CONTRACTIBLE SUBLATTICES IN DATA ANALYSIS

Nguyen Duc Dat
Faculty of Mathematics, Mechanics and Informatics
College of Natural Sciences - VNU

1. INTRODUCTION

Nowadays, the applying of lattice theory to concept analysis and data analysis is of great interests.

In [6] R. Wille has proposed the notion of concept lattice. Concerning the construction of this lattice, some problems as subdirect decomposition, tensorial decomposition etc... have been studied in [7, 8].

Further, the notion of concept lattice is widely used in data analysis, for example, in [3, 4, 5, 9]. P. Luksch and R. Wille [4] have proposed a decomposition of a context (O, O, R) into subcontexts, which are indecomposable.

In this paper, we study the contractible sublattices [1] of a concept lattice B (O, A, R) and with its help, we propose a decomposition of context (O, A, R) into pairwisely disjoint subcontexts and its quotient context determined by these subcontexts.

2. CONCEPTS AND RESULTS

First, we recall some notions from R. Wille [6].

Definition 2.1. *By the symbol (O, A, R) we denote a context, where O, A are arbitrary sets and R is a binary relation between O and A. The elements of O and A are called objects and attributes, respectively. If α Ra for $\alpha \in 0$ and $a \in A$ we say: the object α has the attribute a. If $O' \subseteq O, A' \subseteq A$ and $R' = R \cap O' \times A'$ then (O', A', R') is called a subcontext of (O, A, R).*

The relation R establishes a Galois connection between the power sets of O and A as follows:

$$X^* = \{a \in A | (x, a) \in R, \ \forall x \in X\} \ \ for \ \ X \subseteq O,$$

$$Y^* = \{v \in O | (v, y) \in R, \ \forall y \in Y\} \ \ for \ \ Y \subseteq A.$$

Definition 2.2. *A concept of the context (O, A, R) is defined as a pair (M, N), where $M \subseteq O, N \subseteq A$, such that $M^* = N$ and $N^* = M$. The family of all concepts of (O, A, R) are denoted by B(O, A, R). On B(O, A, R) are defined relation $\leq$ and the lattice operations $\wedge, \vee$ as follows:*

(a) $(M_1, N_1) \leq (M_2, N_2) \ if \ M_1 \subseteq M_2$

(b) $\wedge_{i \in I}(M_i, N_i) = (\cap_{i \in I} M_i, (\cap_{i \in I} M_i)^*)$

(c) $\vee_{i \in I}(M_i, N_i) = ((\cap_{i \in I} N_i)^*, \cap_{i \in I} N_i)$

It is easy to demonstrate that B(O, A, R) is a complete lattice [6].

Definition 2.3. *Lattice B(O, A, R) is called a concept lattice of the context (O, A, R).*

In this paper we consider the set of objects and set of attributes, which are finite, the objects are denoted by a, b, c etc..., the attributes by 1, 2, 3 etc...

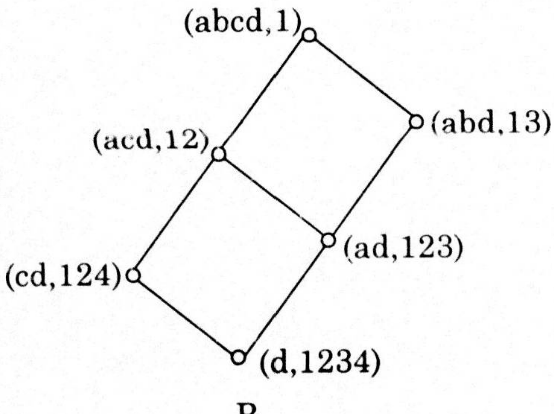
Example. Consider the contexts $C_1 = (O, A, R_1)$ and $C_2 = (O, A, R_2)$ (Fig.1), where $O = \{a, b, c, d\}, A = \{1, 2, 3, 4\}$. These contexts determine the concept lattices $B_1 = B(O, A, R_1)$ and $B_2 = B(O, A, R_2)$ respectively. For denoting a concept, for example, $(M, N) \in B_1$ with $M = \{a, b, d\}, N = \{1, 3\}$, we shall write (abd, 13) instead of $(\{a, b, d\}, \{1, 3\})$.

	1	2	3	4
a	×	×	×	
b	×		×	
c	×	×		×
d	×	×	×	×

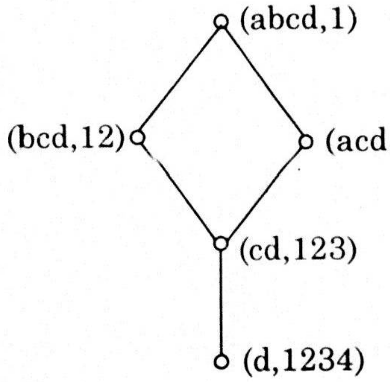
C_1

	1	2	3	4
a	×		×	
b	×	×		
c	×	×	×	
c	×	×	×	×

C_2



B_1



B_2

Fig. 1

Now, we shall deal with the concepts of contractible sublattice [1].

Definition 2.4. *Let L be a lattice.*

1) A sublattice C of L is called convex if $a \leq x \leq b$ with $a, b \in C$ then $x \in C$.

2) If $a, b \in L, a$ is uncomparable with b and $\{c, d\} = \{a \wedge b, a \vee b\}, c \neq d$, then sublattice $\{a, b, c, d\}$ is called a square of L and it is denoted by $< a, b; c, d >$.

3) A proper sublattice C of L with $|C| > 1$ is called a contractible sublattice if C satisfies the following conditions:

a) C is convex.

b) If $< a, b; c, d >$ is a square in L then $c \in C \Leftrightarrow d \in C$.

In [2] there has already been proved:

Proposition 2.5. *Let C be a contractible sublattice of a lattice L and $k \in L \setminus C, c \in C$, then: (P_1) If $k < c$ then $k < x, \ \forall x \in C$.*

(P_2) If $k > c$ then $k > x, \forall x \in C$.

(P_3) If k is uncomparable with c then k is uncomparable with $x, \forall x \in C$.

Definition 2.6. *We say that lattice L has a linear decomposition (or is linearly decomposable) if there exist a chain I with $|I| > 1$ and sublattices $L_i, i \in I$ such that $L = \cup_{i \in I} L_i$ and for $i, j \in I, i < j$ then $a < b, \ \forall a \in L_i, \ \forall b \in L_j$.*

Lemma 2.7. *Let C_1, C_2 be contractible sublattices of L such that one does not contain the other and $C_1 \cap C_2 \neq \phi$, then $C_1 \cup C_2$ is a linearly decomposable sublattice. Further, if $C_1 \cup C_2 \neq L$ then it is a contractible sublattice.*

Proof. Using the properties $(P_1), (P_2), (P_3)$ we come to the conclution of the lemma.◇

Lemma 2.8. *Let $\{C_i | i \in I\}$ be a family of contractible sublattices of L such that $C_i \cap C_j = \phi, \ \forall i, j \in I, i \neq j$. Then on L there exists a congruence such that every $C_i, i \in I$, is an equivalent class and the others are one - element classes.*

Proof. a) We define an equivalence ρ on L, which has the classes as $C_i, i \in I$ and $\{x\}, \ x \in L \setminus \cup_{i \in I} C_i$.

b) Let $a \rho a'$ and $b \rho b'$, we have to prove that $(a \wedge b) \rho (a' \wedge b')$ and $(a \vee b) \rho (a' \vee b')$. When $a = a', b = b'$ or $a, b, a', b' \in C_i$ for some $i \in I$, it is evident. Suppose that $a \neq a', b \neq b'$ and $a, a' \in C_i, \ b, b' \in C_j$ for some $i, j \in I, \ i \neq j$. Denote $c = a \wedge b, \ c' = a' \wedge b'$. If $c \in C_i$ then we get $b \in C_i$ as C_i is contractible, but it contradicts to the relation $C_i \cap C_j = \phi$. Thus, $c \notin C_i$ and hence $c < a'$ according to (P_1).

Analogously, we have $c < b'$ and so $a \leq a' \wedge b' = c'$. By the same argument we also have $c' \leq c$ and thus $c = c'$, i.e. $c \rho c'$.

By duality we have $(a \vee b) \rho (a' \vee b')$. The proof is completed. ◇

Consequence 2.9. *If a lattice L is finite and has no linear decomposition, then L has a congruence ρ such that the quotient lattice L/ρ has no contractible sublattices.*

Proof. Since L is finite, every its contractible sublattice is embeded into a maximal one. Suppose that $C_i, i \in I$ are all maximal contractible sublattices of L then by (2.6) we get $C_i \cap C_j = \phi, \ \forall i, j \in I, i \neq j$, The remain of the proof is implied from (2.8). ◇

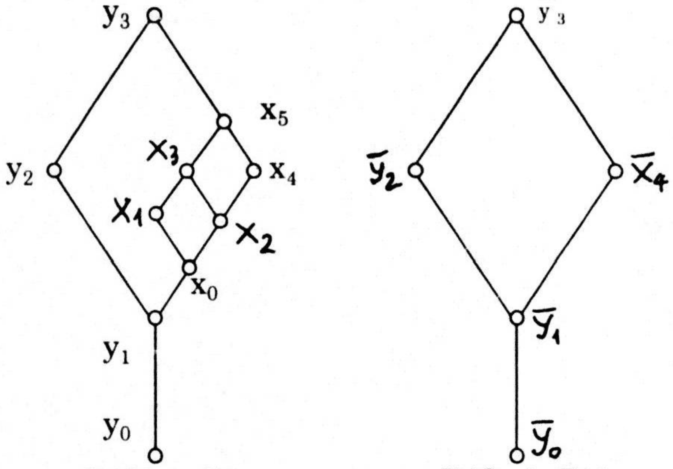
Now, we return to the concept lattices. Consider the following example.

Example 2.10. Let C be a context (Fig.2a) and $B(O, A, R)$ be its concept lattice (fig.2b).

	0	1	2	3	4	5	6
x	×						
a	×	×	×	×			
b	×	×		×			
c	×	×	×		×		
d	×	×	×	×	×		
e	×	×	×	×	×	×	
f	×	×	×	×	×	×	×

$\mathcal{C} = (O, A, R)$

2a



B(O, A, R)

2b

B(O, A, R)/ρ

2c

Fig. 2

Here

$$x_0 = (d\ e\ f,\ 0\ 1\ 2\ 3\ 4) \qquad y_0 = (e\ f,\ 0\ 1\ 2\ 3\ 4\ 5\ 6)$$
$$x_1 = (c\ d\ e\ f,\ 0\ 1\ 2\ 4) \qquad y_1 = (e\ f,\ 0\ 1\ 2\ 3\ 4\ 5)$$
$$x_2 = (a\ d\ e\ f,\ 0\ 1\ 2\ 3) \qquad y_2 = (e\ f\ x,\ 0\ 5)$$
$$x_3 = (a\ c\ d\ e\ f,\ 0\ 1\ 2) \qquad y_3 = (a\ b\ c\ d\ e\ f\ x,\ 0)$$
$$x_4 = (a\ b\ d\ e\ f,\ 0\ 1\ 3)$$
$$x_5 = (a\ b\ c\ d\ e\ f,\ 0\ 1)$$

In $B(O, A, R)$ there exists a contractible sublattice $C = [x_0, x_5]$ (interval). By (2.8), there exists a congruence ρ on $B(O, A, R)$:

$$a \rho b \quad \text{if} \begin{cases} a = b \text{ or} \\ a, b \in C \end{cases}$$

Thus, we have a quotient lattice $B(O, A, R)/\rho$ (Fig. 2c). We want to construct, with a help of the context $\mathcal{C}$, a context defining sublattice C and another context defining $B(O, A, R)/\rho$.

a) Putting $O' = \{a, b, c, d\}$, $A' = \{1, 2, 3, 4\}$ and $R' = R \cap O' \times A'$, we have a new context (O', A', R'), which is a subcontext of $\mathcal{C}$ (given by a small square in Fig.2a).

It is easy to see that $C \cong B(O', A', R')$ (see also the context $\mathcal{C}_1$ and concept lattice B_1 in Fig.1).

b) Putting $O^\# = \{x, O', e, f\}$, $A^\# = \{0, A', 5, 6\}$ and basing $\mathcal{C}$ we construct a context $(O^\#, A^\#, R^\#)$:

	0	A'	5	6
X	×		×	
O'	×	×		
e	×	×	×	
f	×	×	×	×

This context has a concept lattice isomorphic to lattice $B(O, A, R)/\rho$ (see also the context C_2 and concept lattice B_2 in Fig .1). Context $(O^{\#}, A^{\#}, R^{\#})$ is called a quotient context of (O, A, R) by subcontext C.

Now, we will generalize Example 2.10 for an arbitrary contractible sublattice C in a concept lattice $B(O, A, R)$.

In [4], *P.* Luksch and *R.* Wille have proposed notion superalternative for a context (O, O, R) (here $O = A$). In this paper, we define "superalternative" for (O, A, R).

Definition 2.11. *Let* $O_1, O^{'}, O_2$ *and* $A_1, A^{'}, A_2$ *are the pairwisely disjoint sub sets in* O *and* A, *respectively, such that* $O = O_1 \cup O^{'} \cup O_2, A = A_1 \cup A^{'} \cup A_2$. *We say that a pair* $(O^{'}, A^{'})$ *is a pair of superalternatives in* (O, A, R) *if:*

$$x \in O_1 \text{ then } (x, \alpha) \in R, \ \forall \alpha \in A^{'}$$

$$x \in O_2 \text{ then } (x, \alpha) \notin R, \ \forall \alpha \in A^{'}$$

$$\alpha \in A_1 \text{ then } (x, \alpha) \in R, \ \forall x \in O^{'}$$

$$\alpha \in A_2 \text{ then } (x, \alpha) \notin R, \ \forall x \in O^{'}$$

$$x \in O_1, \alpha \in A_1 \text{ then } (r, \alpha) \in R$$

Note 2.12. *If* (O', A') *is a pair of superalternatives, putting* $R' = R \cap O' \times A'$, *we have a context* (O', A', R') *which is a subcontext of* (O, A, R).

Proposition 2.13. *If* C *is a contractible sublattice of* $B(O, A, R)$ *then there exists a pair of superalternatives* (O', A') *such that* $C \cong B(O', A', R')$.

Proof. Suppose that $(M_i, N_i), i \in I$, are all elements of C. Put $M_0 = \cap_{i \in I} M_i$, $N_1 = \cap_{i \in I} N_i$, $N_0 = M_0^*$, $M_1 = N_1^*$ then (M_0, N_0) and (M_1, N_1) are the smallest and greatest element of C, respectively.

Take $(K, R) \in C$ and $(L, S) \in C$ such that $(K, R) \longrightarrow\!\!\!< (M_0, N_0), (M_1, N_1) \longrightarrow\!\!\!< (L, S)$ and $O^{'} = M_1 \setminus K, A^{'} = N_0 \setminus S$ (in the case, where there does not exist (K, R) then we put $O^{'} = M_1$, in the case, where there does not exist (L, S) then $A^{'} = N_0$.

Consider $(M, N) \in B(O, A, R), (M, N) \notin C$. As C is contractible, applying $(P_1)(P_2)(P_3)$, we have 3 possibilities:

(a) $M \subseteq N_0$

(b) $M \supseteq M_1$

(c) (M, N) is uncomparable with (M_1, N_1).

Consider each possibility in detail, by (a) we have: $\forall x \in M, (x, \alpha) \in R, \ \forall \alpha \in A^{'}$, by (b) we have: $\forall x \in M \setminus M_1, (x, \alpha) \notin R, \ \forall \alpha \in A'$ and finally from (c), it implies that $x \in M$, either $x \in M \cap M_1 \subseteq M_0$, or $x \in (N_1 \cap N)^* \supseteq M_1$, i.e. either $(x, \alpha) \in R, \ \forall \alpha \in A'$ or $(x, \alpha) \notin R, \ \forall \alpha \in A^{'}$, respectively.

In conclusion, if x is an object, then

1) $x \in O^{'}$ or

2) $(x, \alpha) \in R, \ \forall \alpha \in A^{'}$ or

3) $(x, \alpha) \notin R, \forall \alpha \in A'$.

Thus, $O = O_1 \cup O' \cup O_2$, where O_1, O', O_2 are pairwisely disjoint and the condition of Definition 2.11 is satisfied.

Analogously, we have also $A = A_1 \cup A' \cup A_2$ such that A_1, A', A_2 are pairwisely disjoint and condition (2) holds.

Finally, consider a pair (x, α) with $x \in O_1, \alpha \in A_1$; as $x \in O_1$ then $(x, \beta) \in R$, $\forall \beta \in A'$., thus $x \in M$, $\forall M \supseteq M_1$. On the other hand, as $\alpha \in A_1$ then $(y, \alpha) \in R$, $\forall y \in O'$, thus $\alpha \in N$, $\forall N \subseteq N_1$. In conclusion, $(x, \alpha) \in R$ and (3) holds.

Now, we take (O', A', R') with $R' = R \cap O' \times A'$ and $B(O', A', R')$. If $(E, F) \in B(O', A', R')$ then, by (1), (2), (3) we have $(O_1 \cup E, A_1 \cup F) \in C$. This correspondence is an isomorphism between $B(O', A', R')$ and C.

The proof is completed. ◇

Proposition 2.14. *If C is a contractible sublattice of $B(O, A, R)$, then there exists a context $(O^\#, A^\#, R^\#)$ such that $B(O, A, R)/\rho \cong B(O^\#, A^\#, R^\#)$, (here ρ is a congruence on $B(O, A, R)$ determined by C).*

Proof. Denote by (O', A', R') the context corresponding C (proposition 2.13.) and put $s = O', t = A', O^\# = (O \backslash O') \cup \{s\}, A^\# = (A \backslash A') \cup \{t\}$. Define $R^\#$ between $O^\#$ and $A^\#$ as follows:

1) $(s, t) \in R^\#$,

2) $x \neq s, \alpha \neq t$, then $(x, \alpha) \in R^\# \Leftrightarrow (x, \alpha) \in R$,

3) $(x, t) \in R^\# \Leftrightarrow x \in O_1$,

4) $(s, \alpha) \in R^\# \Leftrightarrow \alpha \in A_1$.

Take $P = O_1 \cup \{s\}, Q = A_1 \cup \{t\}$, it is easy to deduce that $(P, Q) \in B(O^\#, A^\#, R^\#)$ and $B(O, A, R)/\rho \simeq B(O^\#, A^\#, R^\#)$ such that the equivalence class C of $B(O, A, R)$ is corresponding to (P, Q) of $B(O^\#, A^\#, R^\#)$.

The proposition is proved. ◇

Now, we apply propositions (2.13), (2.14) to study an arbitrary finite context (O, A, R). Consider its concept lattice $B(O, A, R)$.

Case (I). Let $B(O, A, R)$ be linearly indecomposable and have contractible sublattices. Each of these sublattices is embeded into a maximal one. Suppose that $\{C_i | i \in I\}$ is a family of all maximal contractible sublattices. By lemma 2.8 $\{C_i | i \in I\}$ determine a congruence ρ and a quotient lattice $B(O, A, R)/\rho$, the latest has no contractible sublattices.

According to (2.13) and (2.14), there exist subcontexts $C_i = (O_i, A_i, R_i)$ determining $C_i, i \in I$, and context $(O^\#, A^\#, R^\#)$ determining $B(O, A, R)/\rho$.

In this case, we say that (O, A, R) has decomposition by a system of the contexts $C_i, i \in I$ and $C^\#$ where $C_i, i \in I$ are paiwisely disjoint and $C^\#$ is indecomposable.

Example. Consider the lattice in Fig.3a.

Case (II). $B(O, A, R)$ is not linearly decomposable and has contractible sublattices $C_1 ..., C_n$ such that $B \supseteq C_1 \supseteq ... \supseteq C_n$, where the next sublattice is maximal in the previous one.

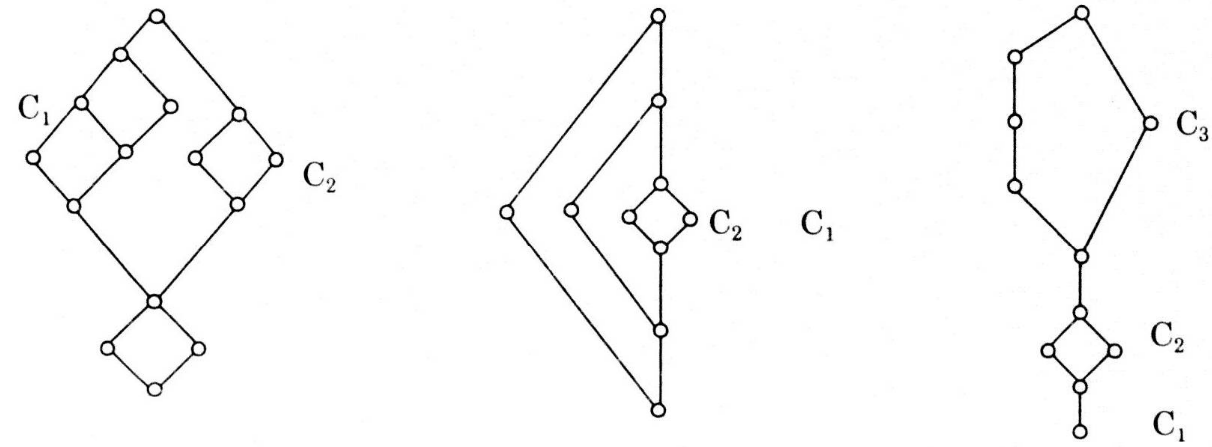
In this case, we have a decomposition $C \supseteq C_1 \supseteq ... \supseteq C_n$.

Example. Consider $B(O, A, R)$ in Fig.3b.

Case (III). $B(O, A, R)$ has a linear decomposition. Suppose that $C_1, C_2 ..., C_n$ are the linear members of $B(O, A, R)$, which are linearly indecomposable. If, for some $1 \leq i \leq n, |C_i| > 1$ then C_i is a contractible sublattice, which is a multi - dimensional part of $B(O, A, R)$.

In this case, (O, A, R) has a convenient decomposition such that $B(O, A, R)/\rho$ is a linear lattice.

Example. Consider $B(O, A, R)$ in Fig. 3c.



$B(O, A, R), C_1 \cap C_2 = \varnothing$ — *3a*

$B(O, A, R) \supset C_1 \supset C_2$ — *3b*

$B(O, A, R) = C_1 \cup C_2 \cup C_3$ — *3c*

Fig.3

REFERENCES

[1.] Nguyen Duc Dat. Some results concerning a Gratzer's problem, *VNU. J.Sci, Nat. Sci, tXI,* N^0 **4** (1995), 64 - 71.

[2.] Nguyen Duc Dat. On lattices L determined by Sub (L) up to isomorphism, *Vietnam J. Math,* N^0 **24** (1996), 357 - 365.

[3.] D.J.N. Van Eijck and Nguyen Quoc Toan. *Understanding the world: from facts to concepts, from concepts to proposition, Centrum voor wiskund en Informatica rapport, computer Science.* Departement of Software Technology, Note CS - N9301, 2 - 1993.

[4.] P. Luksch and R.Wille. *Formal concept analysis of paired comparisons, Classification and Related methods of Data analysis.* In H.H. Bock, editor, North - Holand. 1988, 167 - 176.

[5.] F. Vogt, C. Wachter and R. Wille. *Data analysis based on a conceptual file, Classification, data analysis and knowledge organization,* In H.H.Block and P.Ihm, editors, Springer. 1991, 131 - 140.

[6.] R. Wille. *Restructuring lattice theory: an approach based on hierchies of concepts, Ordered Sets.* In I. Rival, editor. 1982, 445 - 470.

[7.] R. Wille. Subdirect decomposition of concept lattices. *Algebra Universalis* **17**(1983). 275 - 287.

[8.] R. Wille, Tensorial decomposition of concept lattices, Order. **2**(1985), 81 - 95.

[9.] R. Wille. *Lattices in data analysis: how to draw them with a computer, Algorithms and order.* In I. Rival, editor, Reidel. 1989, 33 - 58.

TẠP CHÍ KHOA HỌC ĐHQGHN, KHTN, t.XV, n^{0}5 - 1999

DÀN CON CO ĐƯỢC TRONG BÀI TOÁN PHÂN TÍCH DỮ LIỆU

Nguyễn Đức Đạt

Khoa Toán - Cơ- Tin học

Đại học KH Tự nhiên - ĐHQG Hà Nội

Hiện nay, việc áp dụng lý thuyết dàn vào bài toán phân tích khái niệm và phân tích dữ liệu đang thu hút sự quan tâm của rất nhiều tác giả.

Năm 1982, R. Wille đã đưa ra khái niệm *dàn khái niệm* $B(O, A, R)$ xác định bởi *ngữ cảnh* (O, A, R).

Trong hơn một thập kỷ nay, nhiều công trình nghiên cứu đã đề cập tới cấu trúc cũng như áp dụng của dàn khái niệm.

Trong bài này chúng tôi áp dụng khái niệm *dàn con co được* vào nghiên cứu dàn $B(O, A, R)$ và nhờ nó đã đề xuất một cách phân giải ngữ cảnh (O, A, R) theo các *ngữ cảnh con* rời nhau từng cặp và *ngữ cảnh thương* xác định bởi hệ thống các ngữ cảnh con này.