ON UNIFORM STABILITY OF THE CHARATERISTIC SPECTRUM FOR SEQUENCES OF LINEAR DIFFERENTIAL EQUATION SYSTEM*

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Abstract: *In this paper we give a condition for which the charateristic spectrum of the sequence of linear differential equation systems are stable . This condition is imposed on the coefficeints of systems. The obtained results are applied for studying uniform roughness.*

I. INTRODUCTION

Consider a sequence of systems consisting of linear differential equation

$$\frac{dx}{dt} = A_n(t)x, \quad x \in R^2, \quad n = 1, 2, \cdots, \tag{1}$$

where $A_n(t)$ is a $n \times n$ - matrix continuous on $[t_0, \infty)$ and satisfies the condition

$$\sup_{t \geq t_0} \|A_n(t)\| \leq M < \infty, n = 1, 2, \cdots \tag{2}$$

denote by $\lambda_1^{(n)}, \lambda_2^{(n)} (\lambda_1^{(n)} \leq \lambda_2^{(n)})$ the charateristic spectrum of the system (1).

Let us associate with (1) a sequence of non-linear systems

$$\frac{dx}{dt} = A_n(t)x + f_n(t, x) \tag{3}$$

perturbed by the function $f_n(t, x)$ satisfying the relation

$$\|f_n(t, x)\| \leq \delta_n \|x\|, 0 < \delta_n \leq \delta < \infty. \tag{4}$$

As well known, the above assumptions imply that the charateristic spectrums of the sequence (3) are a bounded set.

Denote by $\overline{\Lambda}_n$ the charateristic spectrum of (3).

* This paper was supported in part by the National Research Program in Natural Scieces, KT 04, 137

Definition. *The charateristic spectrum of (1) is said to be uniformly upper-stable if for any given $\varepsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that the assumption (4) implies*

$$\mu \leq \lambda_2^{(n)} + \varepsilon \tag{5}$$

for all $\mu \in \overline{\Lambda}_n$.

If the assumption (4) implies

$$\mu \geq \lambda_1^{(n)} - \varepsilon, \tag{6}$$

then the charateristic spectrum of (1) is said to be uniformly lower-stable.

If both the inequalities (5)-(6)hold, then the charateristic spectrum of (1) is said to be uniformly stable.

The notion of uniform stability of a charateristic spectrum for the sequence of differential equation systems is used in the study of uniform roughness of this sequence and, in turn, the uniform roughness of the sequence of differential equation systems is used in estimation of number of stable periodic solutions of the differential systems [1].

II. SPECIAL CASE $f_n(t, x) = B_n(t)x$

First of all, we consider the special case where $f_n(t, x)$ is linear in x,that is:

$$f_n(t, x) = B_n(t)x.$$

Then the system (3) is of the form

$$\frac{dx}{dt} = A_n(t)x + B_n(t)x, \quad n = 1, 2, \cdots \|B_n(t)\| \leq \delta_n < \delta \quad \forall t \geq t_0 \tag{7}$$

denote by $\overline{\lambda}_1^{(n)}, \overline{\lambda}_2^{(n)} (\overline{\lambda}_1^{(n)} \leq \overline{\lambda}_2^{(n)})$ the charateristic spectrum of (7).

Applying Perron's transformation

$$x = U_n(t)y, \tag{8}$$

where $U_n(t)$ is an orthogonal matrix, the system (1) is reduced to the triangular one

$$\frac{dy}{dt} = P_n(t)y, \tag{9}$$

where $P_n(t) = U_n^{-1}(t)A_n(t)U_n(t) - U_n^{-1}(t)\dot{U}_n(t)$. It is easy to verify that

$$\|P_n(t)\| \leq M_1, n = 1, 2, \cdots$$

by the transformation (8), the system (7) becomes

$$\frac{dy}{dt} = P_n(t)y + Q_n(t)y, \tag{10}$$

where $Q_n(t) = U_n^{-1}(t)B_n(t)U_n(t)$. Denote by $p_{11}^{(n)}(t), p_{12}^{(n)}(t), p_{22}^{(n)}(t)$ elements of the matrix $P_n(t)$.

We rewrite the system (10) as

$$\frac{dy}{dt} = \tilde{P}_n(t)y + \tilde{Q}_n(t)y, \tag{11}$$

where

$$\tilde{P}_n(t) = \begin{pmatrix} p_{11}^{(n)}(t) & 0 \\ 0 & p_{22}^{(n)}(t) \end{pmatrix},$$

$$\tilde{Q}_n(t) = Q_n(t) + \begin{pmatrix} 0 & p_{12}^{(n)}(t) \\ 0 & 0 \end{pmatrix}.$$

Obviously, if $\|B_n(t)\| < \delta;\ n = 1, 2, \ldots$ we have $\|Q_n(t)\| < \delta;\ n = 1, 2, \ldots$

From the relation $\|P_n(t)\| < M_1$, n=1,2,... and by applying the transformation $u = Sz$ where $S = \begin{pmatrix} \frac{M_1}{\delta} & 0 \\ 0 & \sqrt{\frac{M_1}{\delta}} \end{pmatrix}$ we can verify that

$$\|\tilde{Q}_n(t)\| \le 2\sqrt{M_1\delta}. \tag{12}$$

Since $\tilde{P}_n(t)$ is a diagonal matrix, the solution y(t) of the system (11) with the initial condition $y(t_0) = y_0$ is of the form

$$y(t) = \exp\Big(\int_{t_0}^{t} \tilde{P}_n(\tau)d\tau\Big)\Big[y_0 + \int_{t_0}^{t} \exp\Big(-\int_{t_0}^{\tau} \tilde{P}_n(s)ds\Big)\tilde{Q}_n(\tau)y(\tau)d\tau\Big].$$

Or equivalently,

$$\exp\Big(-\int_{t_0}^{t} \tilde{P}_n(\tau)d\tau\Big)y(t) = y_0 + \int_{t_0}^{t} \exp\Big(-\int_{t_0}^{\tau} \tilde{P}_n(s)ds\Big)\tilde{Q}_n(\tau)y(\tau)d\tau.$$

Hence, we obtain

$$\Big\|\exp\Big(\int_{t_0}^{t} \tilde{P}_n(s)ds\Big)y(t)\Big\| \le \|y_0\| + \int_{t_0}^{t} \Big\|\exp\Big(-\int_{t_0}^{\tau} \tilde{P}_n(s)ds\Big)\tilde{Q}_n(\tau)\exp\Big(\int_{t_0}^{\tau} \tilde{P}_n(s)ds\Big)\Big\| \times \Big\|\exp\Big(-\int_{t_0}^{t} \tilde{P}_n(s)ds\Big)y(\tau)\Big\|d\tau. \tag{13}$$

Denote by $\tilde{q}_{ij}(t)$ (i,j=1,2) elements of the matrix $\tilde{Q}_n(t)$. Then by straightforward calculations, we have

$$\exp\Big(-\int_{t_0}^{t} \tilde{P}_n(\tau)d\tau\Big)\tilde{Q}_n(t)\exp\Big(\int_{t_0}^{t} \tilde{P}_n(\tau)d\tau\Big) =$$

$$= \begin{pmatrix} \tilde{q}_{11}^{(n)}(t) & \tilde{q}_{12}^{(n)}(t)\exp\Big(\int_{t_0}^{t}[p_{22}^{(n)}(\tau) - p_{11}^{(n)}(\tau)]d\tau\Big) \\ \tilde{q}_{21}^{(n)}(t)\exp\Big(\int_{t_0}^{t}[p_{11}^{(n)}(\tau) - p_{22}^{(n)}(\tau)]d\tau\Big) & \tilde{q}_{22}^{(n)}(t). \end{pmatrix}$$

From the proof of Perron's theorem we deduce $p_{ii}^{(n)}(t) = \overline{p}_{ii}^{(n)}(t)$, i=1,2 ; n=1,2,... where $\overline{p}_{ii}^{(n)}(t)$, i=1,2 are diagonal elements of the matrix $U_n^{-1}(t)A_n(t)U_n(t)$. As an orthogonal transfomation in the plane, Perron's matrix $U_n(t)$ in this case is of the form

$$U_n(t) = \begin{pmatrix} \cos\varphi^{(n)}(t) & \sin\varphi^{(n)}(t) \\ \sin\varphi^{(n)}(t) & -\cos\varphi^{(n)}(t) \end{pmatrix},$$

or

$$U_n(t) = \begin{pmatrix} \cos\varphi^{(n)}(t) & -\sin\varphi^{(n)}(t) \\ \sin\varphi^{(n)}(t) & \cos\varphi^{(n)}(t) \end{pmatrix}.$$

where $\varphi^{(n)}(t)$ is the angle between a solution of (1) and the axis x_1. A direct computation shows that

$$p_{11}^{(n)}(t)-p_{22}^{(n)}(t) = \overline{p}_{11}^{(n)}(t)-\overline{p}_{22}^{(n)}(t) = \cos 2\varphi^{(n)}(t)[a_{11}^{(n)}(t)-a_{22}^{(n)}(t)]+\sin 2\varphi^{(n)}(t)[a_{21}^{(n)}+a_{12}^{(n)}(t)],$$

$$p_{22}^{(n)}(t)-p_{11}^{(n)}(t) = \overline{p}_{22}^{(n)}(t)-\overline{p}_{11}^{(n)}(t) = \cos 2\varphi^{(n)}(t)[a_{22}^{(n)}(t)-a_{11}^{(n)}(t)]-\sin 2\varphi^{(n)}(t)[a_{21}^{(n)}+a_{12}^{(n)}(t)].$$

Therefore,

$$p_{22}^{(n)}(t) - p_{11}^{(n)}(t) = \sqrt{\{[a_{11}^{(n)}(t) - a_{22}^{(n)}(t)]^2 + [a_{21}^{(n)} + a_{12}^{(n)}(t)]^2\}}.\cos[2\varphi^{(n)}(t) + \Psi_n(t)].$$

in which

$$tg\Psi_n(t) = \frac{a_{21}^{(n)} + a_{12}^{(n)}(t)}{a_{11}^{(n)}(t) - a_{22}^{(n)}(t)},$$

and

$$p_{22}^{(n)}(t) - p_{11}^{(n)}(t) = \sqrt{\{[a_{11}^{(n)}(t) - a_{22}^{(n)}(t)]^2 + [a_{21}^{(n)} + a_{12}^{(n)}(t)]^2\}}.\cos[2\varphi^{(n)}(t) + \overline{\Psi}_n(t)],$$

where $\overline{\Psi}_n(t) = \Psi_n(t) + \pi$.

Denote

$$\Omega_n(t) = \sqrt{[a_{11}^{(n)}(t) - a_{22}^{(n)}(t)]^2 + [a_{21}^{(n)} + a_{12}^{(n)}(t)]^2} \tag{14}$$

The above reasoning give us

$$\Big\| \exp\Big(-\int_{t_0}^{t} \tilde{P}_n(\tau)d\tau\Big)\tilde{Q}_n(t)\exp\Big(\int_{t_0}^{t} \tilde{P}_n(\tau)d\tau\Big)\Big\| \le M_2\sqrt{\delta}\times$$

$$\{\exp\Big(\int_{t_0}^{t} \Omega_n(\tau)\cos[2\Phi^{(n)}(\tau) - \Psi_n(\tau)]d\tau\Big) + \exp\Big(\int_{t_0}^{t} \Omega_n(\tau)\cos[-\pi + 2\varphi^{(n)}(\tau) - \Psi_n(\tau)]d\tau\Big)\}$$

Assume

$$\int_{t_0}^{t} \Omega_n(\tau)d\tau \le c < \infty, n = 1, 2, ..., \tag{15}$$

then

$$\Big\| \exp\Big(-\int_{t_0}^{t} \tilde{P}_n(\tau)d\tau\Big)Q_n(t)\exp\int_{t_0}^{t} \tilde{P}_n(\tau)d\tau\Big\| \le M_3\sqrt{\delta}. \tag{16}$$

The inqualities (13)-(16) imply that

$$\Big|\exp\big(-\int_{t_0}^{t} p_{11}^{(n)}(\tau)d\tau\big)y_1(t)\Big| \le \exp\big(M_3\sqrt{\delta}\big)(t-t_0), \tag{17}$$

$$\Big|\exp\big(-\int_{t_0}^{t} p_{11}^{(n)}(\tau)d\tau\big)y_2(t)\Big| \le \exp\big(M_3\sqrt{\delta}\big)(t-t_0). \tag{18}$$

From (17)-(18) and properties of triangular systems we deduce

$$\chi[y_1(t)] \le M_3\sqrt{\delta} + \lambda_1^{(n)},$$

$$\chi[y_1(t)] \le M_3\sqrt{\delta} + \lambda_2^{(n)}.$$

Remark that the transfomations used in the above reasoning do not change the charateristic spectrum of differetial equation systems. Therefore, if

$$\delta < (\frac{\epsilon}{M_3})^2,$$

then

$$\chi[y_1(t)] \le \lambda_2^{(n)} + \epsilon. \tag{19}$$

Hence,

$$\overline{\lambda}_2^{(n)} \le \lambda_2^{(n)} + \epsilon. \tag{20}$$

We finish the specal case $f_n(t,x) = B_n(t)x$ by giving a lower bound for $\overline{\lambda}_1^{(n)}$. For this purpose we assume that (1) is regular. Let

$$\gamma_1^{(n)} \ge \gamma_2^{(n)}; \overline{\gamma}_1^{(n)} \ge \overline{\gamma}_2^{(n)}$$

denote the spectra of the adjoint systems corresponding to (1) and (7). Then by Perron's theorem and Lyapunov's inequality we have

$$\lambda_1^{(n)} + \gamma_1^{(n)} = 0, \overline{\lambda}_1^{(n)} + \overline{\gamma}_1^{(n)} \ge 0.$$

Applying (20) to the bigger charateristic exponent it yields

$$\overline{\gamma}_1^{(n)} \le \gamma_1^{(n)} + \epsilon,$$

or

$$-\overline{\lambda}_1^{(n)} \le -\lambda_1^{(n)} + \epsilon.$$

Thus,

$$\overline{\lambda}_1^{(n)} \ge \lambda_1^{(n)} - \epsilon. \tag{21}$$

Suming up, we heve the following:

Lemma.

For ϵ small enough and

$$\int_{t_0}^{t} \Omega_n(\tau)d\tau \leq c < \infty, n = 1, 2, ...,$$

where

$$\Omega_n(t) = \sqrt{[a_{11}^{(n)}(t) - a_{22}^{(n)}(t)]^2 + [a_{21}^{(n)} + a_{12}^{(n)}(t)]^2}.$$

We have

$$\overline{\lambda}_2^{(n)} \leq \lambda_2^{(n)} + \epsilon, n = 1, 2, ...$$

Moreover, if the all systems of (1) are regular, we have also

$$\overline{\lambda}_1^{(n)} \geq \lambda_1^{(n)} - \epsilon \ , n = 1, 2, ...$$

Now, the general case can be reduced to the special one by means of the linear inclusion principle (see[3]).

Theorem. *Under the assumptions of the lemma, the charateristic spectrum of the sequence of systems (1) is uniformly upper-stable .Moreover, if all systems (1) are regular, the charateristic spectrum of the sequence is also uniformly lower-stable and hence it is uniformly stable.*

Proof. Let $x(t)$ be a nontrivial solution of (3). According to the linear inclusion principle in [3], $x(t)$ is a nontrivial solution of linear system

$$\frac{dx}{dt} = A_n(t)x + \Phi_n(t)x. \tag{22}$$

If (4) hold, then

$$\|\Phi_n(t)\| \leq \delta, n = 1, 2, ... \tag{23}$$

Since the system (22) is linear we can apply the above lemma and then (23) gives us

$$\chi[x(.)] \leq \lambda_2^{(n)} + \epsilon$$

If (1) is regular, then we have

$$\chi[x(.)] \geq \lambda_1^{(n)} - \epsilon$$

Remark. *For $\lambda_1^{(n)} = \lambda_2^{(n)}$ this implies the result in [2].*

Now we shall study the uniform roughness of the following sequence of differetial equation system:

$$\frac{dx}{dt} = A_n(t)x, n = 1, 2, ..., \tag{24}$$

where, $A_n(t)$ is an $m \times m$ -matrix which is continuous and bounded on $[t_0, \infty)$.

Definition. *System (24) is said to be uniformly rough if there is a positive number δ, such that for every matrix $B_n(t)$ satisfying the relation:*

$$\|B_n(t)\| < \delta, n = 1, 2, ...$$

the systems

$$\frac{dx}{dt} = [A_n(t) + B_n(t)]x, n = 1, 2, ..., \tag{25}$$

have only nonzero charateristic exponents.

Let $\Lambda_n = \{\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq ... \leq \lambda_m^{(n)}\}$, $n = 1, 2, ...$, be the characteristic spectrum of (24). The following condition is necessary for the uniform roughness of system (24):

Proposition 1. *Suppose that system (24) is uniformly rough. Then there is an interval (α, β), containing zero, such that:*

$$(\alpha, \beta) \cap \Lambda_n = \emptyset$$

for every $n = 1, 2, ...$

Proof: We prove this by contradiction. Suppose there is a sequence of characteristic exponents $\{\lambda_{i_k}^{(k)} \in \Lambda_k\}$ $(1 \leq i_k \leq m)$ such that

$$\lim_{k \to \infty} \lambda_{i_k}^{(k)} = 0.$$

Consider the sequence of systems:

$$\frac{dx}{dt} = [A_k(t) - \lambda_{i_k}^{(k)} I]x, \tag{26}$$

where I is the unit matrix. For suitably large k $\|\lambda_{i_k}^{(k)} I\| < \delta$, but $\lambda_{i_k}^{(k)} - \lambda_{i_k}^{(k)} = 0$ is in the characteristic spectrum of (26). This contracts with uniform rough of (24). $\diamondsuit$

Definition of uniform stability of spectrum for sequence (24) is similar to the one in section 1: inequality (5) is changed by $\mu \leq \lambda_m^{(n)} + \epsilon$.

Proposition 2. . *Assumme that there is an interval (α, β) containing zero, such that*

either $(-\infty, \beta) \cap \Lambda_n = \emptyset$ $\quad n = 1, 2, ...$

or $(\alpha, +\infty) \cap \Lambda_n = \emptyset$ $\quad n = 1, 2, ...$

Moreover, suppose that the characteristic spectrum of (24) is uniformly stable. Then the above system is uniformly rough.

Proof:

The proof follows directly from its hypotheses and definition. $\diamondsuit$

Consider now the case $m = 2$. The proposition 2 and the proved theorem give us:

Corollary. *Suppose that there is an interval (α, β) containing zero and satisfying condition of proposition 2 for the case m=2. Moreover, suppose that:*

$$\int_{t_0}^{t} \sqrt{[a_{11}^{(n)}(\tau) - a_{22}^{(n)}(\tau)]^2 + [a_{21}^{(n)}(\tau) + a_{12}^{(n)}(\tau)]^2}.d\tau \leq c < \infty (n = 1, 2, ...)$$

Then the sequence of systems (1) is uniformly rough.

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TẠP CHÍ KHOA HỌC ĐHQGHN, KHTN, t.XV, n^{0}5 - 1999

SỰ ỔN ĐỊNH ĐỀU CỦA PHỔ ĐẶC TRƯNG CỦA DÃY HỆ PHƯƠNG TRÌNH VI PHÂN TUYẾN TÍNH

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Trong bài báo này, chúng tôi đưa ra một điều kiện ổn định của phổ đặc trưng của một dãy hệ phương trình vi phân tuyến tính. Điều kiện này được đặt lên các hệ số của hệ phương trình tương ứng. Kết quả nhận được được áp dụng cho việc nghiên cứu sự thô đều.