ON ALGEBRAIC CHARACTERIZATIONS OF GENERALIZED RIGHT INVERTIBLE OPERATORS IN LINEAR SPACES

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In this paper we introduce a class of generalized right invertible operators. This class cor all right invertible operators and some well-known operators in Analysis as Projection, Diffe ation, Difference and some classes of algebraic operators. We demonstrate that many fundam properties from the theory of right invertible operators can be extended to generalized rigl vertible operators as Taylor's and Taylor-Gontcharov's formulae and the we apply these resu solve some corresponding equations in generalized right invertible operators.

1. FUNDAMENTAL PROPERTIES OF GENERALIZED RIGHT INVERTIBLE OPERAT

Let X be a linear space over a field of scalars F. Denote by $L(X)$ the set of all linear oper with domains and ranges in X and write: $L_0(X) = \{A \in L(X) : \mathrm{dom} A = X\}$. The set of all invertible operators in $L(X)$ will be denoted by $R(X)$. For a $D \in R(X)$ we denote, respec by R_D and F_D the set of all right inverses and the set of all initial operators for D, i.e.

$$R_D = \{R \in L_0(X) : DR = I\},$$

$$F_D = \{F \in L_0(X) : F^2 = F,\ FX = \ker D\}.$$

Definition 1. An operator $V \in L(X)$ is said to be generalised invertible (GI-operator) if th a $W \in L_0(X)$ such that

$$VWV = V \text{ on } \mathrm{dom}\, V.$$

The set of all GI-operators in $L(X)$ will be denoted by $W(X)$. For a $V \in W(X)$ we d by $\mathcal{W}_V$ the set of all generalized inverses in $L_0(X)$ of V.

Definition 2. An operator $V \in W(X)$ is said to be possessing a right invertibility of deg ($r \in N$) (shortly: V is right invertible of degree r) it there is a $W \in \mathcal{W}_V$ such that

$$\mathrm{Im}\,(VW - I) \subset \ker V^r,$$

where we admit $V^0 = I$ for the case $r = 0$.

The set of all right invertible operators (in $L(X)$) of degree r will be denoted by $R_r(X)$. H by Definitions 1 and 2, it follows

$$R(X) = R_0(X) \subset R_1(X) \subset \cdots \subset R_n(X) \subset W(X);\ n = 0, 1, 2, \ldots$$

tion 3. Every $V \in R_1(X)$ is called a generalized right invertible operator (shortly: *GR*-ble operator). For a $V \in R_1(X)$ we denote by R_V^1 the set of all generalized right inverses y: *GR*-inverses) of V. Moreover, if there is $W \in R_V^1$ such that $\operatorname{Im} W \subset \ker (VW - I)$ then alled an almost right inverse of V. In that case, we write: $W \in R_V^{(1)}$.

sition 1. *Let $D \in R(X)$, $R \in R_D$ and let $V = R^m D^n$, where $n \geq m$, $n, m \in N$. Then* $_V^1$. *Moreover if $n \geq 2m$, then there is an almost right inverse of V.*

Write: $W_0 = R^{n-m}$, where we admit $R^0 = I$ for the case $n = m$. Since $R \in L_0(X)$ we de that $W_0 \in L_0(X)$. Using equalities $D^k R^k = I$, we find

$$V^2 W_0 = R^m D^n R^m D^n R^{n-m} = R^m D^{n-m} D^m = R^m D^n = V \tag{1}$$

$$V W_0 V = R^m D^n R^{n-m} R^m D^n = R^m D^n = V . \tag{2}$$

(1) we get $\operatorname{Im} (V W_0 - I) \subset \ker V$. This and (2) together imply that $V \in R_V^1$.

n the other hand, for the case $n \geq 2m$ we have

$$V W_0^2 = R^m D^n R^{n-m} R^{n-m} = R^m D^m R^{n-m} = R^m D^m R^m R^{n-2m} = W_0 ,$$

$W_0 \subset \ker (V W_0 - I)$. Hence $W_0 \in R_V^{(1)}$.

ple 1. Let $X = C([0, 1], F)$ and let $D = \frac{d}{dt}$, $(Rx)(t) = \int_{t_0}^t x(s)ds$ and $(Fx)(t) = x(t_0)$ $\in [0, 1]$. Write: $V = FD$, then $V \neq 0$ and $V^2 = 0$. It is easy to see that $V \in W(X)$. er, $V \notin R_r(X)$ for $r \in \{0, 1\}$. Indeed, $V \notin R(X)$, i.e. $V \notin R_0(X)$. Suppose that there $W \in L_0(X)$ such that $\operatorname{Im} (V W - I) \subset \ker V$. We find $V = V^2 W = 0$, which contradicts the ion $V \neq 0$.

sition 2. *Let $V \in R_1(X)$ and let $W \in R_V^1$. Then*

$$V^n W^m = \begin{cases} V^{n-1} & \text{if } n > m \geq 1, \\ VW & \text{if } n = m, \\ VW^{m-n+1} & \text{if } m > n \geq 1. \end{cases}$$

If $n = m \geq 2$, we find

$$V^m W^m = V^{n-2} (V^2 W) W^{n-1} = V^{n-1} W^{n-1} = \cdots = V^2 W^2 = (V^2 W) W = V W .$$

$m \geq 1$, we have the equalities

$$V^n W^m = V^{n-m} (V^m W^m) = V^{n-m} (V W) = V^{n-m-1} (V^2 W) = V^{n-m} .$$

inally, for $m > n \geq 1$, we get

$$V^n W^m = (V^n W^m) W^{m-n} = (V W) W^{m-n} = V W^{m-n-1} .$$

roof is complete.

sition 3. *Let $V \in R_1(X)$ and $W \in R_V^1$. Then $V^n \in R_1(X)$ for all $n \in N^+$ and $W^n \in R_{V^n}^1$.*

Proof. The assumptions and Proposition 2 together imply the following equalities

$$V^n = (V\,W\,V)\,V^{n-1} = V\,W\,V^n = V^2\,W\,W\,V^n = V^2\,W^2\,W^n = \cdots = V^n\,W^n\,V^n\,,$$

i.e. $V^n \in W(X)$. On the other hand, also by Proposition 2, we have

$$V^n = V^{n-1}\,V^2\,W = V^n\,(V\,W) = V^n\,(V^2\,W\,W) = V^n\,V^2\,W^2 = \cdots = V^n\,V^n\,W^n\,.$$

Hence $V^n\,(V^n\,W^n - I) = 0$, which shows that $V^n \in R_1(X)$ and $W^n \in R^1_{V^n}$.

Theorem 1. *Let $V \in R_1(X)$ and let $W_0 \in R^1_V$. Then $W \in L_0(X)$ is an GR-inverse of V if only if there is an $A \in L_0(X)$ such that $\operatorname{Im} A \subset \ker V^2$ and*

$$W = W_0 + A - W_0\,V\,A\,V\,W_0\,.$$

Proof. Let W is of the form (1), where $\operatorname{Im} A \subset \ker V^2$ and $W_0 \in R^1_V$. We have

$$V\,W\,V = V + V\,A\,V - V\,W_0\,V\,A\,V\,W_0\,V = V + V\,A\,V - V\,A\,V = V\,,$$

$$V^2\,W = V^2\,W_1 + V^2\,A - V^2\,W_1\,V\,A\,V\,W_1 = V - V^2\,A\,V\,A\,W_1 = V\,.$$

Equalities (5) and (6) together imply $W \in R^1_V$.

Conversely, let W_0, $W \in R^1_V$. Write: $A = W - W_0$. We find $V^2\,A = V^2\,W - V^2\,W$ $V - V = 0$. i.e, $\operatorname{Im} A \subset \ker V^2$. The equalities $V\,W\,V = V$ and $V\,W_0\,V = V$ together in $V\,(W - W_0)\,V = 0$. Hence, we have

$$W_0 + A - W_0\,V\,A\,V\,W_0 = W_0 + (W - W_0) - W_0\,V\,(W - W_0)\,V\,W_0 = W\,,$$

which gives the representation (1).

Theorem 2. *Let A, $B \in L_0(X)$ be given. Then $I + AB \in R_1(X)$ if and only if $I + BA \in R_1($ Moreover, if $W_{AB} \in R^1_{I+AB}$, then*

$$W = I - B\,W_{AB}\,A \in R^1_{I+BA}\,.$$

Proof. Let $I + AB \in R_1(X)$ and $W_{AB} \in R^1_{I+AB}$. Then $(I+AB)^2\,W_{AB} = (I + AB)$ and W def by the formula (1) are well-defined on X. We have the following equalities.

$$\begin{aligned}(I + BA)\,W\,(I + BA) &= (I + BA)\,(I - B\,W_{AB}\,A)\,(I + BA)\\ &= (I + BA)^2 - (I + BA)\,B\,W_{AB}\,A\,(I + BA)\\ &= (I + BA)^2 - B\,(I + AB)\,W_{AB}\,(I + AB)\,A\\ &= (I + AB)^2 - B\,(I + AB)\,A = I + BA\,,\end{aligned}$$

which show that $I + BA \in W(X)$.

On the other hand

$$\begin{aligned}(I + BA)^2\,W_{BA} &= (I + BA)^2\,(I - B\,W_{AB}\,A) = (I + BA)^2 - (I + BA)^2\,B\,W_{AB}\,A\\ &= (I + BA)^2 - B\,(I + AB)^2\,W_{AB}\,A = (I + BA)^2 - B\,(I + AB)\,A = I + B.\end{aligned}$$

ıply that $W \in R^1_{I+EA}$ and $I + BA \in R_1(X)$. The proof is complete.

tion 4. *Let $V \in R_1(X)$ and $W_0 \in R^1_V$. Write: $W_1 = W_0 V W_0$. Then $W_1 \in R^1_V$ and* $= W_1$.

he assumptions $V^2 W_0 = V$ and $V W_0 V = V$ together imply the following equalities

$$V^2 W_1 = V^2 W_0 V W_0 = V (V W_0 V) W_0 = V^2 W_0 = V ,$$

$$W_1 V W_1 = W_0 V W_0 V W_0 V W_0 = W_0 (V W_0 V) W_0 V W_0$$
$$= W_0 V W_0 V W_0 = W_0 V W_0 = V ,$$

ve $W_1 \in R^1_V$ and $W_1 V W_1 = W_1$.

he sequel, we write: $R_V^{(1,0)} = \{W \in R^1_V : W V W = W\}$.

2. RIGHT AND LEFT INITIAL OPERATORS FOR GENERALIZED RIGHT INVERTIBLE OPERATORS

$V \in R_1(X)$, $W \in R_V^{(1,0)}$ and let $\dim \ker V \neq 0$.

on 4. Let $V \in R_1(X)$ and let $W \in R_V^{(0,1)}$. An operator $F_r = F_W^{(r)} \in L_0(X)$ is said to be a tial operator (shortly: *RI* operator) for V corresponding to W if $F_r^2 = F_r$, $\mathrm{Im}\, F_r = \ker V$ $' = 0$ on X. The set of all RI-operators for $V \in R_1(X)$ will be denoted by $F_V^{(r)}$.

on 5. Let $V \in R_1(X)$ and $W \in R_V^{(1,0)}$. Then the operator $F_W^{(I)} = I - V W$ is said to be tial operator for V corresponding to W.

e that, if $V \in R(X)$ then $F_W^{(I)} = 0$ for all $W \in R_V$.

ition 5. *Let $V \in R_1(X)$ and Let $W \in R^1_V$. Then*

$$\mathrm{dom}\, V = W V\, (\mathrm{dom\ V}) \oplus \ker V\, .$$

Jote that for every $x \in \mathrm{dom}\, V$ we have $W V x \in W V\, (\mathrm{dom}\, V)$ and $(I - W V)\, x \in \ker V$ $W V x + (I - W V)\, x$. On the other hand, if $x \in W V\, (\mathrm{dom}\, V) \cap \ker V$, then there is V such that $x = W V v$ and $V x = 0$, simultaneously. Hence $0 = V x = V\, (W V)\, v = V v$ $W\, (V v) = 0$.

ition 6. *Let F be a RI-operator for $V \in R_1(X)$ corresponding to $W \in R_V^{(1,0)}$. Then*

$F v = v$ for all $v \in \ker V$.

$V F = 0$.

$X = \ker F \oplus \ker V$.

For every $v \in \ker V$, by Definition 4, there is $x \in \mathrm{dom} V$ such that $v = V x$. Hence $^2 x = V x = v$.

(ii) Also by Definition 4, for every $x \in X$, we have $F x \in \ker V$. Hence $V F x = 0$, i.e. on X.

(iii) It is easy to see that $F \in R_1(X)$ and $F \in R_F^1$. Hence, by Proposition 5, $X = F(F X) \oplus \ker F = \ker V \oplus \ker F$.

Theorem 3. *Let $V \in R_1(X)$ and let $W \in R_V^{(1,0)}$. Then $F \in L_0(X)$ is an RI-operato corresponding to W if and only if $\operatorname{Im} F \subset \ker V$ and $F = I - W V$ on* $\operatorname{dom} V$.

Proof. Let $F \in F_V^{(r)}$ be an RI-operator for V corresponding to $W \in R_V^{(1,0)}$. For any $x \in$ we find $V x = V W V x = V u$, where $u = W V s$. Hence $x - u \in \ker V$ and by Propositio get $(I - W V) x = x - u = F(x - u) - F x - F u = F x - F W V x = F x$ for $F W = 0$.

Conversely, if $F \in L_0(X)$ such that $\operatorname{Im} F \subset \ker V$ and $F = I - W V$ on $\operatorname{dom} V$, t any $z \in \ker V$ we get $F z = (I - W V) z = z - W V z = z$. Hence $F X = \ker V$ and fc $x \in X$, we find $F x = F(F x)$, i.e. $F^2 = F$. On the other hand, since $\operatorname{Im} W \subset \operatorname{dom} V$, ' $F W = (I - W V) W = W - W V W = 0$. Thus, $F \in F_V^{(r)}$.

Theorem 4 (Taylor-Gontcharov's formula). *Let $V \in R_1(X)$ and let $\{F_\gamma\}_{\gamma \in \Gamma}$ be a fc RI-operator for V corresponding to a family $\{W_\gamma\}_{\gamma \in \Gamma}$ of GB-inverses of V. Then*

$$I = F_{\gamma_0} + \sum_{k=0}^{N-1} W_{\gamma_0} \cdots W_{\gamma_{k-1}} F_{\gamma_k} V^k + W_{\gamma_0} \cdots W_{\gamma_N} V^N \quad on\ \operatorname{dom} V^N .$$

Proof. By induction, for $N = 1$ we have

$$I = F_{\gamma_0} - W_{\gamma_0} V + W_{\gamma_0} V = F_{\gamma_0} + W_{\gamma_0} V .$$

Suppose that (1) in valid for every $n \leq N$. Then for $n = N + 1$, we find

$$\begin{aligned} W_{\gamma_0} \cdots W_{\gamma_N} V^{N+1} &= W_{\gamma_0} \cdots W_{\gamma_{N-1}} (I - F_{\gamma_N}) V^N \\ &= W_{\gamma_0} \cdots W_{\gamma_{N-1}} V^N - W_{\gamma_0} \cdots W_{\gamma_{N-1}} F_{\gamma-N} V^N \\ &= I - F_{\gamma_0} - \sum_{k=0}^{N-1} W_{\gamma_0} \cdots W_{\gamma_{k-1}} F_{\gamma_k} V^k - W_{\gamma_0} \cdots W_{\gamma_{N-1}} F_{\gamma_N} V^N \\ &= I - F_{\gamma_0} - \sum_{k=0}^{N} W_{\gamma_0} \cdots W_{\gamma_{k-1}} F_{\gamma_k} V^k , \end{aligned}$$

which proves (1).

3. ON GENERALIZED RIGHT INVERTIBILITY OF ALGEBRAIC ELEMENTS

Let $F = C$. we say that $A \in L_0(X)$ is algebraic if there exists a non-zero normed pol $P(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ with coefficients in F such that $P(A) = 0$ on X. An a operator A is called of order n if there does not exist a normed polynomial $Q(t)$ of degree such that $Q(A) = 0$ on X. Such a minimal polynomial $P(t)$ is called the characteristic pol of A and is denoted by $P_A(t)$. The set of all algebraic operators in $L_0(X)$ will be denoted b

Let $F = C$ and let S be an algebraic operator in $L_0(X)$ with the characteristic polyn the form

$$P_S(t) = t^N + p_1 t^{N-1} + \cdots + p_{N-1} t + p_N ,$$

em 5. *Let S be an algebraic operator of order N in $L_0(X)$ with the characteristic polynomial of the form (9). Then $S \in R_1(X)$ if and only if $p_{N-1}^2 + p_N^2 \neq 0$.*

Let $S \in R_1(X)$ and let $W \in R_S^1$. Suppose that $p_{N-1} = 0$ and $P_N = 0$. Since $P_S(S) = 0$ $^2 W = S$, we have the following equalities

$$\begin{aligned} 0 = P_S(S)\, W &= (S^{N-2} + p_1 S^{N-3} + \cdots + p_{N-2} I)\, S^2 W \\ &= (S^{N-2} + p_1 S^{N-3} + \cdots + p_{N-2} I)\, S \\ &= S^{N-1} + p_1 S^{N-2} + \cdots + p_{N-2} S\,, \end{aligned}$$

contradicts the assumptions that S is of order N.

onversely, if $P_N \neq 0$, then S is invertible and it is right invertible and GR-invertible, simul-usly. We deal with the case when $p_N = 0$ and $p_{N-1} \neq 0$, simultaneously.

Vrite:

$$W := p_{N-1}^{-2} \Big(\sum_{k=0}^{N-2} p_{N-2}\, p_k\, S^{N-k-1} - \sum_{k=0}^{N-3} p_{N-1}\, p_k\, S^{N-k-2} \Big).$$

ow check that $W \in .R_S^1$. We have the following equalities

$$\begin{aligned} S\,W\,S &= p_{N-1}^{-2} \Big(\sum_{k=0}^{N-2} p_{N-2}\, p_k\, S^{N-k+1} - \sum_{k=0}^{N-3} p_{N-1}\, p_k\, S^{N-k} \Big) \\ &= p_{N-1}^{-2} \Big(p_{N-2}\, S \sum_{k=0}^{N-2} p_k\, S^{N-k} - p_{N-1} \sum_{k=0}^{N-3} p_k\, S^{N-k} \Big) \\ &= p_{N-1}^{-2} \Big(p_{N-2}\, S\, (P_S(S) - p_{N-1}\, S) - p_{N-1} \sum_{k=0}^{N-3} p_k\, S^{N-k} \Big) \\ &= p_{N-1}^{-2} \Big(- p_{N-1}\, p_{N-2}\, S^2 - p_{N-1} \sum_{k=0}^{N-3} p_k\, S^{N-k} \Big) \\ &= -p_{N-1}^{-1}\, (P_S(S) - p_{N-1}\, S) = S\,, \end{aligned}$$

$\in W(X)$ and $W \in \mathcal{W}_S$. On the other hand, we also have $S\,W = W\,S$, which gives $S^2 W = S$. roof is complete.

rem 6. *Let $S \in \overset{\circ}{A}(X) \cap R_1(X)$. . Then there is a unique $W \in R_S^{(1)}$.*

Let S be of the form (9). By Theorem 5, form the assumptions, we have $p_N^2 + p_{N-1}^2 \neq 0$ here is $W \in R_S^{(1)}$. If $p_N \neq 0$, then S is invertible. Let $W \in R_S^{(1)}$ be arbitrary. Since $= S$, we find $W = S^{-2} S^2 W = S^{-2} S = S^{-1}$. So that $W = S^{-1}$, i.e. S is uniquely nined. We now deal with the case $p_N = 0$ and $p_{N-1} \neq 0$, simultaneously. Write: $P(t) =$ $+ p_1 t^{N-3} + \cdots + p_{N-2}$. Then $S = -p_{N-1}^{-1}\, P(S)\, S^2$. Let $W \in R_S^{(1)}$ be arbitrary. We find

$$= S\,W^2 = -p_{N-1}^{-1}\, P(S)\, S^2 W^2 = -p_{N-1}^{-1}\, (S^{N-2} + p_1 S^{N-3} + \cdots + p_{N-3} S + p_{N-2}\, S\,W).$$

n the other hand, the equality $S^2 W = S$ follows $S\,W = -p_{N-1}^{-1}\, P(S)\, S$, which gives W of the form

$$W = -p_{N-1}^{-1}\, (S^{N-2} + p_1 S^{N-3} + \cdots + p_{N-3} S + p_{N-2}\, P(S)\, S)\,.$$

Thus, W is uniquely determined.

Corollary 2. *Let $S \in R_1(X)$ and let $W \in R_S^{(1)}$. Then $S \in \overset{\circ}{A}(X)$ if and only if $W \in \overset{\circ}{A}(X)$.*

Proof. Let $S \in \overset{\circ}{A}(X)$. Then by Theorem 6, W is uniquely determined as a polynomial in S coefficients in F. So that $W \in \overset{\circ}{A}(X)$ ([4]). Conversely, suppose that $W \in \overset{\circ}{A}(X)$ and P_W $t^M + a_1 t^{M-1} + \cdots + a_m$. Then we find $S^{M+1} P_W(W) = 0$ and $S + a_1 S^2 + \cdots + a_M S^{M+}$ which gives $S \in \overset{\circ}{A}(X)$

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ĐẶC TRƯNG ĐẠI SỐ CỦA TOÁN TỬ KHẢ NGHỊCH PHẢI SUY RỘNG TRONG KHÔNG GIAN TUYẾN TÍNH

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Bài báo đề cập đến một lớp các toán tử khả nghịch phải suy rộng. Lớp các toán tử nà gồm tất cả các toán tử khả nghịch phải, một số lớp toán tử quen biết trong giải tích nhı tử chiếu, toán tử vi phân, sai phân và một số dạng toán tử đại số. Các kết quả thu được tỏ rằng rất nhiều tính chất cơ bản của lý thuyết các toán tử đại số có thể mở rộng cho t hợp khả nghịch suy rộng như các công thức khai triển Taylor, Taylor - Goncharow, công thứ diễn nghiệm, ...