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ON THE MEAN ABSOLUTE DEVIATION OF THE RANDOM VARIABLES

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Abstract: *The main object of the study is a measure of dispersion, is named the Mean Absolute Deviation (or MAD, for short) of a random variable X,* $\delta_\mu(X) = E(| X - \mu |)$. *The basic properties of the MAD and some detailed computations on the MAD are established. We also focus on the applications of the MAD in the Limit Theorems, when the role of the standard deviation* $\sigma_\mu(X) = [E(X - \mu)^2]^{1/2}$ *is played by* $\delta_\mu(X)$.

1. INTRODUCTION

Let X be a random variable with finite mean $E(X) = \mu$. The standard deviation of X, denoted by $\sigma_\mu(X) = [E(X - \mu)^2]^{1/2}$, is very well-known in the probabilistic and statistical literature as a measure of dispersion. Especially, its widespread use has been presented in theory of limit theorems, in sampling theory, in the analysis of variance and statistical decision theory (see [1], [2], [3], [6] and [7] for complete bibliography).

On the other hand, although playing a dominant role in functional analysis, the mean absolute deviation (or MAD, for short) of X, denoted by $\delta_\mu(X) = E(| X - \mu |)$, has seen relatively few applications in probability and statistics. In the traditional terminology, $\delta_a(X)$ is said to be the first absolute moment of a random variable X (see [1], [2], [3] and [8] for the definition). Probably, their computational complexities are not convenient to use, especially when the random variables are discrete (see for instance Section 2).

However, from the inequality $\delta_\mu(X) \le \sigma_\mu(X)$ for an arbitrary random variable X (see Proposition 2.5), the question arises as to what happens if the role of the standard deviation $\sigma_\mu(X)$ is played by $\delta_\mu(X)$.

In recent years some results concerning the MAD have been investigated by Pham-Gia THU, Q. P. DUONG and Turkan N. ... in some topics of statistics, econometrics, reliability theory and Bayesian analysis (see [4], [5] and [6] for more details).

The main aim of this note is to present the basic properties of the MAD of a random variable about its mean $\delta_\mu(X)$ and applications in the limit theorems, when the role of the standard deviation $\sigma_\mu(X)$ is played by $\delta_\mu(X)$.

More specifically, in Section 2 we review some of main properties of the $\delta_\mu(X)$ and some illustrative computations on MADs are also presented in this Section. These results are received by using the Lemma 2.1 in Section 2 and they are independent with ones of Pham-Gia THU and Turkan N in [4], [5] and [6]. For making the important role and usefulness of the MADs more apparent, in the Section 3 we will consider some results concerning the limit behaviours of the Bernoulli and Poisson distributed random variables. In addition, some results on Weak Laws of Large Numbers, where the classical conditions are directly imposed on the $\delta_\mu(X)$, are also established.

It is worth pointing out that the received results from the Lemma 3.1 and Theorem 3.3 in last section only are reformulations of the well-known classic Weak Laws of Large Numbers (we refer the readers to [1], [2], [3], [7], [8] and [9]), but we did not really have to use the assumption on independence of the random variables. In addition, the existence of the first absolute moment (the mean absolute deviation) in replacing the second moment (the standard deviation) is to be in concord with some practical problems.

2. SOME COMPUTATIONS ON THE MADs

Throughout this paper we shall continue using the notations are as in [4], [5] and [6]. Let $\delta_a(X) = E(|X - a|)$ denote the MAD (about an arbitrary the point a) of a random variable X. When $a = E(X) = \mu$, we have the MAD about the mean, denoted by $\delta_\mu(X)$, and for $a = Md(X)$ (where $Md(X)$ denotes the median of X) we have the MAD about the median, denoted by $\delta_{Md(X)}$.

At first we will review some basic properties of the $\delta_\mu(X)$ and $\delta_{Md(X)}$.

Proposition 2.1

(i) $\delta_a(X) \geq 0$ *for any random variable* X *and for all points* $a \in \Re$.

(ii) $\delta_\mu(X + a) = \delta_\mu(X)$ *for all* $a \in \Re$.

(iii) $\delta_\mu(cX) = |c| \delta_\mu(X)$ *for any real* c.

(iv) $\delta_\mu(X + Y) \leq \delta_\mu(X) + \delta_\mu(Y)$ *for two arbitrary random variables* X *and* Y.

(v) *For an arbitrary random variable* X

$$\delta_{Md(X)} \leq \delta_\mu(X) \leq \sigma_\mu(X).$$

Proof. (i), (ii), (iii) and (iv) will be proved by using the direct computations from the definition of the $\delta_\mu(X)$. For getting (v) we first observe that the median minimizes the average absolute distance (see [1, p.201] for definition of the median), so we have

$$\delta_{Md(X)} \leq \delta_\mu(X)$$

for all X.

The second part of the inequality, $\delta_\mu(X) \leq \sigma_\mu(X)$, is obtained from the well-known Schwartz inequality (see [7] for more details). ◇

Lemma 2.1. *Let X be a random variable with the distribution function $F_X(x)$. Suppose that the mean $E(X) = \mu$ exits. Then*

$$\delta_\mu(X) = 2\int_{x\geq\mu}(x-\mu)dF_X(x) = 2\int_{x<\mu}(\mu-x)dF_X(x), \tag{1}$$

and for the random variable X is discrete with the distributions $p_k = P\{X = x_k\}; k \geq 1$,

$$\delta_\mu(X) = \sum_k |x_k - \mu| p_k = 2\sum_{k:x_k\geq\mu}(x_k-\mu)p_k = 2\sum_{k:x_k<\mu}(\mu - x_k)p_k. \tag{2}$$

Proof. By virtue of $\int_{-\infty}^{+\infty}(x-\mu)dF_X(x) = 0$ it follows that

$$\delta_\mu(X) = \int_{x\geq\mu}(x-\mu)dF_X(x) + \int_{x<\mu}(\mu-x)dF_X(x) = 2\int_{x\geq\mu}(x-\mu)dF_X(x).$$

In the same manner we can show that

$$\delta_\mu(X) = 2\int_{x<\mu}(\mu-x)dF_X(x).$$

Similar arguments apply to the case the random variable X is discrete with the distributions $p_k = P\{X = x_k\}, k \geq 1$,

$$\delta_\mu(X) = \sum_k |x_k - \mu| p_k = 2\sum_{k:x_k\geq\mu}(x_k-\mu)p_k = 2\sum_{k:x_k<\mu}(\mu - x_k)p_k.$$

and this finishes the proof. ◇

It is to be noticed that seemingly the formulae (1) and (2) are more easier to apply for computations on the MADs than well-known result which is due to Pham-Gia THU and Turkan N. in [5]. This is the main cause that we shall use the formulae (1) or (2) for computing the MADs in the next chapters. The results in next section have been obtained independently, if compare with ones in [4], [5] and [6].

Proposition 2.2. *Let X be a general uniform random variable (see the Table 1 in [6]) with the density function*

$$f(x;a,b) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in (a,b) \\ 0, & \text{if } x \notin (a,b) \end{cases}$$

and the mean $E(X) = \frac{a+b}{2}$. Then we have $\delta_{\frac{a+b}{2}}(X) = \frac{1}{4}(b-a)$.

Proof. By virtue of (1) it is obvious that

$$\delta_{\frac{a+b}{2}}(X) = \frac{b-a}{4}.$$

It is worth noticing that, if $b = \theta + a$ then $\delta_\mu(X) = \frac{\theta}{4}$ is same result in [6]. ◇

Proposition 2.3. *Let X be a power random variable with the density function (see the Table 1 in [6])*

$$f(x;\alpha) = \alpha x^{\alpha-1}, \alpha > 0, 0 \leq x \leq 1.$$

Then

$$\delta_\mu(X) = \frac{2}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

Proof. Applying the formula (1) with the expectation $E(X) = \int_0^1 xf(x;\alpha)dx = \frac{\alpha}{\alpha+1}$, to obtain that

$$\delta_\mu(X) = 2\int_0^{\frac{\alpha}{\alpha+1}} \left(\frac{\alpha}{\alpha+1} - x\right)\alpha x^{\alpha-1}dx = \frac{2}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

The proof is completed and the received result is as in [6] but by short way. ◇

Proposition 2.4. *Let X be an exponential random variable with the density function (see the Table 1 in [6])*

$$f(x;\lambda) = \lambda e^{-\lambda x}, \lambda > 0, x > 0.$$

Then

$$\delta_{\frac{1}{\lambda}}(X) = 2e^{-1}\lambda^{-1} = 2e^{-1}\sigma_{\frac{1}{\lambda}}(X).$$

Proof. We now apply the formula (1) again with the expectation $E(X) = \int_0^{+\infty} \lambda xe^{-\lambda x}dx = \frac{1}{\lambda}$ and the standard deviation $\sigma_{\frac{1}{\lambda}}(X) = \frac{1}{\lambda}$, to obtain that

$$\delta_{\frac{1}{\lambda}}(X) = 2\int_{\frac{1}{\lambda}}^{+\infty} \left(x - \frac{1}{\lambda}\right)\lambda x^2 e^{-\lambda x}dx = 2e^{-1}\lambda^{-1} = 2e^{-1}\sigma_{\frac{1}{\lambda}}(X).$$

The proof is completed and the received result is as in [6] and in [8, p.224]. ◇

Proposition 2.5. *Let X be a gamma distributed random variable with the density function (see Table 1 in [6], Model 6 or [4]),*

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{\frac{-x}{\beta}}}{\beta^\alpha\Gamma(\alpha)},$$

where $\alpha, \beta > 0, x \geq 0$ *and* $\Gamma(\alpha) = \int_0^{+\infty} e^{-x}x^{\alpha-1}dx$ *be the gamma function. Then*

$$\delta_\mu(X) = \frac{2\alpha^\alpha\beta}{e^\alpha\Gamma(\alpha)},$$

where $\mu = E(X) = \alpha\beta$.

Proof. It is clear that

$$\mu = E(X) = \frac{\beta}{\Gamma(\alpha)}\int_0^{+\infty}(\frac{x}{\beta})^{\alpha} e^{\frac{-x}{\beta}} d(\frac{x}{\beta}) = \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\beta.$$

Note that we have used $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Applying (1) and a direct computation shows that

$$\delta_{\mu}(X) = 2\int_0^{\alpha\beta}(\alpha\beta - x)\frac{x^{\alpha-1}e^{\frac{-x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}dx = \frac{2\beta\alpha^{\alpha}}{\Gamma(\alpha)e^{\alpha}}.$$

This completes the proof. ◊

Proposition 2.6. *Let X be a random variable of Pareto type I (see [6: Table 1] or [1; p. 275] for details) with the density function*

$$f(x; x_0, \alpha) = \alpha x_0^{\alpha} x^{-(\alpha+1)}, \alpha > 1, x \geq x_0 > 0.$$

Then

$$\delta_{\mu}(X) = 2x_0(\alpha-1)^{-1}(1-\alpha^{-1})^{\alpha-1},$$

where $\mu = E(X) = \frac{\alpha x_0}{\alpha-1}$.

Proof. It can be verified that

$$\mu = E(X) = \int_{x_0}^{+\infty} \alpha x_0^{\alpha} x^{-\alpha} dx = \frac{\alpha}{\alpha-1}x_0.$$

Taking (1) into account we get

$$\delta_{\mu}(X) = 2\int_{x_0}^{\frac{\alpha}{\alpha-1}x_0}(\frac{\alpha}{\alpha-1}x_0 - x)\alpha x_0^{\alpha} x^{-(\alpha+1)}dx = 2x_0(\alpha-1)^{-1}(1-\alpha^{-1})^{\alpha-1}.$$

The proof is straight-forward. ◊

Proposition 2.7. *Let X be a Poisson distributed random variable with the positive, integer-value mean $E(X) = n, n > 0$. Then*

$$\delta_n(X) = 2ne^{-n}\frac{n^n}{n!} \sim \sqrt{\frac{2}{\pi}n} = \sigma_n(X)\sqrt{\frac{2}{\pi}} \approx 0.79788\sigma_n(X),$$

where the sign $\sim$ is used to indicate that the ratio of the two sides tends to unity as $n \to +\infty$.

Proof. Note that, the variance $\sigma_n^2(X)$ also is the positive integer-value n.

By directly using the formula (2) from Lemma 2.1 we will show that

$$\delta_n(X) = 2\sum_{k=0}^{n}(n-k)e^{-n}\frac{n^k}{k!} = 2ne^{-n}\frac{n^n}{n!}.$$

Using the Stirling's formula $n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$, (see [2; p. 50] for detailed discussions), we get

$$\delta_n(X) = 2ne^{-n}\frac{n^n}{n!} \sim \sqrt{\frac{2}{\pi}n} = \sigma_n(X)\sqrt{\frac{2}{\pi}} \approx 0.79788\sigma_n(X),$$

where the sign $\sim$ is used to indicate that the ratio of the two sides tends to unity as $n \to +\infty$. ◇

For the general case, we note that as $n \to +\infty$,

$$E\left|\frac{X-\mu}{\sigma_\mu(X)}\right| \sim \sqrt{\frac{2}{\pi}},$$

where $\mu = E(X)$ is the mean of X and $\sigma_\mu^2(X)$ is the variance of X.

Proposition 2.8 (see [2, problem 35 p.226]). *Let S_n be the number of success in n Bernoulli trials with the mean $E(S_n) = np$ and the variance $\sigma_{np}^2(S_n) = npq, (0 < p < 1, p + q = 1)$. Then*

$$\delta_{np}(S_n) = E(| S_n - np |) \sim \sqrt{\frac{2npq}{\pi}} = \sqrt{\frac{2}{\pi}}\sigma_{np}(S_n).$$

Proof. A direct computation from the formula (2) of Lemma 2.1 shows that

$$\delta_{np}(S_n) = E(| S_n - np |) = 2\sum_{k=0}^{[np]}(np-k)C_n^k p^k q^{n-k} = 2kqC_n^k p^k q^{n-k},$$

where k is the integer number such that $np < k \leq np + 1$.

By continuity, using again the Stirling's formula, for sufficiently large n, we have

$$\delta_{np}(S_n) = E(| S_n - np |) \sim \sqrt{\frac{2npq}{\pi}} = \sqrt{\frac{2}{\pi}}\sigma_{np}(S_n).$$

This concludes the proof. ◇

The same conclusion can be drawn for this case, as $n \to +\infty$,

$$E\left|\frac{X-\mu}{\sigma_\mu(X)}\right| \sim \sqrt{\frac{2}{\pi}},$$

where $\mu = E(X)$ is the mean of X and $\sigma_\mu^2(X)$ is the variance of X.

Proposition 2.9. *Let X be a normal distributed random variable with the mean $E(X) = \mu$ and the variance $Var(X) = \sigma_\mu^2(X)$. Then*

$$\delta_\mu(X) = \sigma_\mu(X)\sqrt{\frac{2}{\pi}} \approx 0.79788\sigma_\mu(X).$$

Proof. By using (1) from **Lemma 2.1**, an easy computation shows that

$$\delta_\mu(X) = 2\int_\mu^{+\infty} (x-\mu)\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2} dx = \sigma_\mu(X)\sqrt{\frac{2}{\pi}} \approx 0.79788\sigma_\mu(X).$$

In the same manner as above we can see that,

$$E\left|\frac{X-\mu}{\sigma_\mu(X)}\right| = \sqrt{\frac{2}{\pi}},$$

as was to be shown. ◇

3. LIMIT THEOREMS

From above Propositions 2.8 and 2.7 we can now present the following results.

Theorem 3.1 *Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of identically independent binomial distributed random variables with the means $E(X_k) = p, (0 < p < 1)$ and the variance $Var(X_k) = pq, \forall k = 1, 2, \dots, n$. Set $S_n = \sum_{k=1}^n X_k$. Then*

$$E\left|\frac{S_n - E(S_n)}{\sigma(S_n)}\right| = E\left|\frac{S_n - np}{\sqrt{npq}}\right| \sim \sqrt{\frac{2}{\pi}},$$

where the sign $\sim$ is used to indicate that the ratio of the two sides tends to unity as $n \to +\infty$.

This gives

$$\sqrt{\frac{\pi}{2}} E\left|\frac{S_n - E(S_n)}{\sigma(S_n)}\right| \to 1 \quad as \quad n \to +\infty.$$

Proof. It is easily seen that S_n be a number of success of n first Bernoulli trials with $E(S_n) = np$ and $Var(S_n) = npq$. We now apply argument as in Proposition 2.8 again, with X replaced by S_n, to obtain complete proof. ◇

Theorem 3.2 *Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of identically independent Poisson distributed random variables with the means $E(X_k) = \lambda, (\lambda \in \mathcal{Z}^+)$ and the variance $Var(X_k) = \lambda, \forall k = 1, 2, \dots, n$. Set $S_n = \sum_{k=1}^n X_k$. Then*

$$E\left|\frac{S_n - E(S_n)}{\sigma(S_n)}\right| = E\left|\frac{S_n - n\lambda}{\sqrt{n\lambda}}\right| \sim \sqrt{\frac{2}{\pi}},$$

where the sign $\sim$ is used to indicate that the ratio of the two sides tends to unity as $n \to +\infty$. This shows that

$$\sqrt{\frac{\pi}{2}} E\left|\frac{S_n - n\lambda}{\sqrt{n\lambda}}\right| \to 1 \quad as \quad n \to +\infty.$$

Proof. It follows immediately that S_n be a random variable of the Poisson law with the parameter $n\lambda, \lambda > 0, E(S_n) = n\lambda, Var(S_n) = n\lambda$. Analysis similar to that in the proof of Proposition 2.7, with X replaced by S_n, we can finish the proof. ◇

Form now we will formulate some results concerning the weak laws of large numbers when the role of the standard deviation $\sigma_\mu(X) = [E(X-\mu)^2]^{1/2}$ are played by $\delta_\mu(X)$.

Note that the following results are the restatements of the well-known classic weak laws of large numbers (see [1], [2], [3], [7], [8] and [9] for the complete bibliography), but based on the properties of MADs, we did not really have to use the assumption that the random variables are independent.

Lemma 3.1 (Inequality of Chebyshev's style)
Let X be a random variable with finite $\delta_\mu(X)$. Then, for all $\epsilon > 0$

$$P\{| X - E(X) |\geq \epsilon\} \leq \frac{1}{\epsilon}\delta_\mu(X). \tag{3}$$

Proof. The proof is based on the following observation for all $\epsilon > 0$

$$\delta_\mu(X) \geq \int_{|x-\mu|\geq\epsilon} | x - \mu | \, dF_X(x) \geq \epsilon \int_{|x-\mu|\geq\epsilon} dF_X(x) = \epsilon P\{| x - \mu |\geq \epsilon\}.$$

The proof is completed. ◇

Theorem 3.3. (The Weak Law of Large Numbers for arbitrary random variables):
Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of identically random variables (are not necessary independent) with $\delta_\mu(X) < +\infty$. Then, for all $\epsilon > 0$ and $0 < \delta < 1$,

$$P\{| \frac{S_n}{n^{1+\delta}} - \frac{E(S_n)}{n^{1+\delta}} | \geq \epsilon\} \to 0 \qquad as \qquad n \to \infty,$$

where $S_n = \sum_{i=1}^n X_i$.

Proof. By virtue of inequality (3) and Proposition 2.5, for all $\epsilon > 0$ and $0 < \delta < 1$,

$$0 \leq P\{| S_n - E(S_n) |\geq n^{1+\delta}\epsilon\} \leq \frac{\delta_{\mu_n}(S_n)}{n^{1+\delta}\epsilon} \leq \frac{n\delta_\mu(X_1)}{n^{1+\delta}\epsilon} \leq \frac{\delta_\mu(X)}{n^\delta \epsilon}.$$

By getting $n \to +\infty$ we have the complete proof. ◇

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VỀ ĐỘ LỆCH TRUNG BÌNH TUYỆT ĐỐI CỦA CÁC BIẾN NGẪU NHIÊN

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Mục đích chính của bài báo này là nghiên cứu một số tính chất cơ bản của độ lệch tuyệt đối trung bình $\delta_\mu(X)$, thiết lập một số tính toán cụ thể liên quan tới độ lệch tuyệt đối trung bình của một số phân phối quen biết và bước đầu đề cập tới một số ứng dụng của độ lệch tuyệt đố trung bình $\delta_\mu(X)$ trong một số bài toán của lý thuyết xác suất và thống kê khi vai trò của độ lệch tiêu chuẩn $\sigma_\mu(X)$ được thay thế bởi $\delta_\mu(X)$.