LINEAR EQUATIONS WITH POLYINVOLUTIONS

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Abstract. *Let X be a linear space over C. Denote by $L_0(X)$ the set of all linear operators $A \in L(X \to X)$ with dom $A = X$ and by $\tilde{X}$ subalgebra of $L_0(X)$. Let $S_1, \dots, S_m$ be involutions of orders $n_1, \dots, n_m$ respectively, satisfying $S_i.S_j = S_j.S_i$ for any $i, j = 1, 2, \dots, m$. Consider the equation*

$$A(S)x = \sum_{\substack{i_k=\overline{1,n_k}\\ k=\overline{1,m}}} A_{i_1 \dots i_m} S_1^{i_1} \dots S_m^{i_m} x = y, \tag{0}$$

where $A_{i_1 \dots i_m} \in \tilde{X} (i_k = \overline{1, n_k}, k = \overline{1, m})$ and $x, y \in X$.

In this paper we present a method to reduce the equation (0) to the system of equations without any involution. Then we are able to give all solutions of Equation (0) in a closed form.

1. Some fundamental properties of polyinvolution operators

a. An operator $S \in L_0(X)$ is said to be an involution of order n if $S^n = I$ and $S^k \neq I \ \forall k = \overline{1, n-1}$.

Suppose that S is an involution of order n then

$$P_j = \frac{1}{n} \sum_{k=1}^{n} \epsilon^{-kj} S^k \quad (j = \overline{1,n}), \quad \epsilon = exp\frac{2\pi i}{n}$$

are called projections associated with S.

The projections $P_1, \dots, P_n$ satisfy the following properties:

1. $P_i P_j = \delta_{ij} P_j$, where δ_{ij} is the Kronecker symbol.

2. $\sum_{j=1}^{n} P_j = I$.

3. $SP_i = \epsilon^i P_i$.

Hence, it implies that

$$S = \sum_{i=1}^{n} \epsilon^i P_i, \quad S^k = \sum_{i=1}^{n} \epsilon^{ki} P_i,$$

$$X = \bigoplus_{j=1}^{n} X_j, \quad \text{where} \quad X_j = P_j X \quad (j = \overline{1,n}).$$

b. Let $S_1, ..., S_m$ be commutative involution operators of orders $n_1, ..., n_m$, respectively and $P_{kj_k}(j_k = \overline{1, n_k})$ be projections associated with $S_k(k = \overline{1, m})$. We denote

$$\begin{aligned}
&\Gamma = \{(i) = (i_1, ..., i_m)|\ 1 \le i_k \le n_k,\ k = \overline{1,m}\},\\
&S^{(i)} = S_1^{i_1}, ..., S_m^{i_m},\\
&P_{(j)} = P_{1j_1}, ..., P_{mj_m},\ 1 \le i_k \le n_k,\ k = \overline{1,m},\\
&\Lambda = \{(j_i, ..., j_m)|P_{(j)} \ne 0\},\\
&\epsilon^{(i)} = \epsilon_1^{i_1}, ..., \epsilon_m^{i_m},\ \text{where}\ \ \epsilon_k = exp\frac{2\pi i}{n_k}\ \ (k = \overline{1,m}),\\
&\epsilon^{(i)(j)} = \epsilon_1^{i_1 j_1} ... \epsilon_m^{i_m j_m}.
\end{aligned}$$

Proposition. *We have the following relations*

a) $\sum_{(j)\in A} P_{(j)} = I.$

b) $P_{(i)}P_{(j)} = \begin{cases} 0 \ \text{if}\ (i) \ne (j), \\ P_{(j)}\ \text{if}\ (i) = (j); \end{cases} \quad ((i) = (j) \Leftrightarrow i_k = j_k \quad \forall k = \overline{1,m})$

c) $X = \bigoplus_{(j)\in\Lambda} X_{(j)}$ *where* $X_{(j)} = P_{(j)}X.$

Proof: a) From $\sum_{j_k=1}^{n_k} P_{kj_k} = I\ \forall k = \overline{1,m}$ and the commutativity of P_{kj_k} we get

$$I = \prod_{k=1}^{m}\sum_{j_k=1}^{n_k} P_{kj_k} = \sum_{\substack{j_k=\overline{1,n_k}\\ k=\overline{1,m}}} P_{1j_1}, ...,\ P_{mj_m} = \sum_{(j)\in\Lambda} P_{(j)}.$$

b) It naturally holds by the commutativity and the associativity of projections P_{kj_k}.

c) It is implied from a) and b) □

2. The equation (1) can be rewritten in the from

$$A(S)x = \sum_{(i)\in I} A_{(i)}S^{(i)}x = y, \tag{1}$$

where $A_{(i)} = A_{i_1...i_m}$.

We consider the equation (1) under the assumptions: $\forall\ P_{(j)}, P_{(k)}, A_{(i)}$ where $(j), (k) \in \Lambda, (i) \in \Gamma, \exists\ A_{(i)(j)(k)} \in \tilde{X}$ satisfying

$$\begin{cases} P_{(j)}A_{(i)}P_{(k)} = A_{(i)(j)(k)}P_{(k)}\ \ (\text{or} P_{(j)}A_{(i)} = \sum_{(k)\in\Lambda} A_{(i)(j)(k)}P_{(k)}) \\ \\ P_{(j)}A_{(i)(j)(k)}P_{(l)} = 0\ \ \forall (l) \ne (k) \end{cases} \tag{2}$$

Acting on both sides of equation (1) by $P_{(j)}$, we obtain the system

$$P_{(j)} \sum_{(i)\in\Gamma} A_{(i)} S^{(i)} x = P_{(j)} y \quad \forall (j) \in \Lambda$$

$$\Rightarrow \sum_{(i)\in\Gamma} \sum_{(k)\in\Lambda} A_{(i)(j)(k)} P_{(k)} S^{(i)} x = P_{(j)} y \quad \forall (j) \in \Lambda$$

$$\Rightarrow \sum_{(i)\in\Gamma} \sum_{(k)\in\Lambda} A_{(i)(j)(k)} \varepsilon^{(k)(i)} P_{(k)} x = P_{(j)} y \quad \forall (j) \in \Lambda$$

$$\Rightarrow \sum_{(i)\in\Gamma} \sum_{(k)\in\Lambda} A_{(i)(j)(k)} \epsilon^{(k)(i)} x_{(k)} = P_{(j)} y \quad \forall (j) \in \Lambda, \tag{3}$$

where

$$x_{(k)} = P_{(k)} x \in X_{(k)}.$$

Lemma 1. *If the condition (2) is satisfied, then the equation (1) has a solution $x \in X$ if and only if system (3) has a solution $(x_{(k)})_{(k)\in\Lambda} \in \prod_{(k)\in\Lambda} X_{(k)}$. Moreover, if x is a solution of (1) then $(P_{(k)}x)_{(k)\in\Lambda}$ is a solution of (3) and conversely, if $(x_{(k)})_{(k)\in\Lambda} \in \prod_{(k)\in\Lambda} X_{(k)}$ is a solution of (3) then $x = \sum_{(k)\in\Lambda} x_{(k)}$ is a solution of (1).*

Proof: It is obvious that if the equation (1) has a solution x then from (3), $(P_{(k)}x)_{(k)\in\Lambda}$ is a solution of (3).

Conversely, suppose that the system (3) has a solution $(x_{(k)})_{(k)\in\Lambda} \in \prod_{(k)\in\Lambda} X_{(k)}$. We prove that $x = \sum_{(k)\in\Lambda} x_{(k)}$ is a solution of the equation (1).

Indeed, since $x_{(k)} \in X_{(k)}$ $\forall (k) \in \Lambda$ and $X_{(k)} = P_{(k)}X, X = \bigoplus_{(k)\in\Lambda} X_{(k)}$ we have $P_{(k)}x = x_{(k)}$. Furthermore, $(x_{(k)})_{(k)\in\Lambda}$ is a solution of (3), which implies that

$$\forall (j) \in \Lambda, \qquad \sum_{(i)\in\Gamma} \sum_{(k)\in\Lambda} A_{(i)(j)(k)} \epsilon^{(k)(j)} x_{(k)} = P_{(j)} y$$

$$\Rightarrow \sum_{(i)\in\Gamma} \sum_{(k)\in\Lambda} A_{(i)(j)(k)} \dot{\epsilon}^{(k)(x)} P_{(k)} x = P_{(j)} y \quad \Rightarrow \sum_{(i)\in\Gamma} \sum_{(k)\in\Lambda} A_{(i)(j)(k)} S^{(i)} P_{(k)} x = P_{(j)} y$$

$$\Rightarrow \sum_{(i)\in\Gamma} \sum_{(k)\in\Lambda} A_{(i)(j)(k)} P_{(k)} S^{(i)} x = P_{(j)} y \quad \Rightarrow \sum_{(i)\in\Gamma} P_{(j)} A_{(i)} S^{(i)} x = P_{(j)} y$$

$$\Rightarrow \sum_{(j)\in\Lambda} P_{(j)} \sum_{(i)\in\Gamma} A_{(i)} S^{(i)} x = \sum_{(j)\in\Lambda} P_{(j)} y \quad \Rightarrow \sum_{(i)\in\Gamma} A_{(i)} S^{(i)} x = y.$$

Thus,

$$x = \sum_{(k)\in\Lambda} x_{(k)}$$

is a solution of (1). Lemma 1 is proved □

Lemma 2. *Suppose that the condition (2) is satisfied. If the system (3) has solutions and* $(x_{(k)})_{(k)\in\Lambda} \in X^{|\Lambda|}$ *is one its solution in the space* $X^{|\Lambda|}$ *then* $(P_{(k)}x_{(k)})_{(k)\in\Lambda}$ *is also a solution of (3) in the space* $\prod\limits_{(k)\in\Lambda} X_{(k)}$.

Proof: Let $(x_{(k)})_{(k)\in\Lambda} \in X^{|\Lambda|}$ be a solution of system (3), i. e., $\forall (j) \in \Lambda$

$$\sum_{(i)\in\Gamma}\sum_{(k)\in\Lambda} A_{(i)(j)(k)}\epsilon^{(k)(i)}x_{(k)} = P_{(j)}y$$

$$\Rightarrow \sum_{(i)\in\Gamma}\sum_{(k)\in\Lambda} A_{(i)(j)(k)}\epsilon^{(k)(i)}\sum_{(l)\in\Lambda} P_{(l)}x_{(k)} = P_{(j)}y$$

$$\Rightarrow \sum_{(i)\in\Gamma}\sum_{(k)\in\Lambda}\sum_{\substack{(l)\in\Lambda\\(l)\neq(k)}} \epsilon^{(k)(j)}A_{(i)(j)(k)}P_{(l)}x_{(k)}$$

$$+\sum_{(i)\in\Gamma}\sum_{(k)\in\Lambda} \epsilon^{(k)(i)}A_{(i)(j)(k)}P_{(k)}x_{(k)} = P_{(j)}y$$

$$\Rightarrow \sum_{(i)\in\Gamma}\sum_{(k)\in\Lambda}\sum_{\substack{(l)\in\Lambda\\(l)\neq(k)}} \epsilon^{(k)(i)}A_{(i)(j)(k)}P_{(l)}x_{(k)}$$

$$= P_{(j)}y - \sum_{(i)\in\Gamma}\sum_{(k)\in\Lambda} \epsilon^{(k)(i)}A_{(i)(j)(k)}P_{(k)}x_{(k)}.$$

From Condition (2), it follows that the left side of the last equality belongs to ker $P_{(j)}$ and its right side belongs to $X_{(j)}$. Note that Ker $P_{(j)} = \bigoplus\limits_{(i)\neq(j)} X_{(i)} \Rightarrow \ker P_{(j)} \cap X_{(j)} = \{0\}$ we get

$$P_{(j)}y - \sum_{(i)\in\Gamma}\sum_{(k)\in\Lambda} \epsilon^{(k)(x)}A_{(i)(j)(k)}P_{(k)}x_{(k)} = 0$$

$$\Rightarrow \sum_{(i)\in\Gamma}\sum_{(k)\in\Lambda} \epsilon^{(k)(i)}A_{(i)(j)(k)}P_{(k)}x_{(k)} = P_{(j)}y,$$

i.e., $(P_{(k)}x_{(k)})_{(k)\in\Lambda}$ is a solution of system (3) □

Combining two results just obtained yields the following

Theorem. *Suppose that Condition (2) is satisfied. The equation (1) has solutions if and only if the system (3) has solution. Moreover, if* x *is a solution of (1) then* $(P_{(k)}x)_{(k)\in\Lambda}$ *is a solution of (3) and conversely, if* $(x_{(k)})_{(k)\in\Lambda}$ *is a solution of (3) then* $x = \sum\limits_{(k)\in\Lambda} P_{(k)}x_{(k)}$ *is a solution of (1).*

Remark. If $A_{(i)}$ $((i) \in \Gamma)$ are commutative with the operators S_k $(k = \overline{1,m})$ then the system (3) becomes the independent system

$$\sum_{(i)\in\Gamma} A_{(i)}\epsilon^{(j)(i)}x_{(j)} = P_{(j)}y \quad \forall (j) \in \Lambda.$$

3. Examples.

Example 1. Consider the Volterra - Carleman integral equation of the form

$$\varphi(x,y,t) - \sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^t K_{ij}(x,y,t,\tau)\varphi[\alpha_i(x),\beta_j(y),\tau]d\tau = g(x,y,t), \tag{4}$$

where

1) $g(x,y,t), K_{ij}(x,y,t,\tau)$ are continuous functions $\forall i = \overline{1,n},\quad j = \overline{1,m}$.

2) $\alpha(x), \beta(y)$ are Carleman transformations of order n and m respectively on R, *i.e.*,
$\alpha_{(k+1)}(x) = \alpha[\alpha_k(x)], \alpha_k(x) \neq x \quad \text{if} \quad 1 \leq k < n, \quad \alpha_n(x) = x,$
$\beta_{(k+1)}(y) = \beta[\beta_k(y)], \beta_k(y) \neq y \quad \text{if} \quad 1 \leq k < m, \quad \beta_m(y) = y.$

3) $K_{ij}(x,y,t,\tau)$ are invariant under the transformations $\alpha(x), \beta(y)$.

We define the operators $V, W, A_{ij} \in L_o(x)$ as follows:

$$(V\varphi)(x,y,t) = \varphi[\alpha(x),y,t],$$
$$(W\varphi)(x,y,t) = \varphi[x,\beta(y),t],$$
$$(A_{ij}\varphi)(x,y,t) = \int_0^t K_{ij}(x,y,t,\tau)\varphi(x,y,\tau)d\tau.$$

Then the equation (4) can be rewritten in the form

$$(I - \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}V^iW^j)\varphi = g.$$

We note that V, W are commutative involution operators of order n and m respectively and the operators A_{ij} also commute with $V, W (i = \overline{1,n}, j = \overline{1,m})$.

Denote by $P_\nu(\nu = \overline{1,n})$ and $Q_\mu(\mu = \overline{1,m})$ the projections associated with the V, W respectively.

From the above obtained results, in order to study of Equation (4), we can study the system of the independent equations

$$\varphi_{\nu\mu}(x,y,t) - \int_0^t M_{\nu\mu}(x,y,t,\tau)\varphi_{\nu\mu}(x,y,\tau)d\tau = g_{\nu\mu}(x,y,t) \ \ \forall \nu = \overline{1,n}, \mu = \overline{1,m},$$

where

$$M_{\nu\mu}(x,y,t,\tau) = \sum_{i=1}^{n}\sum_{j=1}^{m} \epsilon_1^{\nu i}\epsilon_2^{\mu j}K_{ij}(x,y,t,\tau),$$
$$\epsilon_1 = exp\frac{2\pi i}{n}, \quad \epsilon_2 = exp\frac{2\pi i}{m},$$
$$g_{\nu\mu}(x,y,t) = (P_\nu Q_\mu g)(x,y,t).$$

Example 2: Consider the Fredholm - Carleman integral equation of form

$$\varphi(x,t) = \sum_{i=1}^{n}\sum_{j=1}^{2}\int_{-1}^{1} K_{ij}(x,t,\tau)\varphi[\alpha_i(x),\beta_j(\tau)]d\tau = g(x,t), \tag{5}$$

where

1) $g(x,y,t)$ and $K_{ij}(x,t,\tau)$ are continuous functions $x \in R; t,\tau \in [-1,1]$.

2) $\alpha(x)$ is Carleman transformation of order n.

3) $\beta(t) = -t, \quad \beta_2(t) = \beta(\beta(t)) = t$.

We define the operators V, W, A_{ij} as follows:

$$(V\varphi)(x,t) = \varphi[\varphi(x),t],$$
$$(W\varphi)(x,t) = \varphi(x,-t),$$
$$(A_{ij}\varphi)(x,t) = \int_{-1}^{1} K_{ij}(x,t,\tau)\varphi(x,\tau)d\tau, \quad \forall i = \overline{1,n}; j = 1,2.$$

Then the equation (5) can be rewritten in the form:

$$\left(I - \sum_{i=1}^{n}\sum_{j=1}^{2} A_{ij}V^iW^j\right)\varphi = g.$$

We get

$$V^n = I, W^2 = I, VW = WV.$$

Denote by $P_\nu(\nu = \overline{1,n})$ the projections associated with V and

$$Q_1 = \frac{1}{2}(I - W), \quad Q_2 = \frac{1}{2}(I + W),$$
$$\varepsilon = \exp\frac{2\pi i}{n}.$$

We prove that the condition (2) is satisfied. Indeed, putting

$$A_{ij\nu s\mu r} = \frac{1}{2n}\sum_{k=1}^{n}\sum_{l=1}^{2}(-1)^{(r-s)l}\varepsilon^{(\mu-\nu)k}V^kW^lA_{ij}V^{n-k}W^{2-l},$$

then

$$\begin{cases} P_\nu Q_s A_{ij} P_\mu Q_r = A_{ij\nu s\mu r}P_\mu Q_r, \\ P_\nu Q_s A_{ij\nu s\mu r}P_\theta Q_\eta = 0 \quad \text{if} \quad \mu \neq \theta \quad \text{or} \quad \eta \neq r \end{cases}$$

$\forall A_{ij}(i = \overline{1,n}; j = 1,2)$ and $\forall P_\nu, P_\mu(\nu,\mu = \overline{1,n})$ $Q_s, Q_r(s,r = 1,2)$.

The rest is to show that $A_{ij\nu s\mu r}$ are the integral operators. We have

$$(A_{ij\nu s\mu r}\varphi)(x,t) = \frac{1}{2n}\sum_{k=1}^{n}\sum_{l=1}^{2}(-1)^{(r-s)l}\varepsilon^{(\mu-\nu)k}\int_{-1}^{1}K_{ij}[\alpha_k(x),\beta_l(t),\tau]\varphi[x,\beta_{2-l}(\tau)]d\tau.$$

Putting

$$\sigma = \beta_{2-l}(\tau) \Rightarrow \tau = \beta_l(\sigma), d\tau = (-1)^l d\sigma,$$

the right side of the last equality can also be written as

$$\frac{1}{2n}\sum_{k=1}^{n}\sum_{l-1}^{2}(-1)^{(r-s)l}\epsilon^{(\mu-\nu)k}\int_{-1}^{1}K_{ij}[\alpha_k(x),\beta_l(t),\beta_l(\sigma)]\varphi(x,\sigma)d\sigma.$$

Putting

$$K_{ij\nu s\mu r}(x,t,\tau) = \frac{1}{2n}\sum_{k=1}^{n}\sum_{l=1}^{2}(-1)^{(r-s)l}\epsilon^{(\mu-\nu)k}K_{ij}[\alpha_k(x),\beta_l(t)\beta_l(\tau)].$$

We get

$$(A_{ij\nu s\mu r}\varphi)(x,t) = \int_{-1}^{1} K_{ij\nu s\mu r}(x,t,\tau)\varphi(t,\tau)d\tau$$

Thus, instead of studying the equation (5) we can study the following one

$$\begin{aligned} \varphi_{\nu s}(x,t) - \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{\mu=1}^{n}\sum_{r=1}^{2}\int_{-1}^{1}\frac{1}{2n}\sum_{k=1}^{n}\sum_{l=1}^{2}(-1)^{(r-s)l+jr}\epsilon^{(\mu-\nu)k+i\mu}\times \\ \times K_{ij}[\alpha_k(x),\beta_l(t),\beta_l(\tau)]\varphi_{\mu r}(x,\tau)d\tau \\ = P_\nu Q_s g(x,t) \quad \forall \nu=\overline{1,n}; s=1,2 \end{aligned} \tag{6}$$

Equation (5) has solutions if and only if the system (6) has solutions. Moreover, if $(\varphi_{\nu s})_{\nu=\overline{1,n},s=1,2}$ are solutions of (6), then

$$\varphi = \sum_{\nu=1}^{n}\sum_{s=1}^{2} P_\nu Q_s \varphi_{\nu s},$$

is a solution of (5).

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PHƯƠNG TRÌNH TUYẾN TÍNH VỚI CÁC TOÁN TỬ ĐA PHỐI HỢP

Trần Thị Tạo
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X là một không gian tuyến tính trên trường C. $L_0(X)$ là tập tất cả các toán tử tuyến tính A trên X với dom $A = X$. $\tilde{X}$ là đại số con của $L_0(X)$. Giả sử $S_1, \ldots, S_m$ là các toán tử đối hợp cấp $n_1, \ldots, n_m$ tương ứng đôi một giao hoán với nhau.

Xét phương trình

$$A(S)x = \sum_{\substack{i_k=\overline{1,n} \\ k=\overline{1,m}}} A_{i_1 \ldots i_m} S_1^{i_1} \ldots S_m^{i_m} x = y, \tag{*}$$

trong đó

$$A_{i_1 \ldots i_m} \in \tilde{X}(i_k = \overline{1,n_k}, k=\overline{1,m}) \quad \text{và} \quad x,y \in X.$$

Nội dung của bài báo này là đưa phương trình (*) về hệ phương trình không còn toán tử đối hợp mà tính giải được của nó khả thi hơn nhiều, đồng thời cho mối liên hệ giữa cấu trúc nghiệm của phương trình (*) với cấu trúc nghiệm của hệ đó.