ON LINEAR MULTIPOINT BOUNDARY - VALUE PROBLEMS FOR INDEX-2 DIFFERENTIABLE - ALGEBRAIC EQUATIONS

Nguyen Van Nghi

Faculty of Mathematics
College of Natural Sciences - VNU

Abstract. *This paper deals with multipoint BVPs for linear index-2 DAEs. It has been shown that the results obtained by [3] for transferable DAEs can be extended to linear time varying index-2 systems.*

I. INTRODUCTION

Consider the following multipoint boundary - value problem (BVP) for linear differential - algebraic equations (DAEs):

$$Lx := A(t)\,x' + B(t)\,x = q(t), \;\; t \in J := [t_0, T] \tag{1.1}$$

$$\Gamma x := \int_{t_0}^{T} d\eta(t)\,x(t) = \gamma, \tag{1.2}$$

where A, $B \in C(J, \mathbb{R}^{n\times n})$ are continuous matrix - valued functions, $\eta \in BV(J, \mathbb{R}^{n\times n})$ is a matrix - valued function of bounded variations, $q(t) \in C := C((J, \mathbb{R}^n)$ and $\gamma \in \mathbb{R}^n$ are given function and vector respectively.

By the Riesz theorem, the left hand side of (1.2) represents a general form of linear bounded operators from C to $\mathbb{R}^n$.

In what follows, we assume that DAE (1.1) with the pair $\{A, B\}$ is tractable with index 2, i.e., (see [1, 2]):

1) There exists a continuously differentiable projector - function $Q \in C^1(J, \mathbb{R}^{n\times n})$, i.e., $Q^2(t) = Q(t)$, such that $\operatorname{Im} Q(t) = \operatorname{Ker} A(t)$ for every $t \in J$.
2) The matrix $A_1(t) = A_0(t) + B_0(t)\,Q(t)$, with $A_0 := A$, $B_0 := B - AP'$, is singular and the matrix $A_2(t) := A_1(t) + B_1(t)\,Q_1(t)$, where $Q_1(t)$ denotes a projection onto the nullspace $\operatorname{Ker} A_1(t)$, $B_1 := (B_0 - A_0(PP_1)')P$, is nonsingular for all $t \in [t_0, T]$.

Denote by P and P_1 the operators $I - Q$ and $I - Q_1$ respectively. Obviously, P and P_1 are also projector functions satisfying relations: $P \in C^1(J, \mathbb{R}^{n\times n}), PQ = QP = P_1Q_1 = Q_1P_1 = 0$. Since (1.1) can be reformulated as $A[(Px)' - P'x] + Bx = q$, we should look for solutions belonging to the Banach space

$$\mathcal{X} := \{x \in C(I, \mathbb{R}^n) \,:\, Px \in C^1(J, \mathbb{R}^n)\},$$

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with the norm $\|x\| := \|x\|_\infty + \|(Px)'\|_\infty$.

Let $Q_1 \in C^1(J, \mathbb{R}^{n\times n})$ and without loss of generality, we can suppose that Q_1 is a canonical projection satisfying $Q_1 Q = 0$. It follows from the last relation that $PP_1 x = PP_1 Px \in C^1(J, \mathbb{R}^n)$. Let $Y(t)$ be a fundamental solution of the following ODE:

$$Y' = [(PP_1)' - PP_1 A_2^{-1} B]Y;\ Y(s,s) = I.$$

Denote by $X(t,s)$ the matrix $M(t)\, Y(t,s)\, P(s)\, P_1(s)$, where $M(t) := I + Q[QQ_1(PQ_1)' - P_1 A_2^{-1} B]PP_1$, then $X(t,s)$ is a solution of the homogeneous IVP:

$$A(t)X' + B(t)X = 0;\ P(s)P_1(s)[X(s,s) - I] = 0.$$

It has been proved that Ker $X(t,s) =$ Ker $P(s)\, P_1(s)$ for all $t, s \in [t_0, T]$. Moreover, the IVP:

$$A(t)x' + B(t)x = q(t);\ P(t_0)P_1(t_0)(x(t_0) - x_0) = 0,$$

has a unique solution of the form (cf. [2]):

$$x(t) = X(t,t_0)x_0 + X(t,t_0)\int_{t_0}^{t} X(t_0,\tau)\, h(\tau)\, d\tau + \overline{q}(t),$$

where

$$h(t) = PP_1 A_2^{-1} q + (PP_1)' PQ_1 A_2^{-1} q, \tag{1.3}$$

and

$$\overline{q}(t) := (PQ_1 + QP_1)A_2^{-1} q + QQ_1(PQ_1 A_2^{-1} q)' - QQ_1(PQ_1)' PQ_1 A_2^{-1} q. \tag{1.4}$$

For investigating multipoint BVP (1.1), (1.2), the technique described in [3] can be applied. Since proofs of most statements in this article can be carried out in similar ways as in [1], they will be omitted.

II. REGULAR MULTIPOINT BVP

We denote by D the shooting matrix $\int_{t_0}^{T} d\eta(t)\, X(t,t_0)$ and by $\mathcal{R}_0$ the following subset of $\mathbb{R}^n$:

$$\mathcal{R}_0 := \left\{ \int_{t_0}^{T} d\eta(t)\, x(t) \ :\ x \in C \right\}.$$

Theorem 2.1. *Problem (1.1), (1.2) is uniquely solvable on $\mathcal{X}$ for any $q \in C$ and $\gamma \in \mathcal{R}_0$ if and only if the shooting matrix D satisfies conditions:*

$$Ker\, D = Ker\, A(t_0) \oplus Ker\, A_1(t_0) \equiv Ker\, P(t_0)\, P_1(t_0), \tag{2.1}$$

$$Im\, D = \mathcal{R}_0. \tag{2.2}$$

In particular, we can consider the following multipoint condition:

$$\Gamma x := \sum_{i=1}^{m} D_i\, x(t_i) = \gamma, \tag{2.3}$$

where $t_0 \le t_1 < t_2 < \cdots < t_m \le T$ and $D_i \in \mathbb{R}^{n\times n}$ $(i = \overline{1,m})$ are given constant matrices.

Corollary 2.1. *Problem (1.1), (2.3) is uniquely solvable on $\mathcal{X}$ for any $q \in C$ and $\gamma \in Im(D_1, D_2, \dots, D_m)$ if and only if the shooting matrix $D = \sum_{i=1}^{m} D_i X(t_i, t_0)$ satisfies conditions:*

$$Ker\, D = Ker\, A(t_0) \oplus Ker\, A_1(t_0) \equiv Ker\, P(t_0)\, P_1(t_0),$$

$$Im\, D = Im\,(D_1, D_2, \dots, D_m).$$

III. IRREGULAR MULTIPOINT BVP

In this section, we suppose that condition (2.1) and/or (2.2) are not satisfied.

Consider a linear bounded operator $\mathcal{L}$ acting from $\mathcal{X}$ to $\mathcal{Y} := C^1_{(2)} \times \mathbb{R}^n$ defined by:

$$\mathcal{L}x := \begin{pmatrix} Lx \\ \Gamma x \end{pmatrix},$$

where $C^1_{(2)} := \{q \in C \, : \, Q_1 A_2^{-1} q \in C^1\}$ and $\|q\| := \|q\|_\infty + \|(Q_1 A_2^{-1} q)'\|_\infty$. Since $\operatorname{Ker} P(t_0)\, P_1(t_0) \subset \operatorname{Ker} D$, for the sake of simplicity we can suppose that

$$\dim Ker\, P(t_0)\, P_1(t_0) = \nu < \dim Ker\, D = p.$$

Let $\{\omega_i^0\}_1^\nu$ be an orthonormal basis of $\operatorname{Ker} P(t_0)\, P_1(t_0)$, $\omega_i^{0T} \omega_j^0 = \delta_{ij}$, where the superscript T denotes the transposition. Let $\{\omega_i^0\}_1^p$ be an extension of $\{\omega_i^0\}_1^\nu$ to an orthonormal basis of $\operatorname{Ker} D$. Define a column matrix $\Phi(t) := (\varphi_{\nu+1}(t), \dots, \varphi_p(t))$ with $\varphi_i(t) = X(t, t_0)\, \omega_i^0$ $(i = \overline{\nu+1, p})$ and put $M := \int_{t_0}^T \Phi^T(t)\Phi(t)\, dt$. It is easy to prove that M is nonsingular and $\mathcal{X}$ can be decomposed into a direct sum of closed subspaces: $\mathcal{X} = \operatorname{Ker} \mathcal{L} \oplus \operatorname{Ker} \mathcal{U}$, where $(\mathcal{U}x)(t) := \Phi(t)\, M^{-1} \int_{t_0}^T \Phi^T(s)\, x(s)\, ds$, and $\operatorname{Ker} \mathcal{L} = \{x = \Phi(t)\, a \, : \, a \in \mathbb{R}^{p-\nu}\} = \operatorname{Span}(\{\omega_i^0\}_{\nu+1}^p)$. Further, denote by $\{\omega_i\}_1^p$ be an orthonormal basis of $\operatorname{Ker} D^T$ and let $W_0 := (\omega_{\nu+1}^0, \dots, \omega_p^0)$, $W := \begin{pmatrix} \omega_1^T \\ \vdots \\ \omega_p^T \end{pmatrix}$ be $n \times (p - \nu)$ and $p \times n$ matrices respectively.

Theorem 3.1. *The following statements hold:*

i) $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ *is a bounded linear Noëther operator,*

$$\operatorname{ind} \mathcal{L} := \dim \operatorname{Ker} \mathcal{L} - \operatorname{codim} \operatorname{Im} \mathcal{L} = -\nu =$$
$$= -\dim \{\operatorname{Ker} A(t_0) \oplus \operatorname{Ker} A_1(t_0)\} \equiv -\dim \operatorname{Ker} P(t_0)\, P_1(t_0).$$

ii) Problem (1.1), (1.2) is solvable on $\mathcal{X}$ if and only if the given data $\{q, \gamma\}$ satisfy condition:

$$W(\gamma - \int_{t_0}^T d\eta(t) f(t, t_0)) = 0,$$

where $f(t, t_0) := X(t, t_0) \int_{t_0}^t X(t_0, \tau)\, h(\tau)\, d\tau + \overline{q}(t)$ and $h(t)$, $\overline{q}(t)$ are difined by (1.3), (1.4) respectively.

iii) A general solution of (1.1), (1.2) is of the form:

$$x(t) = X(t, t_0)(\overline{x}_0 + W_0 \alpha) + f(t, t_0) + \Phi(t) a,$$

where $\overline{x}_0 = \widehat{D}^{-1}(\gamma - \int_{t_0}^{T} d\eta(t)\, f(t,t_0))$, $\alpha = -M^{-1}\int_{t_0}^{T} \Phi^T(t)\{X(t,t_0)\overline{x}_0 + f(t,t_0)\}\, dt$, $a \in \mathbb{R}^{p-\gamma}$ *is an arbitrary vector and* $\widehat{D}$ *denote the restriction of* D *onto* $\operatorname{Im} D^T$.

4. Examples

Consider system (1.1) with the following data:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & -1 & -1 \\ t-t^2 & 0 & -t \\ 1-t & 1 & 0 \end{pmatrix}; \quad q \in C(J, \mathbb{R}^3); \quad J := [0,\, 1]. \tag{4.1}$$

A simple computation shows that:

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{pmatrix};$$

$$Q_1 = \begin{pmatrix} 1-t & 1 & 0 \\ t-t^2 & t & 0 \\ 1-t & 1 & 0 \end{pmatrix}; \quad Q_1 Q = 0; \quad A_2 = \begin{pmatrix} 1 & 0 & -1-t \\ 0 & 1 & -t^2 \\ 0 & 0 & 1 \end{pmatrix};$$

$$P_1 = \begin{pmatrix} t & -1 & 0 \\ t^2-t & 1-t & 0 \\ t-1 & -1 & 1 \end{pmatrix}; \quad X(t) = \begin{pmatrix} 0 & -e^t & 0 \\ 0 & (1-t)e^t & 0 \\ 0 & te^t & 0 \end{pmatrix}.$$

i/ Suppose that:

$$d\eta(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} dt\,; \quad \gamma := (0,\, \gamma_2,\, 0)^T. \tag{4.2}$$

In this case the shooting matrix is of the form:

$$D = \int_0^1 d\eta(t)\, X(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e-2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\operatorname{Ker} D = \operatorname{Ker} P(0)\, P_1(0) = \operatorname{Span}\{(1,\,0,\,0)^T;\ (1,\,0,\,0)^T\};$$
$$\operatorname{Im} D = \mathcal{R}_0 = \operatorname{Span}\{(0,\,1,\,0)^T\},$$

it follows from Theorem 2.1 that Problem (1.1), (1.2) with data (4.1), (4.2) is uniquely solvable for every $q \in C(J, \mathbb{R}^3)$ and $\gamma_2 \in \mathbb{R}$.

ii/ Now let

$$d\eta(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} dt; \quad \gamma := (\gamma,\, \gamma_2,\, 0)^T. \tag{4.3}$$

The shooting matrix D is defined as:

$$D = \int_0^1 d\eta(t)\, X(t) = \begin{pmatrix} 0 & 1-e & 0 \\ 0 & e-2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\operatorname{Ker} D = \operatorname{Ker} P(0)\, P_1(0) = \operatorname{Span}\{(1,0,0)^T; (0,0,1)^T\}$, but $\operatorname{Im} D = \operatorname{Span}\{(1-e, e-2, 0)^T\} \neq \mathcal{R}_0 = \operatorname{Span}\{(1,\, 0,\, 0)^T;\, (0,\, 1,\, 0)^T\}$. Therefore, condition (2.2) of Theorem 2.1 is not valid. Using Theorem 3.1, part (ii), we come to the following necessary and sufficient condition for the existence of solutions of (1.1), (1.2) with data (4.1), (4.3):

$$\begin{aligned}
&(e-2)\int_0^1 e^t\Big\{\int_0^t (1-\tau)e^\tau\big[(\tau-1)q_1(\tau) + (\tau^2-1)q_3(\tau)\big]d\tau\Big\}dt + \\
&+ (1-e)\int_0^1 (1-t)e^t\Big\{\int_0^t (1-\tau)e^\tau\big[(\tau-1)q_1(\tau) + (\tau^2-1)q_3(\tau)\big]d\tau\Big\}dt + \\
&+ (2-e)\int_0^1 \{(1-t)q_1(t) + q_2(t) + q_3(t)\}dt + \\
&+ (1-e)\int_0^1 \{(1-t^2)q_1(t) + tq_2(t) + tq_3(t)\}dt \\
&= (2-e)\gamma_1 + (1-e)\gamma_2.
\end{aligned}$$

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VỀ BÀI TOÁN BIÊN NHIỀU ĐIỂM ĐỐI VỚI PHƯƠNG TRÌNH VI PHÂN ĐẠI SỐ CHỈ SỐ 2

Nguyễn Văn Nghi

Khoa toán Đại học Khoa học Tự nhiên - ĐHQG Hà Nội

Bài báo đề cập đến bài toán biên nhiều điểm đối với phương trình vi phân đại số chỉ số 2. Kết quả chính của bài báo là chỉ ra rằng kết quả nhận được bởi [3] đối với chỉ số 1 có thể mở rộng lên cho phương trình chỉ số 2.