ON SOLVABILITY IN A CLOSED FORM OF A CLASS OF SINGULAR INTEGRAL EQUATIONS WITH ROTATION

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Abstract. *In this paper we shall give some algebraic characterizations of the operator $\tilde{S}_{n,k}$ of the form (2) and study solvability in a closed form of singular integral equation of the form (1).*

By algebraic method we reduce the equation (1) to the system of singular integral equations and then obtain all Solutions in a closed form.

Suppose that $\Gamma = \{t : |t| = 1\}, D^+ = \{z : |z| < 1\}, D^- = \{z : |z| > 1\}$, are respectively the boundary, interior and exterior of the unit disk on the complex plan

Consider the singular integral equation of the form

$$\varphi(t) + \frac{b(t)}{\pi i} \int_\Gamma \frac{\tau^{n-1-k} t^k}{\tau^n + t^n} \varphi(\tau) d\tau = f(t), \tag{1}$$

where $\varphi(t), f(t), b(t) \in H^\mu(\Gamma) \quad (0 < \mu \leq 1)$.

Define

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau,$$

$$(\tilde{S}_{n,k}\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\tau^{n-1-k}\, t^k}{\tau^n + t^n} \varphi(\tau) d\tau, \quad 0 \leq k \leq n-1, \quad 1 \leq n \in N \tag{2}$$

$$(W\varphi)(t) = \varphi(\varepsilon_1 t), \quad \varepsilon_1 = exp(\frac{\pi i}{n}), \quad \varepsilon_k = \varepsilon_1^k.$$

It is easy to check that

$$SW = WS, \quad \tilde{S}_{n,k} W = W\tilde{S}_{n,k}, \quad \tilde{S}_{n,k} S = S\tilde{S}_{n,k}. \tag{3}$$

Denote

$$P = \frac{1}{2}(I + S), Q = \frac{1}{2}(I - S),$$

$$P_j = \frac{1}{2n} \sum_{\nu=1}^{2n} \varepsilon_j^{2n-1-\nu} W^{1+\nu}, \quad j = \overline{1, 2n}. \tag{4}$$

Then

$$P^2 = P, Q^2 = Q, PQ = QP = 0,$$

$$P_i P_j = \delta_{ij} P_j, \quad i,j = \overline{1,2n}, \tag{5}$$

$$I = P_1 + P_2 + ... + P_{2n}, \tag{6}$$

$$W^k = \varepsilon_1^k P_1 + \varepsilon_2^k P_2 + ... + \varepsilon_{2n}^k P_{2n}, \tag{7}$$

$$X = X^+ \bigoplus X^- = \bigoplus_{j=1}^{2n} X_j,$$

where $X^+ = PX, \ X^- = QX, \ X_j = P_j X \ (j = \overline{1,2n}), \delta_{ij}$ is the Kronecker symbol.

Lemma 1. *Let $\tilde{S}_{n,k}$ be defined as in (2). Then*

$$\tilde{S}_{n,k} = SP_k - SP_{n+k} \ (k = \overline{0, n-1}), \tag{8}$$

where we put $P_0 = P_{2n}$.

Proof: From the identity

$$\frac{\tau^{n-1-k} t^k}{\tau^n + t^n} = \frac{\tau^{2n-1-k} t^k}{\tau^{2n} - t^{2n}} - \frac{\tau^{n-1-k} t^{n+k}}{\tau^{2n} - t^{2n}}.$$

We obtain

$$(\tilde{S}_n \varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\tau^{n-1-k} t^k}{\tau^n + t^n} \varphi(\tau) d\tau$$

$$= \frac{1}{\pi i} \int_\Gamma \frac{\tau^{2n-1-k} t^k}{\tau^{2n} - t^{2n}} \varphi(\tau) d\tau - \frac{1}{\pi i} \int_\Gamma \frac{\tau^{n-1-k} t^{n+k}}{\tau^{2n} - t^{2n}} \varphi(\tau) d\tau$$

$$= (SP_k \varphi)(t) - (SP_{n+k} \varphi)(t).$$

(see [2])

Lemma 2. *Every operator $\tilde{S}_{n,k}$ is an algebraic operator with the characteristic polynomial*

$$P_{\tilde{S}_{n,k}}(\lambda) = \begin{cases} \lambda^3 - \lambda & \text{for } n > 1, \\ \lambda^2 - 1 & \text{for } n = 1. \end{cases}$$

Proof. Let $n = 1$, from (3), (5), (6) and (8), we get

$$\tilde{S}_{1,0}^2 = (SP_0 - SP_1)^2 = P_0 + P_1 = I,$$

It is easy to check that $P_{\tilde{S}_{1,0}}(\lambda) = \lambda^2 - 1$.

Let $n > 1$,from (3), (5) and (8), we get

$$\tilde{S}_{n,k}^3 = \tilde{S}_{n,k}^2 \tilde{S}_{n,k}$$

$$= (P_k + P_{n+k})(SP_k - SP_{n+k})$$

$$= SP_k - SP_{n+k} = \tilde{S}_{n,k}.$$

To finish the proof it suffices to show that for every polynomial $Q(\lambda) = \alpha\lambda^2 + \beta\lambda + \gamma$ such that $Q(\tilde{S}_{n,k}) = 0$ we can follow $\alpha = \beta = \gamma = 0$ $(\alpha, \beta, \gamma \in C)$. Indeed, from (5) ve have

$$\begin{cases} 0 = P_k Q(\tilde{S}_{n,k}) = (\alpha + \gamma)P_k + \beta SP_k, \\ 0 = P_{n+k} Q(\tilde{S}_{n,k}) = (\gamma - \alpha)P_{n+k} - \beta SP_{n+k}. \end{cases}$$

From the last equalities, we get $\alpha = \beta = \gamma = 0$.

Lemma 3. *[2] If the function $K(\tau, t)$ can be extended to D^+ in such a manner tlat $K(\tau, t)$ is analytic in both variables in D^+ and is continuous in $\overline{D^+}$. Then*

1.
$$\int_\Gamma K(\tau, t)\varphi(\tau)dt \in X^+ \quad \text{for every } \varphi \in X;$$

2.
$$\int_\Gamma K(\tau, t)\varphi^+(\tau)d\tau = 0 \quad \text{for every } \varphi^+ \in X^+.$$

In the following, for every function $a(t) \in X$, we shall write

$$(K_a\varphi)_{(t)} = a(t)\varphi(t).$$

Lemma 4. *[2] Let $a(t) \in X$ be fixed. Then for every $k, j \in \{1, 2..., 2n\}$ the following identity fields*

$$P_k K_a P_j = K_{a_{kj}} P_j = P_k K_{a_{kj}},$$

where

$$a_{kj}(t) = \frac{1}{2n} \sum_{\nu=1}^{2n} \varepsilon_{\nu+1}^{j-k} a(\varepsilon_{\nu+1} t).$$

Now we deal with the equation of the form (1). Rewrite this equation as follows

$$\varphi(t) + b(t)(SP_k\varphi)(t) - b(t)(SP_{n+k}\varphi)(t) = f(t). \tag{9}$$

Lemma 5. *The equation (9) is equivalent to the following system*

$$\begin{cases} (P_k\varphi)(t) + \tilde{b}(t)(SP_k\varphi)(t) - \tilde{b}_1(t)(SP_{n+k}\varphi)(t) = (P_k f)(t), \\ (P_{n+k}\varphi)(t) + \tilde{b}_1(t)(SP_k\varphi)(t) - \tilde{b}(t)(SP_{n+k}\varphi)(t) = (P_{n+k} f)(t), \\ \varphi(t) = f(t) - b(t)(SP_k\varphi)(t) + b(t)(SP_{n+k}\varphi)(t), \end{cases} \tag{10}$$

where

$$\tilde{b}(t) = \frac{1}{2n} \sum_{\nu=1}^{2n} b(\varepsilon_\nu t), \quad \tilde{b}_1(t) = \frac{1}{2n} \sum_{\nu=1}^{2n} (-1)^\nu b(\varepsilon_\nu t).$$

Proof: According to Lemma 3, we have

$$\begin{aligned} P_k K_b P_k &= K_{b_{k,k}} P_k = K_{\tilde{b}} P_k, \\ P_{n+k} K_b P_{n+k} &= K_{b_{n+k,n+k}} P_{n+k} = K_{\tilde{b}} P_{n+k}, \\ P_k K_b P_{n+k} &= K_{b_{k,n+k}} P_{n+k} = K_{\tilde{b}_1} P_{n+k}, \\ P_{n+k} K_b P_k &= K_{b_{n+k,k}} P_k = K_{\tilde{b}_1} P_k. \end{aligned}$$

Hence, (9) is equivalent to the system

$$\begin{cases} \varphi(t) + b(t)(SP_k\varphi)(t) - b(t)(SP_{n+k}\varphi)(t) = f(t), \\ (P_k\varphi)(t) + \tilde{b}(t)(SP_k\varphi)(t) - \tilde{b}_1(t)(SP_{n+k}\varphi)(t) = (P_k f)(t), \\ (P_{n+k}\varphi)(t) + \tilde{b}_1(t)(SP_k\varphi)(t) - \tilde{b}(t)(SP_{n+k}\varphi)(t) = (P_{n+k} f)(t). \end{cases}$$

Moreover, it has been prove that the last system is equivalent to (10). Hence, in order to solve the equation (9) it suffices to solve the following system

$$\begin{cases} \varphi_k(t) + \tilde{b}(t)(S\varphi_k)(t) - \tilde{b}_1(t)(S\varphi_{n+k})(t) = (P_k f)(t), \\ \varphi_{n+k}(t) + \tilde{b}_1(t)(S_{\varphi_k})(t) - \tilde{b}(t)(S\varphi_{n+k})(t) = (P_{n+k} f)(t). \end{cases} \quad (11)$$

in the space $X_k \times X_{n+k}$

Lemma 6. *If $(\varphi_k, \varphi_{n+k})$ is a solution of System (11) in $X \times X$ then $(P_k\varphi_k, P_{n+k}\varphi_{n+k})$ is a solution of System (11) in $X_k \times X_{n+k}$.*

Proof Suppose that $(\varphi_k, \varphi_{n+k})$ is a solution of System (11) in $X \times X$. Acting to both side of system (11) by operators P_k, P_{n+k}, respectively, by virtue of Lemma 4, we get

$$\begin{cases} (P_k\varphi_k)(t) + \tilde{b}(t)(SP_k\varphi_k)(t) - \tilde{b}_1(t)(SP_{n+k}\varphi_{n+k})(t) = (P_k\varphi)(t), \\ (P_{n+k}\varphi_{n+k})(t) + \tilde{b}_1(t)(SP_k\varphi_k)(t) - \tilde{b}_1(t)(SP_{n+k}\varphi_{n+k})(t) = (P_{n+k} f)(t). \end{cases}$$

Hence, $(P_k\varphi_k, P_{n+k}\varphi_{n+k})$ is a solution of system (11) in $X_k \times X_{n+k}$ □

Due to results of Lemma 5 and Lemma 6 we obtain the following result.

Lemma 7. *The equation (9) is solvable in X if and only if the system (11) is solvable in $X \times X$. Moreover, every solution of (9) can be determined by the formula*

$$\varphi(t) = f(t) - b(t)(SP_k\tilde{\varphi})(t) + b(t)(SP_{n+k}\tilde{\varphi})(t),$$

where $\tilde{\varphi}(t) = (P_k\varphi_k)(t) + (P_{n+k}\varphi_{n+k})(t)$, $(\varphi_k, \varphi_{n+k})$ is a solution of system (11) in $X \times X$.

Theorem 1. *Suppose that $d(t)a(\tau)$ is a continuous function in $(\tau, t) \in \Gamma \times \Gamma$ which admits an analytic prolongation in both variables onto D^+, where*

$$a(t) = \tilde{b}(t) + \tilde{b}_1(t), \quad d(t) = \tilde{b}(t) - \tilde{b}_1(t). \quad (12)$$

Then the equation (9) admits all solution in a closed form.

Proof: Due to the results of Lemma 7, it suffices to show that the system (11) admits all solution in a closed form. The system (11) is equivalent to the following system

$$\begin{cases} \varphi_k(t) + \varphi_{n+k}(t) + a(t)[S(\varphi_k - \varphi_{n+k})](t) = (P_k f(t) + (P_{n+k} f)(t), \\ \varphi_k(t) - \varphi_{n+k}(t) + d(t)[S(\varphi_k + \varphi_{n+k})(t)] = (P_k f)(t) - (P_{n+kf}(t)). \end{cases} \quad (\text{.}3)$$

where $a(t), d(t)$ are defined by (12).

Put

$$\psi_1(t) = \varphi_k(t) + \varphi_{n+k}(t), \quad \psi_2(t) = \varphi_k(t) - \varphi_{n+k}(t),$$

$$g_1(t) = (P_k f)(t) + (P_{n+k} f)(t) \quad , g_2(t) = (P_k f)(t) - (P_{n+k} f)(t).$$

We can write the system (13) in the form

$$\begin{cases} \psi_1(t) + (K_a S\psi_2)(t) = g_1(t), \\ \psi_2(t) + (K_d S\psi_1)(t) = g_2(t). \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi_1(t) + (K_a S\psi_2)(t) = g_1(t), \\ \psi_2(t) - (K_d S K_a \psi_2)(t) = g_2(t) - d(t)(Sg_1)(t). \end{cases} \quad (14)$$

In order to solve the System (14), we have only to solve the equation

$$\psi_2(t) - \Big[K_d S K_a S\psi_2\Big](t) = g_3(t), \quad (15)$$

where $g_3(t) = g_2(t) - d(t)(Sg_3)(t)$.
Rewrite (15) as the following

$$\psi_2^+(t) - \psi_2^-(t) - \Big[K_d S K_a(\psi_2^+ + \psi_2^+)\Big](t) = g_3(t), \quad (16)$$

where

$$\psi_2^+(t) = (P\psi_2)(t), \quad \psi_2^+(t) = -(Q\psi_2)(t).(\psi_2^+ \in X^+, \psi_2^- \in X^-).$$

By our assumption for $(\tau - t)^{-1} d(t) a(\tau)$, by the Lemma 3, we have

$$(K_d S K_a \psi_2^+)(t) = 0. \quad (17)$$

From (16) and (17), we get

$$\psi_2^+(t) - (K_d S K_a \psi_2^-)(t) - \psi_2^-(t) = g_3(t). \quad (18)$$

It is easy to see that

$$\phi^+(t) := \psi_2^+(t) - (K_d S K_a \psi_2^-)(t) \in X^+,$$

$$\phi^-(t) := \psi_2^-(t) \in X^-. \quad (19)$$

Hence, the equation (18) is just a Riemann boundary problem

$$\phi^+(t) - \phi^-(t) = g_3(t). \quad (20)$$

The equation (20) has the solution

$$\begin{cases} \phi^+(t) = \frac{1}{2} g_3(t) + \frac{1}{2}(Sg_3)(t), \\ \phi^-(t) = \frac{-1}{2} g_3(t) + \frac{1}{2}(Sg_3)(t). \end{cases} \quad (21)$$

From (19) and (21), we obtain

$$\psi_2(t) = \psi_2^+(t) - \psi_2^-(t)$$

$$= \phi^+(t) - \phi^-(t) + (K_d S K_a \phi^-)(t)$$

$$= g_3(t) - \frac{-1}{2}(K_d S K_a g_3)(t) + \frac{1}{2}(K_d S K_a S g_3)(t).$$

The theorem is proved by a similar argument as above, we prove a dual statement, namely we have

Theorem 2. *Suppose that* $(\tau - 1)^{-1}.d(t)a(\tau)$ *is a continuous function in* $(\tau, t) \in \Gamma \times \Gamma$ *which admits an analytic prolongation in both variables on to* D^-, *where* $a(t), d(t)$ *are defined by (12). Then the equation (9) admits all solution in a closed form.*

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REFERENCES

1. F.D. Gakhov. *Boundary value problems*, Oxford 1966 (3rd Russian complemented and corrected edition, Moscow, 1977).
2. Ng.V.Mau. Generalized algebraic elements and linear singular integral equation with transformed arguments, *WPW*, Warszawa 1989.
3. Ng.V.Mau, Ng.M.Tuan. On solutions of integral equations with analytic kernel and rotations, *Annales Polonici Mathematici* LXIII, **3**, 1996.
4. D. Przeworska-Rolewicz. *Equations with transformed arguments, An algebraic approach*. Amsterdam - Warzawa 1973.
5. D. Przeworska-Rolewicz, S. Rolewicz. *Equations in Linear Space*. Amsterdam - Warzawa 1968.

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VỀ TÍNH GIẢI ĐƯỢC Ở DẠNG ĐÓNG CỦA MỘT LỚP PHƯƠNG TRÌNH TÍCH PHÂN KỲ DỊ VỚI PHÉP QUAY

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Bài báo này sẽ đề cập đến vài đặc trưng đại số của toán tử $\tilde{S}_{n,k}$ dạng (2) và nghiên cứu tính giải được ở dạng đóng của phương trình tích phân kỳ dị dạng (1).

Bằng phương pháp đại số sẽ đưa phương trình (1) về hệ phương trình tích phân kỳ dị và sau đó thu được tất cả các nghiệm ở dạng đóng.