ON LENGTH FUNCTIONS DEFINED BY A SYSTEM OF PARAMETERS IN LOCAL RINGS

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I. INTRODUCTION

Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring and M be a finitely generated A- module with dim $M = d$. We denote $Q_M(\underline{x})$ the submodule of M defined by

$$Q_M(\underline{x}) = \bigcup_{n>0}((x_1^{n+1}, ..., x_d^{n+1})M : x_1^n...x_d^n),$$

where $\underline{x} = (x_1, ..., x_d)$ is a system of parameters of M.

Note that the submodule $Q_M(\underline{x})$ is used for studying the monomial conjecture with respect to the system of parameters $\underline{x}$ (see [7, 8]). Recall that the monomial conjecture holds true for the system of parameters $\underline{x}$ if $x_1^n...x_d^n \notin (x_1^{n+1}, ..., x_d^{n+1})M$ for all $n > 0$. Therefore, the monomial conjecture holds true for $\underline{x}$ if and only if $Q_M(\underline{x}) \neq M$. On the other hand, it was shown in [4] that $Q_M(\underline{x}) = (x_1, ..., x_d)M$ provided M is Cohen - Macaulay module. Conversely, if there is a system of parameters $\underline{x}$ such that $Q_M(\underline{x}) = \underline{x}M$ then M is Cohen - Macaulay module. This fact suggest us to study the length $l_A(M/Q_M(\underline{x}))$. The purpose of this note is to study the following function of $\underline{n}$

$$q_{M,\underline{x}}(\underline{n}) = l_A(M/Q_M(\underline{x}, \underline{n})),$$

where $\underline{n} = (n_1, ..., n_d)$ is a d-tuple of positive integers and $\underline{x}(\underline{n}) = (x_1^{n_1}, ..., x_d^{n_d})$. Then, a natural question is whether $q_{M,\underline{x}}(\underline{n})$ is a polynomial of $n_1, ..., n_d$ for $\underline{n}$ sufficiently large ($\underline{n} \gg 0$) ? or it is equivalent to ask whether the function

$$J_{M,\underline{x}}(\underline{n}) = n_1 ... n_d e(\underline{x}, M) - q_{M,\underline{x}}(\underline{n})$$

is a polynomial for $\underline{n} \gg 0$?

We will give in this note some basic properties of the function $q_{M,\underline{x}}(\underline{n})$ in Section 2 and some properties of the function $J_{M,\underline{x}}(\underline{n})$ in Section 3.

II. BASIC PROPERTIES OF $q_{M,\underline{x}}(\underline{n})$

Throughout this note, we denote by $(A, \mathfrak{m})$ a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$ and by M a finitely generated A-module with dim $M = d$. Let

$\underline{x} = (x_1, ..., x_d)$ be a system of parameters of M. Then the submodule $Q_M(\underline{x})$ of M is defined by

$$Q_M(\underline{x}) = \bigcup_{n>0}((x_1^{n+1}, ..., x_d^{n+1})M : x_1^n ... x_d^n),$$

and for $\underline{n} = (n_1, ..., n_d)$, we put

$$Q_M(\underline{x}, \underline{n}) = Q_M(\underline{x}(\underline{n})).$$

The functions $q_{M,\underline{x}}(\underline{n})$ and $J_{M,\underline{x}}(\underline{n})$ are defined by

$$q_{M,\underline{x}}(\underline{n}) = l_A(M/Q_M(\underline{x}, \underline{n})),$$

and

$$J_{M,\underline{x}}(\underline{n}) = n_1...n_d e(\underline{x}, M) - l_A(M/Q_M(\underline{x}, \underline{n})).$$

Therefore, we can consider $q_{M,\underline{x}}(\underline{n})$ and $J_{M,\underline{x}}(\underline{n})$ as functions of $\underline{n}$.

Lemma 2.1. *Let $\underline{x} = (x_1, ..., x_d)$ be a system of parameters of M. Then the following statements are true.*

i) Let N be an Arinian submodule of M. Then $\underline{x}$ is a system of parameters of $\overline{M} = M/N$ and $q_{\overline{M},\underline{x}}(\underline{n}) = q_{M,\underline{x}}(\underline{n})$.

ii) Put $\overline{M}_1 = M/(o : x_1)$. Then $\underline{x}$ is a system of parameters of $\overline{M}_1$ and $q_{\overline{M}_1,\underline{x}}(\underline{n}) = q_{M,\underline{x}}(\underline{n})$.

Proof: i) From the property of the system of parameters, $\underline{x}$ is a system of parameters of $\overline{M}$. Let

$$\mathfrak{m}N \supseteq \mathfrak{m}^2 N \supseteq ... \supseteq \mathfrak{m}^t N \supseteq ...$$

be a descending chain of submodule of N. Since N is an Artinian A- module then $\mathfrak{m}^k N = \mathfrak{m}^{k+1} N$ for a positive integer k. Since

$$\bigcap_{n>0} \mathfrak{m}^n N - 0,$$

we have $\mathfrak{m}^k N = 0$.

Consider the map

$$\Phi : M/Q_M(\underline{x}, \underline{n}) \to \overline{M}/Q_{\overline{M}}(\underline{x}, \underline{n}),$$

defined by $\Phi(u + Q_M(\underline{x}, \underline{n})) = \overline{u} + Q_{\overline{M}}(\underline{x}, \underline{n})$ for any $u \in M$. Since M is an A- module Noetherian it should be note that there exists $n_0 \gg 0$, such that for $\underline{n} = (n_1, ..., n_d)$, we have

$$Q_M(\underline{x}, \underline{n}) = (x_1^{n_1(n_0+1)}, ..., x_d^{n_d(n_0+1)})M : x_1^{n_1 n_0} ... x_d^{n_d n_0},$$

and

$$Q_{\overline{M}}(\underline{x}, \underline{n}) = (x_1^{n_1(n_0+1)}, ..., x_d^{n_d(n_0+1)})\overline{M} : x_1^{n_1 n_0} ... x_d^{n_d n_0}.$$

Thus, it is easy to show that Φ is well defined and it is surjective. Therefore $ker\phi = 0$.

Furthermore, we can choose $n_0 \geq k$, hence it is also injective. Therefore $q_{M,\underline{x}}(\underline{n}) = q_{\overline{M},\underline{x}}(\underline{n})$.

ii) Can be proved similarly as (i) □

Lemma 2.2. *Suppose that $\widehat{A}$ is the $\mathfrak{m}$-adic completion of A and $\widehat{M}$ is the $\mathfrak{m}$- adic completion of M. Then*

$$q_{M,\underline{x}}(\underline{n}) = q_{\widehat{M},\underline{x}}(\underline{n})$$

for all $\underline{n} = (n_1, ..., n_d)$.

Proof. Since the natural homomorphism $A \to \widehat{A}$ is absolutly flat, then $\underline{x}$ is a system of parameters of $\widehat{M}$ and

$$\widehat{Q_M(\underline{x}, \underline{n})} = Q_{\widehat{M}}(\underline{x}, \underline{n}).$$

Therefore we have

$$q_{M,\underline{x}}(\underline{n}) = l_A(M/Q_M(\underline{x}, \underline{n})) = l_A(\widehat{M}/\widehat{Q_M(\underline{x}, \underline{n})}) = l_A(\widehat{M}/Q_{\widehat{M}(\underline{x},\underline{n})}) = q_{\widehat{M},\underline{x}}(\underline{n}) \quad \square$$

Lemma 2.3. *If $\underline{n} \geq \underline{m}$ (i.e. $n_i \geq m_i, i = 1, ..., d$) then $Q_M(\underline{x}, \underline{n}) \subseteq Q_M(\underline{x}, \underline{m})$.*

Proof. Let α be a positive integer. We put

$$Q_M(\alpha) = \bigcup_{n>0}((x_1^{\alpha(n+1)}, x_2^{n+1}, ..., x_d^{n+1})M : x_1^{\alpha n} x_2^n ... x_d^n).$$

Since $Q_M(\underline{x})$ is independent of the order of the sequence $\underline{x}$, we have only to show that

$$Q_M(\alpha) \subset Q_M(\alpha - 1) \subset ... \subset Q_M(1),$$

with $\alpha \geq 2$. In fact, M is Noetherian then there exist $n_0 \gg 0$ such that

$$Q_M(\alpha) = (x_1^{\alpha(n_0+1)}, x_2^{n_0+1}, ..., x_d^{n_0+1})M : x_1^{\alpha n_0} x_2^{n_0} ... x_d^{n_0},$$

and

$$Q_M(\alpha - 1) = (x_1^{(\alpha-1)(2n_0+1)}, x_2^{2n_0+1}, ..., x_d^{2n_0+1})M : x_1^{(\alpha-1)(2n_0)} x_2^{2n_0} ... x_d^{2n_0}.$$

For any element $a \in Q_M(\alpha)$

$$(x_1^{(\alpha-1)2n_0} x_2^{2n_0} ... x_d^{2n_0})a = (x_1^{\alpha n_0 - 2n_0} ... x_d^{n_0})(x_1^{\alpha n_0} ... x_d^{n_0})a$$

$$= (x_1^{\alpha n_0 - 2n_0} ... x_d^{n_0})(x_1^{\alpha(n_0+1)} y_1 + x_2^{n_0+1} y_2 + ... + x_d^{n_0+1} y_d)$$

for some $y_1, ..., y_d \in M$. It follows that

$$(x_1^{(\alpha-1)2n_0} x_2^{2n_0} ... x_d^{2n_0})a = x_1^{(\alpha-1)(2n_0+1)} z_1 + x_2^{2n_0+1} z_2 + ... + x_d^{2n_0+1} z_d$$

for some $z_1, ..., z_d \in M$. Therefore, $a \in Q_M(\alpha - 1)$ $\square$

Corollary 2.4. *The function $q_{M,\underline{x}}(\underline{n})$ is ascending, i.e., $q_{M,\underline{x}}(\underline{n}) \geq q_{M,\underline{x}}(\underline{m})$ for $\underline{n} \geq \underline{m}$, ($n_i \geq m_i$ for all $i = 1, ... d$).*

Proof. For $\underline{n} \geq \underline{m}$, we consider the map

$$\varphi : M/Q_M(\underline{x}, \underline{n}) \to M/Q_M(\underline{x}, \underline{m}),$$

defined by

$$\varphi(a + Q_M(\underline{x}, \underline{n})) = a + Q_M(\underline{x}, \underline{m}),$$

for any $a \in M$. By Lemma 2.3. the map φ is well defined and it is surjective. Hence

$$l_A(M/Q_M(\underline{x}, \underline{m})) \leq l_A(M/Q_M(\underline{x}, \underline{n})) \ \square$$

Theorem 2.5. $q_{M,\underline{x}}(\underline{n}) \leq n_1 ... n_d \, e(\underline{x}, M)$.

Proof. We only need to show that $q_{M,\underline{x}}(1) \leq e(\underline{x}, M)$. We prove this inequality by induction on d.

If $d = 1$, by Lemma 2.1 (i), we may assume that depth $M > 0$. Since depth $M =$ dim$M$ then M is an A-module Cohen-Macaulay. Hence, we get $l_A(M/x_1M) = e(x_1, M)$ and $Q_M(x_1, 1) = x_1M$. So we have done for the case $d = 1$.

For $d > 1$ and the assertion is true for all A-modules of dimension $< d$. By Lemma 2.1, (ii), we may assume that depth $M > 0$ and x_1 is a non-zero divisor of M. Let $\overline{M} = M/x_1M$. We get dim $\overline{M} = d - 1$ and $\underline{x}' = (x_2, ..., x_d)$ is a system of parameters of $\overline{M}$. Consider the map

$$\Phi : \overline{M}/Q_{\overline{M}}(\underline{x}', 1) \to M/Q_M(\underline{x}, 1),$$

defined by

$$\Phi(\overline{a} + Q_{\overline{M}}(\underline{x}', 1)) = a + Q_M(\underline{x}, 1),$$

for any element $a \in M$. The map Φ is well defined and it is an epimorphism. We obtain

$$l_A(M/Q_M(\underline{x}, 1)) \leq l_A(\overline{M}/Q_{\overline{M}}(\underline{x}', 1)).$$

Applying the induction hypothesis, we get

$$l_A(\overline{M}/Q_{\overline{M}}(\underline{x}', 1)) \leq e(\underline{x}', \overline{M}).$$

Since x_1 is a non-zero divisor of M then $e(\underline{x}', \overline{M}) = e(\underline{x}, M)$. Therefore, $l_A(M/Q_M(\underline{x}, 1) \leq e(\underline{x}, M)$ and the theorem is proved $\square$

III. THE FUNCTION $J_{M,\underline{x}}(\underline{n})$

Recall that the function $q_{M,\underline{x}}(\underline{n})$ is a polynomial when $\underline{n}$ is large enough ($n \gg 0$) if and only if

$$J_{M,\underline{x}}(\underline{n}) = n_1...n_d e(\underline{x}, M) - l_A(M/Q_M(\underline{x}, \underline{n})),$$

is a polynomial for $n \gg 0$.

Proposition 3.1. *Suppose that* $\underline{x} = (x_1, ..., x_d)$ *is a system of parameters of* M *and* $\underline{n} = (n_1, ..., n_d)$. *Then* $J_{M,\underline{x}}(\underline{n}) \le n_1...n_d J_{M,\underline{x}}(1)$.

Proof. Let α be a positive integer and $\underline{x}(\alpha) = (x_1^\alpha, x_2, ..., x_d)$. By Lemma 2.3, we obtain

$$Q_M(\alpha) \subseteq Q_M(\alpha - 1) \subseteq ... \subseteq Q_M(1), \tag{1}$$

for $\alpha \ge 2$. Consider the map

$$\varphi : M/Q_M(\alpha) \to M/Q_M(\alpha - 1),$$

defined by

$$\varphi(a + Q_M(\alpha)) = a + Q_M(\alpha - 1),$$

for any element $a \in M$. By (1), it is easy to show that the map φ is well defined and it is an epimorphism and

$$Ker(\varphi) = Q_M(\alpha - 1)/Q_M(\alpha).$$

Consider the map

$$\Psi : M/Q_M(1) \to Ker(\varphi)$$

defined by

$$\Psi(a + Q_M(1)) = x_1^{\alpha-1} a + Q_M(\alpha),$$

for any element $a \in M$. Since $x_1^{\alpha-1} Q_M(\alpha) \subseteq Q_M(1)$, we can verify that the map Ψ is well defined and it is a monomorphism. Since φ is surjective and Ψ is injective, we obtain

$$\begin{aligned} l_A(M/Q_M(\alpha)) &= l_A(M/Q_M(\alpha - 1)) + l_A(Ker(\varphi)) \\ &\ge l_A(M/Q_M(\alpha - 1)) + l_A(M/Q_M(1)). \end{aligned}$$

Applying the induction hypothesis, we get

$$l_A(M/Q_M(\alpha - 1)) \ge (\alpha - 1)) l_A(M/Q_M(1)).$$

Hence

$$l_A(M/Q_M(\alpha)) \ge \alpha \ l_A(M/Q_M(1)).$$

Because the proof is independent of the order of the sequence $\underline{x}$, finally, we have

$$l_A(M/Q_M(\underline{x}, \underline{n})) \ge n_1...n_d \ l_A(M/Q_M(\underline{x}, 1)).$$

Hence

$$J_{M,\underline{x}}(\underline{n}) = n_1...n_d \ e(\underline{x}, M) - l_A(M/Q_M(\underline{x}, \underline{n})) \le n_1...n_d J_{M,\underline{x}}(1).$$

The proposition is proved. □

Theorem 3.2. *The function $J_{M,\underline{x}}(\underline{n})$ is ascending, i.e,*

$$J_{M,\underline{x}}(\underline{m}) \leq J_{M,\underline{x}}(\underline{n}),$$

when $\underline{m} \leq \underline{n}$.

Proof. For every $\sigma \in S_d$, we have

$$Q_M(\underline{x}, \underline{n}) = Q_M(\underline{x}^\sigma, \underline{n}),$$

where $\underline{x}^\sigma = (x_{\sigma(1)}, ..., x_{\sigma(d)})$. Hence, we only need to prove the theorem in the case $m_1 = n_1, ..., m_{d-1} = n_{d-1}$ and $m_d \leq n_d$. We do it by induction on d. In the case $d = 1$, we get

$$J_{M,\underline{x}}(\underline{m}) = J_{M,\underline{x}}(\underline{n}) = 0.$$

For $d > 1$, by Lemma 2.1, (ii), we can assume that depth $M > 0$ and x_1 is a non-zerodivisor. Let $\overline{M} = M/x_1^{n_1}M$.

Consider the map

$$\Psi_1 : \overline{M}/Q_{\overline{M}}(\underline{x}', \underline{m}') \to M/Q_M(\underline{x}, \underline{m}),$$

defined by

$$\Psi_1(\overline{a} + Q_{\overline{M}}(\underline{x}', \underline{m}')) = a + Q_M(\underline{x}, \underline{m}),$$

for any element $\overline{a} \in \overline{M}$, where $\underline{m}' = (m_2, ..., m_d), \underline{x}' = (x_2, ..., x_d)$; and

$$\Psi_2 : \overline{M}/Q_{\overline{M}}(\underline{x}', \underline{n}') \to M/Q_M(\underline{x}, \underline{n}),$$

defined by

$$\Psi_2(\overline{u} + Q_{\overline{M}}(\underline{x}', \underline{n}')) = u + Q_M(\underline{x}, \underline{n}),$$

for any element $\overline{u} \in \overline{M}$, where $\underline{n}' = (n_2, ..., n_d)$. We can see that these maps are well defined and they are surjective. So we get

$$l_A\big(\overline{M}/Q_{\overline{M}}(\underline{x}', \underline{m}')\big) = l_A(Ker(\Psi_1)) + l_A\big(M/Q_M(\underline{x}, \underline{m})\big),$$

and

$$l_A\big(\overline{M}/Q_{\overline{M}}(\underline{x}', \underline{n}')\big) = l_A(Ker(\Psi_2)) + l_A\big(M/Q_M(\underline{x}, \underline{n})\big).$$

It follows that

$$J_{M,\underline{x}}(\underline{m}) = J_{\overline{M},\underline{x}'}(\underline{m}') + l_A(Ker(\Psi_1)),$$

and

$$J_{M,\underline{x}}(\underline{n}) = J_{\overline{M},\underline{x}'}(\underline{n}') + l_A(Ker(\Psi_2)).$$

Applying the induction hypothesis, we obtain

$$J_{\overline{M},\underline{x}'}(\underline{m}') \leq J_{\overline{M},\underline{x}'}(\underline{n}').$$

Let $m_d + s = n_d$, we have

$$x_d^s Q_{\overline{M}}(\underline{x}', \underline{m}') \subseteq Q_{\overline{M}}(\underline{x}', \underline{n}'). \tag{2}$$

Consider the map

$$\Phi : \overline{M}/Q_{\overline{M}}(\underline{x}', \underline{m}') \to \overline{M}/Q_{\overline{M}}(\underline{x}', \underline{n}'),$$

defined by

$$\Phi(\overline{u} + Q_{\overline{M}}(\underline{x}', \underline{m}')) = x_d^s \overline{u} + Q_{\overline{M}}(\underline{x}', \underline{n}'),$$

for any element $\overline{u} \in \overline{M}$. By (2), the map Φ is well defined and it is an injection. Let Φ_1 be the map of Φ restricted into the set Ker (Ψ_1). We can easily check that Φ_1 mapping of the set Ker (Ψ_1) into Ker (Ψ_2) is also injective. Therefore, $l_A(ker(\Psi_1)) \leq l_A(ker(\Psi_2))$. It follows that

$$J_{M,\underline{x}}(\underline{m}) \leq J_{M,\underline{x}}(\underline{n}),$$

as required □

For dim $M \leq 2$, we have following result.

Theorem 3.3. *If dim $M \leq 2$ then the function $J_{M,\underline{x}}(\underline{n})$ is a constant for $\underline{n} \gg 0$.*

Proof. In the case $d = 1$, by Lemma 2.1, we can assume that depth $M > 0$. Since depth M = dim M then M is an A-module Cohen-Macaulay. Hence, $Q_M(x_1, n_1) = x_1^{n_1} M$ and

$$l_A(M/Q_M(x_1, n_1)) = l_A(M/x_1^{n_1} M) = e(x_1^{n_1}, M) = n_1 e(x_1, M).$$

Therefore $J_{M,x_1}(n_1) = 0$.

In the case $d = 2$, by Lemma 2.1 and Lemma 2.2, without any loss of the generality, we can assume that $A = \widehat{A}$. Let $M_n = M/x_1^n M$, we have dim $M_n = 1$. For any positive integer n we set $\underline{x}(n) = (x_1^n, x_2^n)$ and $\underline{x}'(n) = (x_2^n)$ to be a system of parameters of M_n. There is an exact sequence of A-modules and A-homomorphism

$$0 \to Ker(\varphi) \to M_n/Q_{M_n}(x_2^n) \xrightarrow{\varphi} M/Q_M(\underline{x}(n)) \to 0 \tag{3}$$

where φ is defined by

$$\varphi(\overline{u} + Q_{M_n}(x_2^n)) = u + Q_M(\underline{x}(n)),$$

for any $\overline{u} \in M_n$. Following [1], we can choose x_1 so that

$$Ker(\varphi) \cong H^1_{\mathfrak{m}}(M)/x_1^n H^1_{\mathfrak{m}}(M),$$

and the length of $H^1_{\mathfrak{m}}(M)/x_1^n H^1_{\mathfrak{m}}(M)$ is finite and independent of n when n is large enough. By (3), it follows that

$$l_A(M_n/Q_{M_n}(x_2^n)) = l_A(ker(\varphi)) + l_A(M/Q_M(\underline{x}(n))).$$

We get

$$J_{M,\underline{x}}(n) = n^2 e(\underline{x}, M) - l_A(M/Q_M(\underline{x}(n)))$$
$$= e(x_2^n, M_n) - l_A(M_n/Q_{M_n}(x_2^n)) + l_A(H^1_{\mathfrak{m}}(M)/x_1^n H^1_{\mathfrak{m}}(M))$$
$$= l_A(H^1_{\mathfrak{m}}(M)/x_1^n H^1_{\mathfrak{m}}(M)).$$

is a constant for $n \gg 0$. Applying Theorem 3.2, the theorem is proved □

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VỀ NHỮNG HÀM ĐỘ DÀI XÁC ĐỊNH BỞI HỆ THAM SỐ TRONG VÀNH ĐỊA PHƯƠNG

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Trong bài này chúng tôi định nghĩa hai hàm độ dài $q_{M,\underline{x}}(\underline{n})$ và $J_{M,\underline{x}}(\underline{n})$ theo d - biến $\underline{n} = (n_1, ..., n_d)$ liên kết với hệ tham số $\underline{x} = (x_1, ..., x_d)$ của A - môđun M. Một số tính chất của những hàm này được nêu ra.