the former blue cells by white ones. This means that we set

$$x'_{i_t j_t} = 1, \ x'_{i_t j_{t+1}} = 0, \ t = 0, 1, \ldots, k, \ x'_{i_k j_0} = 0,$$
$$x'_{ij} = x_{ij}, \ \forall (i,j) \notin \mathcal{C}.$$

Lemma 2. *Suppose that x' is obtained from x by Transformation B on a cycle of altternating white and blue cells, then $t^{x'} = t^x$.*

The rows and columns numbering. The procedure of rows and columns numbering is defined as follows. First of all, we assign 0 to each column j which is full ($t_j^x = t^x$)). If column j is numbered, we assign number j to each row i which has not been numbered and has $x_{ij} = 1$ ((i,j) is a blue cell). Then, if row i is numbered, we assign number i to each column j which has not been numbered and has $a_{ij} - x_{ij} = 1$ (this is equivalent to $a_{ij} = 1, x_{ij} = 0$, i.e. (i,j) is a white cell) and so on. The above procedure must stop after at most $m + n$ times of rows and columns numbering.

If a deficient column, e.g. column j_0 with $t_{j_0}^x \leq t^x - 2$, is numbered there must be a sequence of alternating white and blue cells joining a some full column and j_0. Such a sequence of cells can be determined as follows. Suppose that column j_0 with $t_{j_0}^x \leq t^x - 2$ is assigned with number i_0 ((i_0, j_0) is a white cell) and row i_0 is assigned with number $j_1 \neq j_0$ ((i_0, j_1) is a blue cell). If column j_1 is assigned with number not equal to 0., for instance, $i_1 \neq i_0$ ((i_1, j_1) is a white cell), and row i_1 is assigned with number $j_2 \neq j_0, j_1$ ((i_1, j_2) is a blue cell). If column j_2 is assigned with number not equal to 0, we continue searching. As the number of columns in the table is finite (equal to n), finally we must find out a column $j_k \neq j_t, t = 0, 1, \ldots, k-1$, assigned with number 0, i.e. j_k is a full column and the required sequence is

$$\mathcal{C} = \{(i_0, j_0), (i_0, j_1), \ldots, (i_{k-1}, j_{k-1}), (i_{k-1}, j_k), \ (k \geq 1)\},$$

where (i_t, j_t), $t = 0, 1, \ldots, k-1$, are white cells, while (i_t, j_{t+1}), $t = 0, 1, \ldots, k-1.$, are blue cells. We have

Proposition 2. *Let x be a feasible solution of (P). If there exists a sequence of alternating white and blue cells joining a full column and a deficient column, then x can be changed to a new feasible solution x' which is better or has smaller number of full columns than x.*

Proof: Let $\mathcal{C}$ be a sequence joining a full column j_k and a deficient column j_0. We apply Transformation A on $\mathcal{C}$. Arguing as in the proof of Lemma 1, we obtain the relations (11) - (13).

As j_0 is a deficient column, from (11) - (13) it follows that if j_k is a unique full column with respect to x then $t^{x'} = t^x - 1$, i.e. new feasible solution x' is better than the current solution x. In the opposite case, we have $t^{x'} = t^x$, i.e. x' is not worse than x, but has at least one full column fewer than x' (as j_k will not be a full column with respect to x') □

Proposition 3. *Let x be a feasible solution of (P). If there is no sequence of alternating white and blue cells joining a full column and a deficient column, then x is an optimal solution of (P).*

Proof: We argue by contradiction, by supposing that there is a feasible solution $y = \{y_{ij}\}$ which is better than $x = \{x_{ij}\}$, i.e.

$$t^y < t^x, \tag{14}$$

where t^x, t^y are defined by (5). We shall show that this leads to a contradiction. Indeed, from (14) it follows that there exists an index j such that $t^y_j < t^x_j$. According to (6) we have

$$\sum_{j=1}^{n} t^x_j = \sum_{j=1}^{n} t^y_j = \sum_{i=1}^{m} p_i = p,$$

and consequently there must be at least a column j_0 so that

$$t^x_{j_0} < t^y_{j_0} \leq t^y < t^x. \tag{15}$$

Since the numbers in the above inequalities are integers it follows that $t^x_{j_0} \leq t^x - 2$, i.e. j_0 is a deficient column with respect to x. From the first inequality in (15) and the definition (5) of $t^x_{j_0}$ and $t^y_{j_0}$ it follows that there is a row i_0 such that $0 = x_{i_0 j_0} \neq y_{i_0 j_0} = 1$ (i.e. (i_0, j_0) is a white cell with respect to x). Moreover, as both x and y satisfy (2), we must have

$$\sum_{j=1}^{n} x_{i_0 j} = \sum_{j=1}^{n} y_{i_0 j} = p_{i_0}.$$

This means that there is column j_1 such that (i_0, j_1) is a blue cell:

$$x_{i_0 j_1} = 1,\ y_{i_0 j_1} = 0.$$

If $t^x_{j_1} \leq t^y_{j_1}$, there exists row i_1 such that (i_1, j_1) is a white cell:

$$x_{i_1 j_1} = 0,\ y_{i_1 j_1} = 1,$$

and also by (2) there must be column j_2 such that (i_1, j_2) is a blue cell:

$$x_{i_1 j_2} = 1,\ y_{i_1 j_2} = 0.$$

Continuing this process will lead to one of the following cases:

a) A column j_r with $t^x_{j_r} > t^y_{j_r}$ is reached. In this case we have a sequence of alternating white and blue cells of the form

$$(i_0, j_0), (i_0, j_1), \ldots, (i_{r-1}, j_{r-1}), (i_{r-1}, j_r), \quad (r \geq 1), \tag{16}$$

joining column j_r and column j_0. Let us distinguish two possibilities:

a1) $t^x_{j_r} = t^x$, i.e. j_r is a full column . In this event sequence of cells (16) joins full column j_r and deficient column j_0. This is a contradiction to the hypothesis of the proposition. So, this possibility can not occur.

a2) $t^x_{j_r} < t^x$, i.e. j_r is not a full column. Applying Transformation A on sequence (16), a new feasible solution x' with $t^{x'} = t^x$ will be obtained (by virtue of Lemma 1) and the number of different components of x' and y will decrease by at least two.

b) A cycle is found

$$(i_0, j_0), (i_0, j_1), \ldots, (i_s, j_s), (i_s, j_0), (i_0, j_0), \quad (s \geq 1). \tag{17}$$

Applying Transformation B on cycle (17), a new feasible solution x' with $t^{x'} = t^x$ is obtained (by virtue of Lemma 2) and the number of different components of x' and y will decrease by at least four.

If x' still differs from y, the above process will be repeated with x replaced by x'. As the number of different components of x and y strictly reduces when Transformation A or B is applied, after a finite number of repetitions we must have $\hat{x} = y$, at the same time $t^{\hat{x}} = t^x$, i.e. $t^y = t^{\hat{x}} = t^x$. This is contradicts to (14). □

III. FINITE ALGORITHM FOR PROBLEM (P)

From the above results we are now in a position to develop an algorithm for solving (P).

Step 0: Create a table consisting of m rows and n columns. Each row corresponds to a student and each column corresponds to a subject. Make a cell (i, j) *black* if $a_{ij} = 0$ (black cells will be not changed through the course of solving the problem).

Step 1: Construct an initial feasible solution. For each row i, from 1 to m, we write 1 in white cells of the row from left to right until having p_i ones (the remaining cells are assigned 0), then go to the next row. As a result, we obtain an initial feasible solution $x^1 = \{x^1_{ij}\}$ of (P). It may also be started with any feasible solution of (P). Set $k = 1$ and go to step 2.

Step 2: Test for optimality. For the obtained feasible solution x^k we adopt the convention that cells with 1 are called *blue cells*, and cells (not black) with 0 are called *white cells*. Determine

$$t^k_j \equiv t^{x^k}_j = \sum_{i=1}^{m} x^k_{ij}, \; j = 1, 2, \ldots, n.$$

$$t^k \equiv t^{x^k} = \max_{1 \leq j \leq n} t^{x^k}_j = \max_{1 \leq j \leq n} t^k_j.$$

Column j is said to be a *full column* if $t^k_j = t^k$, called a *deficient column* if $t^k_j \leq t^k - 2$. If no deficient column exists then by virtue of Proposition 1, x^k is an optimal solution of (P). Otherwise, perform rows and columns numbering as described in section 2. If there is no deficient column that is numbered then x^k is also optimal (by virtue of Proposition 3). In the opposite case, we must have a sequence C of form (10) that consists of alternating white and blue cells and joins a full column j_k and a deficient column j_0. Go to step 3.

In the course of numbering when a deficient column is numbered, we go immediately to step 3 to improve the solution.

Step 3: Solution improvement. Apply Transformation A on the sequence C obtained in step 2. As a result we get a new feasible solution x' which either is better ($t^{x'} < t^k$) or has fewer number of full columns than x^k (Proposition 2). Set $x^{k+1} = x'$ and $k \leftarrow k+1$, then return to step 2.

Proposition 4. *The above algorithm terminates after a finite number of steps.*

Proof: If the algorithm is not terminated at step 2 then after each improvement in step 3, either a better feasible solution or a solution with fewer number of full columns than previous one is obtained. As the objective function of the problem can take only a finite number of positive integer values and as the number of columns in the problem is also finite (equal to n), the above steps can not be infinitely extended □

Illustrative example. Solve problem (P) whose data are as follows: $m = 4, n = 5, p_1 = 2, p_2 = 3, p_3 = 3, p_4 = 2$ and

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Sum up elements of A in each row and column:

$$a_1 = 3, a_2 = a_3 = a_4 = 4;\ b_1 = b_2 = b_3 = b_4 = b_5 = 3 \text{ and } p = 10.$$

Carrying out Step 1 of the algorithm, we obtain an initial feasible solution of (P):

$$x^1 = \begin{pmatrix} 1 & 1 & \times & 0 & \times \\ 1 & 1 & 1 & \times & 0 \\ \times & 1 & 1 & 1 & 0 \\ 1 & \times & 1 & 0 & 0 \end{pmatrix} \begin{matrix} 1 \\ 1 \\ 2 \\ 1 \end{matrix}$$
$$\begin{matrix} 0 & 0 & 0 & 1 \end{matrix}$$

(Black cells are marked by $\times$).

Step 2. Summing up all elements in each column of x^1, we obtain:

$$t_1^1 = t_2^1 = t_3^1 = 3, t_4^1 = 1, t_5^1 = 0 \text{ and } t^1 = 3.$$

Columns 1, 2, 3 are full, columns 4, 5 are deficient. Columns 1, 2, 3 are first numbered with 0. We search column 1 in x^1 for a 1 (blue cell) and find it in rows 1, 2, 4, so these rows are numbered with 1 (subscript of column 1). Then, in column 2 there is a 1 in row 3 (not yet numbered), so that this row is numbered with 2 (subscript of column 2). All the rows have been numbered, columns 4 and 5 are not yet numbered. We now search numbered row 1 for a 0 (white cell) and find it in column 4 (not yet numbered), so column 4 is numbered with 1 (subscript of row 1). At this point, deficient column 4 is numbered with 1 (row 1), row 1 is numbered with 1 (column 1). Column 1 is full. Thus, we obtain the sequence of cells: (1, 1) – (1, 4) joining full column 1 and deficient column 4.

Step 3. Changing x^1 on just found sequence of cells, we obtain new feasible solution

$$x^2 = \begin{array}{c}\begin{pmatrix} 0 & 1 & \times & 1 & \times \\ 1 & 1 & 1 & \times & 0 \\ \times & 1 & 1 & 1 & 0 \\ 1 & \times & 1 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 1 & 0 & 0 & & 2 \end{matrix}\end{array} \begin{matrix} 2 \\ 2 \\ 2 \\ 3 \\ \end{matrix}$$

Return to step 2. Summing up elements in each column of x^2, we obtain

$$t_1^2 = 2, t_2^2 = t_3^2 = 3, t_4^2 = 2, t_5^2 = 0 \text{ and } t^2 = 3.$$

Columns 2, 3 are full, column 5 is deficient. Columns 2, 3 are first numbered with 0. We search column 2 in x^2 for a 1 (blue cell) and find it in rows 1, 2, 3, so these rows are numbered with 2 (subscript of column 2). In full column 3 there is a 1 in row 4 (not yet numbered), so that this row is numbered with 3 (subscript of column 3). We search numbered row 1 for a 0 (white cell) and find it in column 1 (not yet numbered), so column 1 is numbered with 1 (subscript of row 1). Then, we search numbered row 2 for a 0 (white cell) and find it in column 5 (not yet numbered), so column 5 is numbered with 2 (subscript of row 2). At this point, deficient column 5 is numbered with 2 (row 2), row 2 is numbered with 2 (column 2). Column 2 is full. Thus, we obtain the sequence of cells: $(2,2)-(2,5)$ joining full column 2 and deficient column 5.

Step 3. Changing x^2 on just found sequence of cells, we obtain new feasible solution

$$x^3 = \begin{array}{c}\begin{pmatrix} 0 & 1 & \times & 1 & \times \\ 1 & 0 & 1 & \times & 1 \\ \times & 1 & 1 & 1 & 0 \\ 1 & \times & 1 & 0 & 0 \end{pmatrix} \\ \begin{matrix} & 2 & 0 & & 3 \end{matrix}\end{array} \begin{matrix} \\ 3 \\ 3 \\ 3 \\ \end{matrix}$$

Return to step 2. Summing up elements in each column of x^3, we obtain

$$t_1^3 = t_2^3 = 2, t_3^3 = 3, t_4^3 = 2, t_5^3 = 1 \text{ and } t^3 = 3.$$

Column 3 is full, column 5 is deficient. Column 3 is first numbered with 0.. We search column 3 in x^3 for a 1 (blue cell) and find it in rows 2, 3, 4, so these rows are numbered with 3 (subscript of column 3). We search numbered row 2 for a 0 (white cell) and find it in column 2 (not yet numbered), so column 2 is numbered with 2 (subscript of row 2). Then, we search numbered row 3 for a 0 (white cell) and find it in column 5 (not yet numbered), so column 5 is numbered with 3 (subscript of row 3). At this point, deficient column 5 is numbered with 3 (row 3), row 3 is numbered with 3 (column 3). Column 3 is full. Thus, we obtain the sequence of cells: $(3,3)-(3,5)$ joining full column 3 and deficient column 5.

Step 3. Changing x^3 on just found sequence of cells, we obtain new feasible sollution

$$x^4 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Return to step 2. Summing up elements in each column of x^4, we obtain

$$t_1^4 = t_2^4 = t_3^4 = t_4^4 = t_5^4 = 2 \text{ and } t^4 = 2.$$

Now all the columns are full, there is no deficient column, so that x^4 is an optimal solution with the objective function value is $t^* = t^4 = 2$.

Reoptimization techniques can be used when the number of students or subjects is changed. These matters will be investigated in the forthcoming paper.

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PHƯƠNG PHÁP HỮU HẠN GIẢI MỘT LỚP BÀI TOÁN TỐI ƯU PHI TUYẾN

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