VNU JOURNAL OF SCIENCE, Nat., Sci., & Tech., T.XIX, N$_0$1, 2003

AN EVOLUTIONARY APPROACH TO FUZZY RELATION EQUATIONS WITH CONSTRAINTS

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Abstract. *Fuzzy relation equations play an important role in areas such as fuzzy system analysis, design of fuzzy controllers, and fuzzy pattern recognition. In this paper, we define the fuzzy relation equation with constraints and propose an evolutionary algorithm for determining an approximate solution of this equation.*

1. Introduction

The notion of fuzzy relation equation was first studied by Sanchez (1976). Since then, many further studies have been done by other researchers (see [5, 6, 7, 8]). Fuzzy relation equations play an important role in areas such as fuzzy system analysis, design of fuzzy controllers, decision making processes, and fuzzy pattern recognition.

The notion of fuzzy relation equations is associated with the concept of composition of fuzzy relations. Let A be a fuzzy set in the input space U and R be a fuzzy relation in the input -output product space UxV. The composition of fuzzy set A and fuzzy relation R, denoted by AoR, is defined as a fuzzy set B in the output space V.

$$AoR = B, \tag{1}$$

whose membership function is

$$\mu_B(y) = \max_{x \in U} \mu_A(x) * \mu_R(x, y), \tag{2}$$

where * is the t-norm operator. Because the t-norm can take a variety of formulas, for each t-norm we obtain a particular composition. The two most commonly used compositions in numerous applications are the so-called max-min composition and max-product composition, which are defined as follows:

- The max-min composition

$$\mu_B(y) = \max_{x \in U} \min[\mu_A(x), \mu_R(x, y)]$$

- The max-product composition

$$\mu_B(y) = \max_{x \in U} \mu_A(x).\mu_R(x, y).$$

The equation A o R = B is a so-called fuzzy relations equation. If we view R as a fuzzy system, then given a fuzzy set A to a fuzzy system R, we can compute the system's output B by (2). The two basis problems concerning the fuzzy relation equation are as follows:

- Problem P1: Given the input fuzzy set A in U and the output fuzzy set B in V, determine the fuzzy relation R such that A o R = B.
- Problem P2: Given the fuzzy relation R and the output B, determine the input A such that A o R = B.

Therefore, solving the fuzzy relation equation A o R = B means solving the above two problems. In this paper we are only interested in the problem P1. Since the solutions for the problem P1 may not exist, we first need to check the solvability of these equations or the existence of their solutions.

Theorem 1.1. Problem P1 has solutions if and only if the height of the fuzzy set A is greater than or equal to the height of the fuzzy set B, that is

$$\sup_{x \in U} \mu_A(x) \geq \mu_B(x) \qquad \text{for all } y \in B.$$

The proof of this theorem can see in [2].

In order to solve problem P1, one introduces the φ-operator. The φ-operator is an operator φ: [0,1] x [0,1] → [0,1] defined by

$$a \varphi b = \sup \{c \in [0,1] | a * c \leq b$$

where * denotes t-norm operator.

If the t-norm operator is specified as minimum, the φ-operator becomes the so-called α-operator:

$$a \alpha b = \begin{cases} 1 & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases}$$

For fuzzy sets A in U and B in V, using the φ-operator we can define the fuzzy relation R* in UxV which is defined as

$$\mu_R(x, y) = \mu_A(x) \varphi \mu_B(y).$$

We denote this fuzzy relation by AφB.

The following theorem is demonstrated (see [2,3]).

Theorem 1.2. If the solution of problem P1 exists, then the largest R (in the sense of fuzzy set theoretic inclusion) that satisfies the fuzzy relation equation AoR = B is $R^* = A\varphi B$.

However, in many cases, an exact solution of problem P1 may not exist. Therefore, $R^* = A\varphi B$ may not be solution. If an exact solution does not exist, what we can do is to determine approximate solutions. Wang L. X. proposed the method of determining approximate solution through neural network training (see [10]).

The further details of fuzzy relation equations can be found in [2, 3].

2. Fuzzy relation equations with constraints

The approximate reasoning in fuzzy systems is based on the rule of generalized Modus Ponens. This inference rule states that given two fuzzy propositions "if x is A then y is B" and "x is A'" we should infer a new proposition "y is B'" such that the closer the A' to A, the closer the B' to B, where A and A' are fuzzy sets in space U, B and B' are fuzzy sets in space V. The fuzzy proposition "if x is A then y is B" is interpreted as a fuzzy relation R in UxV. The fuzzy set B' in the conclusion of generalized Modus Ponens rule is determined as B' = A' o R. In the literature, many different interpretations of fuzzy if-then rules are proposed, for example, Lukasiewicz implication, Zadeh implication, Mamdani implication, etc. We wish determine the fuzz relation R interpreting fuzzy proposition "if x is A then y is B" such that the closer the A' to A, the closer the B' = A' o R to B.

The notion of fuzzy relation equation with constraints is stated as follows. Given the fuzzy sets A and A_i (i = 1, ...,k) in space U and the fuzzy set B in space V, we should determine a fuzzy relation R^* in product UxV such that the following requirements are satisfied:

$$A \circ R^* = B. \tag{3}$$

If we denote $A_i \circ R^* = B_i$ (i = 1,..., k) then

$$d(B_i, B) = \alpha\, d(A_i, A), \tag{4}$$

where α is constant, $\alpha>0$, and d(.,.) is the distance between fuzzy sets. The distance d(C, D) between the fuzzy set C and the fuzzy set D is defined as follows

$$d(C, D) = \left(\int |\mu_C(x) - \mu_D(y)|^p dx\right)^{1/p}, \qquad p \geq 1 .$$

For p = 1 one has the Hamming distance and p = 2 yields the Euclidean distance. In the cases the space U is finite we can simply define

$$d(C, D) = \sum_{x \in U} |\mu_C(x) - \mu_D(x)| .$$

Hence, our problem is to determine the fuzzy relation R^* which satisfies (3) and (4), given the fuzzy sets A, A_i (i = 1, ...,k) in space U and the fuzzy set in space V.

3. An evolutionary approach to fuzzy relation equations with constraints

It is very difficult to determine the exact solution of fuzzy relation equations with constraints. In this section, we propose an evolutionary scheme for determining the approximate solution of fuzzy relation equation with constraints by using an evolution strategy. Evolution strategies are algorithms which imitate the principles of natural evolution as method to solve parameter optimization problems (see [1, 4, 9]). We reformulate our problem in form of an optimization problem. Given the fuzzy sets A and A_i (i = 1, ..., k) in U and the fuzzy set B in V. Assume that R is a fuzzy relation in UxV. Denote

$$A \circ R = B',$$

$$A_i \circ R = B_i' \qquad (i = 1, \dots, k).$$

For each fuzzy relation R, we define a real value f(R) as follows

$$f(R) = d(B', B) + \sum_{i=1}^{k} \left| d(B_i', B) - \alpha\, d(A_i, A) \right|.$$

Our problem now is to determine the fuzzy relation R such that f(R) is minimum.

To apply the evolution strategy to the above problem, we first need to have suitable representations for fuzzy sets and fuzzy relations. Assume that the spaces U and V consist of finite number of elements, $U = \{u_1, \dots, u_m\}$, $V = \{v_1, \dots, v_n\}$. Then, each fuzzy set A in U is represented as a vector $A = (a_1, \dots, a_m)$, where a_i is membership degree of u_i to the fuzzy set A, that is $a_i = \mu_A(a_i)$. Analogically, the fuzzy set B in V has the representation $B = (b_1, \dots, b_n)$. Each fuzzy relation R is represented as a matrix of order mxn $R = (r_{ij})$, where $r_{ij} = \mu_R(u_i,v_j)$. Under these assumptions, when given the fuzzy sets A, A_i (i = 1, ..., k), B and the fuzzy relation R we can easily compute the fuzzy sets $B' = A \circ R$ and $B_i' = A_i \circ R$ (i = 1, ..., k), where we can employ the max- min composition or the max-product composition. Hence, we can compute the value of objective function f(R).

The idea of evolution strategy for our problem is as follows. Each individual is represented as a pair (R, Σ), where $R = (r_{ij})$ is a matrix of order mxn with $r_{ij} \in [0,1]$ (i = 1, ..., m; j = 1, ..., n), $\Sigma = (\sigma_{ij})$ is a (mxn)- matrix of standard deviations σ_{ij}

Each population consists of N individuals, all individuals in the population have the same mating probabilities. In each iterative step, two randomly selected parents:

$$(R_1, \Sigma_1) = \left((r^1_{ij}), (\sigma^1_{ij})\right)$$

and

$$(R_2, \Sigma_2) = \left((r^2_{ij}), (\sigma^2_{ij})\right)$$

produce an offspring

$$(R, \Sigma) = \left((r_{ij}), (\sigma_{ij})\right),$$

where $r_{ij} = r^1_{ij}$ ar $r_{ij} = r^2_{ij}$ with equal probability and if $r_{ij} = r^k_{ij}$ then $\sigma_{ij} = \sigma^k_{ij}$ $(k = 1,2)$.

The mutation operator is performed on the offspring (R, Σ) which as generated by the above crossover operator. Applying the mutation to the offspring (R, Σ), we obtain the new offspring (R', Σ):

$$R' = (r'_{ij}),$$

$$r'_{ij} = r_{ij} + N(0, \sigma_{ij}), \ (i = 1, \dots, m;\ j = 1, \dots, n),$$

where $N(0, \sigma_{ij})$ is a normally distributed random value with expectation zero and standard deviation σ_{ij}.

We now represent the scheme of evolutionary algorithm for determining the approximate solution of fuzzy relation equation with constraints.

Algorithm

1. Generate a population of N individuals (R, Σ), where $R = (r_{ij})$ is a matrix of order mxn, each r_{ij} is randomly taken from the interval [0, 1], $\Sigma = (\sigma_{ij})$ is a mxn – matrix of standard deviations.

2. (Iterative step)

- Randomly select two parents from N individuals

$$(R_1, \Sigma_1) = ((r^1_{ij}), (\sigma^1_{ij})) \text{ and}$$

$$(R_2, \Sigma_2) = ((r^2_{ij}), (\sigma^2_{ij})),$$

These parents produce an offspring

$$(R, \Sigma) = ((r_{ij}), (\sigma_{ij})),$$

where $r_{ij} = r^1_{ij}$ or $r_{ij} = r^2_{ij}$ with equal probability and if $r_{ij} = r^1_{ij}$ then $\sigma_{ij} = \sigma^1_{ij}$, if $r_{ij} = r^2_{ij}$ then $\sigma_{ij} = \sigma^2_{ij}$.

- Applying the mutation to the offspring (R, Σ), we obtain the new offspring (R', Σ)

$$R' = (r'_{ij}),$$

$$r'_{ij} = r_{ij} + N(0, \sigma_{ij}), \ (i = 1, \dots, m;\ j = 1, \dots, n),$$

where $N(0, \sigma_{ij})$ is a normally distributed random value with expectation zero and standard deviation σ_{ij}. If all r'_{ij} stay within the interval [0, 1], the new individual (R', Σ) is added to the population.

- Eliminate the weakest individual from N+1 individuals (original N individuals plus one offspring).

Conclusion

We have defined the notion of fuzzy relation equation with constraints, and proposed the evolutionary algorithm for determining an approximate solution of this equation. This evolutionary algorithm can be applied to determine the approximate solution of the problem P1 in case an exact solution of problem P1 does not exist.

References

1. Back T., Hoffmeister F, and Shwefel H. P. A survey of Evolution Strategies, *in Proceedings of the fourth International Conference on Genetic Algorithm*, Morgan Kangmann, Can Matco, 1991.
2. Chin- Teng Lin, C. S. George Lee, *Neural Fuzzy Systems*. Prentice- Hall, Inc., 1996.
3. Li- Xin Wang, *A course in Fuzzy Systems and Control*. Prentice- Hall, Inc., 1997.
4. Michalewicz Z, *Genetic Algorithms + Data Structures = Evolution Programs*, Springer, 1996.
5. Pedrycz W, Fuzzy relational equations with generalized connectives and their applications. *Fuzzy sets and Systems*, **10**(1983), 185-201.
6. Pedrycz W, s- t Fuzzy relational equations. *Fuzzy sets and Systems*, **59**(1993), 189- 196.
7. Sanchez E, Resolution of composite fuzz relation equations. *Information and Control*, **30**(1976), 38- 49.
8. Sanchez E, Solution of fuzzy equations with extended operators. *Fuzzy Sets and Systems*, **12**(1983), 237- 248.
9. Schwefel H. P, Evolution Strategies: A Family of Non- Linear OPtimization Techniques Based on Imitating Some Principles of Organic Evolution. *Annals of Operations Research*. Vol 1(1984), 165- 167.
10. Wang L. D, Solving fuzzy relational equations through network training. *Proc. 2nd IEEE Inter. Conf. on Fuzzy Systems*. San Francisco, 1993, 956- 960.

TẠP CHÍ KHOA HỌC ĐHQGHN, KHTN & CN, t.XIX, N$_0$1, 2003

MỘT GIẢI PHÁP TIẾN HOÁ CHO PHƯƠNG TRÌNH QUAN HỆ MỜ VỚI CÁC RÀNG BUỘC

Đinh Mạnh Tường

Khoa Công nghệ, ĐHQG Hà Nội

Khái niệm phương trình quan hệ lần đầu tiên được đề xuất và nghiên cứu bởi Sanchez (xem [7]). Phương trình quan hệ mờ đóng vai trò quan trọng trong nhiều lĩnh vực, chẳng hạn phân tích các hệ mờ, thiết kế các hệ điều khiển mờ, nhận dạng mẫu mờ. Trong bài báo này chúng tôi xác định khái niệm phương trình quan hệ mờ với các ràng buộc và đề xuất một thuật toán tiến hoá để tìm nghiệm xấp xỉ của phương trình này.