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ON THE STABILITY OF A CHARACTERIZATION OF EXPONENTIAL DISTRIBUTION BY GEOMETRIC COMPOUNDING

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I. INTRODUCTION

Let $X_1, X_2, ...$ be nonnegative i.i.d.rv's (independent identically distributed random variables) with $F(x) = P(X_j < x)$, $\varphi(t) = Ee^{itX_j}$, $m = EX_j < +\infty$ and let V be independent of $X_j, j = 1, 2, ...$ with geometric distribution, i.e.,

$$P(V = k) = p.q^{k-1}, \ \ k = 1, 2, ..., (0 < p < 1, q = 1 - p).$$

The random variable $Z = X_1 + X_2 + ... + X_V$ is called geometric composed variable of $X_{j's}$.

The notation $G_p(x)$ will mean $P(pZ < x)$; the notation $\varphi_{pZ}(t)$ will mean Ee^{itpZ}. $F_0(x)$ and $\varphi_0(t)$ will denote distribution function and characteristic function respectively of the exponential distribution. In [4], Renyi characterized the exponential distribution proving the following assertions:

(i) $lim_{p\to 0}\overline{G}_p(x) = e^{-x}$; $(\overline{G}_p(x) = P(p.Z \geq x))$,

(ii) $\overline{G}_p(x) = \overline{F}(x) \Leftrightarrow F(x) = e^{-x}$; $(\overline{F}(x) = 1 - F(x))$.

In [2], we estimated the stable degree of this characterization in the case of the distribution function $F(x)$ being ϵ - exponential, i.e., $\exists\, T(\epsilon) > 0, T(\epsilon) \to 0$ when $\epsilon \to 0$, such that

$$| \varphi(t) - \varphi_0(t) | \leq \epsilon, \forall\, t : | t | \leq T(\epsilon) \qquad \text{(1a)}$$

with metrics

$$\lambda(F_1, F_2) = \min_{T>0} \max\{\max_{|t|\leq T} \frac{1}{2} \mid \varphi_1(t) - \varphi_2(t); \frac{1}{T}\},$$

$$\rho(F_1, F_2) = \sup_x \mid F_1(x) - F_2(x) \mid$$

for $F_1(x), F_2(x)$ are two distribution function and $\varphi_1(t), \varphi_2(t)$ are characteristic functions respectively.

In this paper we shall consider the stability of Renyi's characterization in the case of the distribution function $G_p(x)$ being ϵ- exponential, i.e., $\exists\ T(\epsilon) > 0, T(\epsilon) \to 0$ when $\epsilon \to 0$, such that

$$| \varphi_{pZ}(t) - \varphi_0(t) | \le \epsilon \ \ \forall t : | t | \le T(\epsilon), \tag{1b}$$

with the uniform metric ρ.

2. STABILITY THEOREMS

Lemma.

a) Z and p.Z have the finite moment of first degree

$$EZ = \frac{m}{p}, \quad E(pZ) = m. \tag{1}$$

b)

$$\varphi_{pZ}(t) = \frac{p.\varphi(p.t)}{1 - p.\varphi(p.t)}, \quad t \in R. \tag{2}$$

c)

$$| \varphi(t) - 1 | \ \le m. | t |, \ \forall t \in R. \tag{3}$$

Proof:

a) We have:

$$\begin{aligned} EZ &= E[E(Z|_V)] = E[(\sum_{k=1}^{\infty} E(Z|_{V=k}) I_{\{V=k\}}(\omega)] \\ &= \sum_{k=1}^{\infty} E[X_1 + ... + X_k].p.q^{k-1} = \sum_{k=1}^{\infty} m.k.p.q^{k-1} = mp \sum_{k=1}^{\infty} k.q^{k-1} \\ &= mp.\frac{d}{dq}(\frac{1}{1-q}) = mp\frac{1}{(1-q)^2} = \frac{m}{p}. \end{aligned}$$

Hence, we get (1).

b) By a similar way, it follows that

$$\begin{aligned} \varphi_Z(t) &= Ee^{itZ} = E[E(e^{itZ}|_V)] = E[\sum_{k=1}^{\infty} E(e^{itZ}|_{V=k}.I_{\{V=k\}}(\omega)] \\ &= \sum_{k=1}^{\infty} Ee^{it(X_1+...+X_k)}p.q^{k-1} = \sum_{k=1}^{\infty} \varphi^k(t).p.q^{k-1} = p\varphi(t).\sum_{k=0}^{\infty}[q.\varphi(t)]^k = \frac{p\varphi(t)}{1-q\varphi(t)}. \end{aligned}$$

Hence:

$$\varphi_{pZ}(t) = \varphi_Z(pt) = \frac{p\varphi(pt)}{1 - q\varphi(pt)}.$$

So, we get (2).

c) It is easily seen that

$$| e^{i\alpha} - 1 | \leq \alpha, \forall\, \alpha \in R. \tag{4}$$

Therefore, we get the following estimation:

$$\begin{aligned} | \varphi(t) - 1 | &= \left| \int_{-\infty}^{+\infty} (e^{itx} - 1) dF(x) \right| \leq \int_{-\infty}^{+\infty} | e^{itx} - 1 | \, dF(x) \\ &\leq | t | \int_{-\infty}^{+\infty} | x | \, dF(x) = m. | t |, \quad \forall\, t \in R, \end{aligned}$$

(where we used (4) with $\alpha = t.x$.) This completes the proof of the Lemma. ◇

Theorem 1. *If* $| \varphi_{pZ}(t) - \varphi_0(t) | < \epsilon$, $\forall\, t : | t | \leq T$ *(for* $0 < \epsilon < p/q$ *and for some* $T > 0$*), then we have*

$$| \varphi(t) - \varphi_0(t) | < \frac{\epsilon}{p - q.\epsilon}, \quad \forall\, t : | t | \leq p.T. \tag{5}$$

Proof:

Denote: $r(t) = \varphi_{pZ}(t) - \varphi_0(t)$.

It follows readily from the hypothesis that

$$\begin{aligned} | r(t) | &\leq \epsilon, \quad \forall\, t : | t | \leq T, \\ | r(t/p) | &\leq \epsilon, \quad \forall\, t : | t | \leq p.T. \end{aligned} \tag{6}$$

From (2) we have

$$\varphi(u) = \frac{\varphi_{pZ}(u/p)}{p + q\varphi_{pZ}(u/p)}, \quad \forall\, u \in R. \tag{7}$$

Therefore, from (7) we get the following estimations

$$\begin{aligned} | \varphi(u) - \varphi_0(u) | &= \left| \frac{\varphi_{pZ}(u/p)}{p + q\varphi_{pZ}(u/p)} - \frac{1}{1 - iu} \right| = \left| \frac{\varphi_{pZ}(u/p) - iu\varphi_{pZ}(u/p) - p - q\varphi_{pZ}(u/p)}{(1 - iu)(p + q\varphi_{pZ}(u/p))} \right| \\ &= \left| \frac{(p - iu)\varphi_{pZ}(u/p) - p}{(1 - iu)(p + q\varphi_{pZ}(u/p))} \right| = \left| \frac{[r(u/p) + \varphi_0(u/p)](p - iu) - p}{(1 - iu)(p + q\varphi_{pZ}(u/p))} \right| \\ &= \left| \frac{(p - iu).r(u/p)}{(1 - iu)(p + q\varphi_{pZ})(u/p))} \right| = \frac{| p - iu |}{| 1 - iu |} \cdot \frac{| r(u/p) |}{| p + q\varphi_{pZ}(u/p) |} \leq \frac{| r(u/p) |}{p + q\varphi_{pZ}(u/p)} \end{aligned}$$

Consequently,

$$| \varphi(u) - \varphi_0(u) | \leq \frac{| r(u/p) |}{| p + q\varphi_{pz}(u/p) |}, \quad \forall\, u \in R. \tag{8}$$

Notice that $| v | \geq \max\{| \Im m\, v |, | \Re e\, v |\}$ for any complex number $v \in C$. Hence,

$$
\begin{aligned}
&| p + q\varphi_{pz}(u/p) | = | p + q[r(u/p) + \varphi_0(u/p)] | = | p + q[r(u/p) + \frac{1}{1 - i.u/p}] | \\
&= \left| p + \frac{pq}{p - iu} + qr(u/p) \right| = \left| p + \frac{qp^2}{p^2 + u^2} + qr(u/p) + iu\frac{pq}{p^2 + u^2} \right| \\
&\geq \left| \Re e \, \{p + \frac{qp^2}{p^2 + u^2} + qr(u/p) + iu \cdot \frac{pq}{p^2 + u^2} \right| = \left| p + \frac{qp^2}{p^2 + u^2} + q.\Re e \, r(u/p) \right| \\
&\geq p + \frac{qp^2}{p^2 + u^2} - q. | \Re e \, r(u/p) | > p - q | \Re e \, r(u/p) | \geq p - q.\epsilon > 0 \quad \forall u : | u | \leq p.T.
\end{aligned}
$$

Consequently,

$$
| p + q\varphi_{pZ}(u/p) | > p - q\epsilon > 0, \quad \forall u : | u | \leq p.T \tag{9}
$$

By (8) and (9) we get

$$
| \varphi(u) - \varphi_0(u) | \leq \frac{| r(\frac{u}{p}) |}{| p + q\varphi_{pZ}(u/p) |} < \cdot \frac{\epsilon}{p - q\epsilon}, \quad \forall u : | u | \leq p.T.
$$

This completes the proof of Theorem 1. ◇

Theorem 2. *Assume that $G_p(x)$ is ϵ - exponential distribution function with the number $T(\epsilon)$ in (1b) satisfying the condition $T(\epsilon) = O(\epsilon^{-\alpha})$ for some $\alpha > 0$ (when $\epsilon \to 0$). Then we have*

$$
\rho(F, F_0) \leq K_1.\epsilon^\alpha - K_2\epsilon.\ln \epsilon, \tag{10}
$$

where $0 < \epsilon < \min\{1, \frac{p}{q}\}$ and $K_1 > 0, K_2 > 0$ are constant numbers independent of ϵ.

Proof:

At first, since $F_0(x)$ is exponential distribution function then $\sup | F_0'(x) | = 1$. Using Essen's inequality (see [3]) with $T = p.T(\epsilon)$ we get the following estimation:

$$
\begin{aligned}
\rho(F, F_0) &\leq \frac{1}{\pi} \int_{-p.T(\epsilon)}^{p.T(\epsilon)} \left| \frac{\varphi(t) - \varphi_0(t)}{t} \right| dt + \frac{24}{\pi p.T(\epsilon)}.sup_x \, | F_0'(x) | \cdot \\
&= \frac{1}{\pi} \int_{-p.T(\epsilon)}^{p.T(\epsilon)} \left| \frac{\varphi(t) - \varphi_0(t)}{t} \right| dt + \frac{24}{\pi p.T(\epsilon)} \leq \\
&\leq \frac{1}{\pi} \int_{|t| \leq \delta} \left| \frac{\varphi(t) - \varphi_0(t)}{t} \right| dt + \int_{\delta < |t| \leq p.T(\epsilon)} \left| \frac{\varphi(t) - \varphi_0(t)}{t} \right| dt + \frac{24}{\pi p.T(\epsilon)} \\
&\leq \frac{1}{\pi}[J_1 + J_2] + \frac{24}{\pi p.T(\epsilon)}, \quad \forall \delta : 0 < \delta < p.T(\epsilon).
\end{aligned} \tag{11}
$$

Where

$$
J_1 = \int_{|t| \leq \delta} \left| \frac{\varphi(t) - \varphi_0(t)}{t} \right| dt; \quad J_2 = \int_{\delta < |t| \leq p.T(\epsilon)} \left| \frac{\varphi(t) - \varphi_0(t)}{t} \right| dt.
$$

In order to estimate J_1, we put

$$J_1^* = \int_{|t|\leq\delta} \left|\frac{\varphi(t)-1}{t}\right| dt; \quad J_2^* = \int_{|t|\leq\delta} \left|\frac{\varphi_0(t)-1}{t}\right| dt.$$

Using Lemma above, it yields

$$J_1^* \leq m \int_{|t|\leq\delta} dt = 2\delta.m,$$

$$J_2^* \leq m_0. \int_{|t|\leq\delta} dt = 2\delta.m_0 = 2\delta \quad (m_0 = \int_{-\infty}^{+\infty} x dF_0(x) = 1).$$

Therefore,

$$J_1 \leq J_1^* + J_2^* \leq 2\delta(1+m). \tag{12}$$

On the other hand, according to Theorem 1, we have

$$J_2 \leq \frac{\epsilon}{p-q\epsilon} \int_{\delta<|t|\leq pT(\epsilon)} \frac{1}{|t|} dt = \frac{2\epsilon}{p-q\epsilon} \int_{\delta}^{p.T(\epsilon)} \frac{dt}{t} = \frac{2\epsilon}{p-q\epsilon}. \ln \frac{p.T(\epsilon)}{\delta}.$$

Choosing $\delta = \epsilon^\beta$ with $\beta > \max\{1, \alpha\}$, we get

$$J_2 \leq \frac{2\epsilon}{p-q\epsilon} \ln \frac{p.T(\epsilon)}{\epsilon^\beta}. \tag{13}$$

By (11), (12) and (13), it follows that

$$\rho(F, F_0) \leq \frac{2}{\pi}(1+m)\epsilon^\beta + \frac{24}{\pi p.T(\epsilon)} + \frac{2\epsilon}{(p-q\epsilon)\pi} \cdot \ln \frac{p.T(\epsilon)}{\epsilon^\beta}. \tag{14}$$

According to the hypothesis $T(\epsilon) = 0(\epsilon^{-\alpha})$ we have $T(\epsilon) = K(\epsilon).\epsilon^{-\alpha}$, where $K(\epsilon) = \frac{T(\epsilon)}{\epsilon^{-\alpha}} \to K > 0$ when $\epsilon \to 0$. Then, for sufficiently small ϵ we have

$$\begin{aligned} \epsilon \ln \frac{p.T(\epsilon)}{\epsilon^\beta} &= \epsilon \ln K(\epsilon) - (\alpha+\beta)\epsilon \ln \epsilon \leq -\epsilon \ln \epsilon - (\alpha+\beta)\epsilon \ln \epsilon \\ &\leq -(\alpha+\beta+1)\epsilon \ln \epsilon, \end{aligned} \tag{15}$$

$2/\pi(p-q\epsilon) < 3(\pi p)$ (for sufficiently small ϵ).

Since $K(\epsilon) \to K > 0(\epsilon \to 0)$ then $K(\epsilon) > \frac{K}{2}$ (for sufficiently small ϵ). Consequently,

$$\frac{24}{\pi p.T(\epsilon)} = \frac{24}{\pi p \cdot K(\epsilon)}.\epsilon^\alpha < \frac{48}{\pi p.K} \cdot \epsilon^\alpha. \tag{17}$$

By (14), (15), (16) and (17) we get the following estimation

$$\begin{aligned} \rho(F, F_0) &= \frac{2}{\pi}(1+m)\epsilon^\alpha + \frac{48}{\pi p.K}.\epsilon^\alpha - \frac{3(\alpha+\beta+1)}{\pi p} \cdot \epsilon \ln \epsilon \\ &= K_1.\epsilon^\alpha - K_2.\epsilon \ln, \epsilon, \end{aligned}$$

where $K_1 = \frac{2}{\pi}(1+m) + \frac{48}{\pi p.K} > 0$; $K_2 = \frac{3(\alpha+\beta+1)}{\pi p} > 0$.

The proof of Theorem 2 is completed. ◇

Remarks: The right hand side of (10), i.e the expression $K_1\epsilon^\alpha - K_2\epsilon.\ln\epsilon$

- tends to 0 when $\epsilon \to 0$,
- is the same degree as ϵ^α, if $0 < \alpha < 1$,
- is the same degree as $\epsilon\ln\epsilon$, if $\alpha \geq 1$.

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VỀ TÍNH ỔN ĐỊNH CỦA MỘT ĐẶC TRƯNG PHÂN PHỐI MŨ BỞI VIỆC PHỨC HỢP HÌNH HỌC

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Giả sử $X_1, X_2, \ldots$ là các biến ngẫu nhiên không âm, độc lập, cùng phân phối, có kỳ vọng hữu hạn và giả sử V là biến ngẫu nhiên độc lập với tất cả các biến $X_1, X_2, \ldots$ và có phân phối hình học tham số p

Biến $Z = X_1 + \ldots + X_V$ được gọi là biến phức hợp hình học của các biến $X_1, X_2, \ldots$

Renyi ([4]) đã chỉ ra phân phối mũ của pZ là đặc trưng cho việc các X_j có phân phối mũ.

Trong [2] tính ổn định của kết quả trên đã được xét khi X_1 có phân phối ϵ - mũ với các metric λ và ρ.

Trong bài báo này chúng tôi đã xét tính ổn định của kết quả trên của Renyi khi $p.Z$ có phân phối ϵ - mũ với metric ρ.